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PERIODIC SOLUTIONS OF SOME SEMILINEAR WAVE EQUATIONS ON BALLS AND ON SPHERES

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Dedicated to the memory of Karol Borsuk

1. Introduction

The existence of weak radially symmetric solutions for the problem

$$\begin{aligned} &u_{tt} - \Delta u + g(u) = f(t,x), && (t,x) \in \mathbb{R} \times B_a^n, \\ &u(t,x) = 0, && (t,x) \in \mathbb{R} \times S_a^{n-1}, \\ &u(t+T,x) = u(t,x), && (t,x) \in \mathbb{R} \times B_a^n, \end{aligned}$$

has been recently considered by Smiley [11], using the alternative method. Here,

$$\Delta = \Big(\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}\Big),$$
$$B_a^n = \{x \in \mathbb{R}^n,\ \|x\| < a\}$$

and

$$S_a^{n-1} = \{x \in \mathbb{R}^n,\ \|x\| = a\}, \qquad \text{with } \|x\| = \Big(\sum_{i=1}^{n} x_i^2\Big)^{1/2}.$$

In [11] the nonlinear term g is required to be Lipschitz continuous and strictly monotone. The ratio a/T is a rational number, and some restrictions are imposed on the Lipschitz and monotonicity constants. Those restrictions are not

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sharp, even in the special case $n = 1$, $T = 2\pi$, $2a = \pi$, which corresponds to the problem of a periodically forced semilinear vibrating string with an additionnal symmetry with respect to the mid-point of the string. In this situation however, sharp existence conditions are known (see e.g. [3] and its references) and a natural question is the possibility of extending those sharp existence conditions to the case where $n > 1$.

In doing so we have realized that the spectral properties of the linear wave operator with periodic-Dirichlet boundary conditions on the ball depended upon the parity of space dimension n, a feature overlooked in [11].

Indeed, in contrast to the case of odd n, this operator has a compact resolvent when n is even, and hence the nonlinear problem can be approached by techniques of nonlinear functional analysis relying upon compactness.

In particular the combination of sharp estimates for the inverse of the linear part and the Banach fixed point theorem can be replaced by the use of usual Leray-Schauder's degree argument.

As a consequence, when n is even, the monotonicity condition upon g can be dropped and we can deal with a more general nonlinear term $g(t, x, u)$ for which the nonresonance conditions need only to hold asymptotically and in a non-uniform way with respect to t and x. In the case of n odd, we use an existence theorem in [7] (for $n = 1$ or $n = 3$), and a recent existence result of the authors in [3] (for $n \geq 5$) to improve the results of Smiley [11] by providing sharp conditions on the monotonicity constants of $g(t, x, u)$, which insure the existence and uniqueness of the solution.

With the assumption of radial symmetry, the above problem can be written in the more general form, with $r = ||x||$,

$$
(1) \qquad \begin{aligned}
&u_{tt} - u_{rr} - \tfrac{1}{r}(n-1)u_r + g(t, r, u) = 0, && (t, r) \in]0, T[\times]0, a[, \\
&u(t, a) = 0, && t \in]0, T[, \\
&u(0, r) - u(T, r) = u_t(0, r) - u_t(T, r) = 0, && r \in]0, a[.
\end{aligned}
$$

By a solution of (1) we mean, as in [11], a *weak solution* in the following sense. Let D denote the class of radially symmetric functions $\phi \in C^\infty(\mathbb{R} \times B_a^n, \mathbb{R})$, which are $T-$periodic in time for each $x \in B_a^n$, and have compact support in B_a^n for each $t \in \mathbb{R}$.

Let H denote the space of functions $u : [0, T] \times B_a^n \to \mathbb{R}$, which are radially symmetric and belong to $L^2([0, T] \times B_a^n)$. Equipped with the usual L^2-norm and inner product $(\cdot, \cdot)$, H is a Hilbert space. We say that $u \in H$ is a *weak solution* of (1) provided that

$$
\int_0^T \int_0^a [u(\phi_{tt} - \phi_{rr} - \tfrac{1}{r}(n-1)\phi_r) + g(t, r, u)\phi]\, r^{n-1}\, dr\, dt = 0,
$$

for every $\phi \in D$.

To summarize the main results of the problem (1), which we state for $T = 2\pi$ and $2a = \pi$, let us recall that the spectrum of the associated linear problem

$$\begin{aligned} &u_{tt} - u_{rr} - \tfrac{1}{r}(n-1)\,u_r - \lambda u = 0, && (t,r) \in]0, 2\pi[\times]0, \tfrac{\pi}{2}[, \\ &u(t, \tfrac{\pi}{2}) = 0, && t \in]0, 2\pi[, \\ &u(0,r) - u(2\pi, r) = u_t(0,r) - u_t(2\pi, r) = 0, && r \in]0, \tfrac{\pi}{2}[\end{aligned}$$

is made of isolated eigenvalues, which accumulate only at $+\infty$ and $-\infty$.

Let $\lambda < \mu$ be two consecutive eigenvalues, $\alpha_\pm$ and $\beta_\pm$ be in $L^\infty(J)$, with $J =]0, 2\pi[\times]0, \frac{\pi}{2}[$, and such that

$$\begin{aligned} \lambda \le \alpha_+(t,r) \le \beta_+(t,r) \le \mu, \\ \lambda \le \alpha_-(t,r) \le \beta_-(t,r) \le \mu, \end{aligned}$$

a.e. on J, with

$$\int_J [(\alpha_+ - \lambda)(v^+)^2 + (\alpha_- - \lambda)(v^-)^2] > 0, \qquad \text{for all } v \in \ker(L - \lambda I)\backslash\{0\},$$

and

$$\int_J [(\mu - \beta_+)(w^+)^2 + (\mu - \beta_-)(w^-)^2] > 0, \qquad \text{for all } w \in \ker(L - \mu I)\backslash\{0\},$$

where L denotes the abstract realization in H of the radial symmetric wave operator with periodic-Dirichlet conditions on J, and

$$u^+ = \tfrac{1}{2}(|u| + u), \quad u^- = \tfrac{1}{2}(|u| - u).$$

Then, when n is even, the existence of a weak solution for (1) is insured when

$$\begin{aligned} \alpha_+(t,r) \le \liminf_{u\to+\infty} u^{-1}g(t,r,u) \le \limsup_{u\to+\infty} u^{-1}g(t,r,u) \le \beta_+(t,r), \\ \alpha_-(t,r) \le \liminf_{u\to-\infty} u^{-1}g(t,r,u) \le \limsup_{u\to-\infty} u^{-1}g(t,r,u) \le \beta_-(t,r) \end{aligned}$$

hold uniformly a.e. in $(t,r) \in J$ (Theorem 1). For $n = 1$ or $n = 3$, the existence holds when g satisfies the jumping nonlinearities above with λ and μ nonzero and $\operatorname{sign}\lambda \cdot g(t,r,\cdot)$ is nondecreasing (Theorem 2) although, for n odd ($n \ge 5$), the existence of an unique weak solution for (1) holds when g satisfies

$$\lambda < \beta_0 \le \frac{g(t,r,u) - g(t,r,v)}{u - v} \le \beta_1 < \mu,$$

a.e. in $(t,r) \in J$ and all $u, v \in \mathbb{R}$, with β_0 and β_1 two real constants.

The same arguments can be used to prove the existence of weak solution $u \in L^2(S^1 \times S^n)$ for the equation

$$u_{tt} - \Delta_n u = g(t,x,u), \qquad t \in S^1,\ x \in S^n,$$

where S^n is the n-dimensional sphere, Δ_n denotes the Laplace operator on S^n, and $g(t,x,u)$ is a continuous real function defined on $S^1 \times S^n \times \mathbb{R}$.

Indeed, the spectral properties of the linear spherical wave operator depend upon the parity of the space dimension n. If n is odd, the results are similar to those in the case $n = 1$, but if n is even the corresponding linear operator is Fredholm of index zero and has a compact resolvent.

The existence of nontrivial solutions of the equation above has been considered by Chang-Hong [4] and Benci-Fortunato [2] using variational methods.

We say that $u \in L^2(S^1 \times S^n)$ is a weak solution of the spherical wave equation above, provided that

$$\int_{S^1 \times S^n} u(\phi_{tt} - \Delta_n \phi) - g(t,x,u)\phi = 0,$$

for every $\phi \in C^2(S^1 \times S^n)$.

We recall in the Appendix the statements of the abstract existence theorems we use.

2. The linear eigenvalue problem

Let us first reproduce, for the reader's convenience, some interesting results of [11] on

$$(2)\quad \begin{aligned} &\phi_{tt} - \phi_{rr} - \tfrac{1}{r}(n-1)\phi_r = \lambda\phi, && (t,r) \in]0,T[\times]0,a[,\\ &\phi(t,a) = 0, && t \in]0,T[,\\ &\phi(0,r) - \phi(T,r) = \phi_t(0,r) - \phi_t(T,r) = 0, && r \in]0,a[. \end{aligned}$$

By a classical method of separation of variables, we set $\phi(t,r) = \tau(t)\rho(r)$ and derive that ρ must satisfy the equations

$$(3)\quad r^2\rho'' + (n-1)r\rho' + r^2\mu^2\rho = 0, \qquad 0 < r < a,$$

$$(4)\quad \rho(a) = 0, \qquad \rho \text{ bounded on [0,a]},$$

where $\mu^2 = \lambda + k^2$ for any integer $k \geq 0$, the corresponding functions τ_k being linear combinations of $\cos(2kt\pi/T)$ and $\sin(2kt\pi/T)$. The change of variables $\psi(r) = r^{(n-2)/2}\rho(r)$ transforms (3) (4) into

$$(5)\quad r^2\psi'' + r\psi' + [\mu^2 r^2 - \left(\tfrac{n-2}{2}\right)^2]\psi = 0, \qquad 0 < r < a,$$

$$(6)\quad \psi(a) = 0, \quad \psi(r) = 0, \qquad (r^{(n-2)/2}) \text{ as } r \to 0^+.$$

For the Bessel equation of order $\nu = (n-2)/2$, this is the classical eigenvalue problem. Let $J_\nu(x)$ denote the Bessel function of the first kind of order $\nu = (n-2)/2$ (cf. [1], [12]). Then $y = J_\nu(x)$ satisfies

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0, \qquad x > 0,$$

and $J_\nu(x) = 0(x^\nu)$ as $x \to 0^+$. It is well know that J_ν has an infinite sequence of distinct positive zeros $(x_{\nu,j})_{j=1}^\infty$ tending to infinity. The eigenvalues from problem (2) are seen to be $\mu_{n,j}^2 = (\alpha_{n,j}/a)^2$ where $\alpha_{n,j}$ is the j^{th} positive zero of J_ν with $\nu = (n-2)/2$.

The corresponding eigenfuctions are $\psi_{n,j}(r) = J_\nu(\alpha_{n,j}r/a)$. Hence problem (2) has eigenvalues and eigenfunctions

$$\lambda_{j,k}^n = \left(\frac{\alpha_{n,j}}{a}\right)^2 - \left(\frac{2k\pi}{T}\right)^2, \tag{7}$$
$$\phi_{j,k}^n(t,r) = \left\{\begin{matrix} \cos(2kt\pi/T) \\ \sin(2kt\pi/T) \end{matrix}\right\} r^{(2-n)/2} J_{(n-2)/2}(\alpha_{n,j}\, r/a),$$

for $k \geq 0$ and $j \geq 1$. It is clear that for each $n \geq 1$, the sequence $(\lambda_{j,k}^n)$ is unbounded from above and below.

Notice that for $n = 1$, the problem is equivalent to the periodic-Dirichlet problem on $[0,T] \times [-a,a]$ with an additional constraint of symmetry at the mid-point 0 of $[-a,a]$. Such a problem has been widely considered when $T = 2\pi$ and $a = \frac{\pi}{2}$ and we shall consider this case for arbitary values of n.

To motivate our results for n even, we first consider the more simple case where $n = 2$. Then the $\alpha_{2,j}$ are the zeros of $J_0(x)$. In particular it is known (cf. [1], [12]) that

$$(j - \tfrac{1}{4})\pi < \alpha_{2,j} < (j - \tfrac{1}{8})\pi, \qquad \text{for all } j \geq 1. \tag{8}$$

We have the following result.

LEMMA 1. *Suppose that $n = 2, T = 2\pi$, and $2a = \pi$. Then the eigenvalues $\lambda_{j,k}^2$ $(j \geq 1,\ k \geq 0)$ have finite multiplicity.*

PROOF. If λ is an eigenvalue,

$$\lambda = \left(\frac{2\alpha_{2,j}}{\pi}\right)^2 - k^2 \qquad (j \geq 1,\ k \geq 0)$$

and

$$|\lambda| = \left|\frac{2\alpha_{2,j}}{\pi} - k\right| \left|\frac{2\alpha_{2,j}}{\pi} + k\right|;$$

from (8) we see that $|2\alpha_{2,j}/\pi - k|$ is minimum for $k = 2j$. Thus,

$$-\frac{1}{2} < \frac{2\alpha_{2,j}}{\pi} - 2j < -\frac{1}{4};$$

so $|2\alpha_{2,j}/\pi - 2j| > \frac{1}{4}$ and $|\lambda| > \frac{1}{4}|2\alpha_{2,j}/\pi + k|$. We deduce that there are only finitely many pairs (j,k), $(j \geq 1,\ k \geq 0)$, which satisfy the last inequality and complete the proof. ☐

REMARK 1. (see [11]). If $n = 2$, $T = 2\pi$ and $2a = \pi$, we find that,

$$|\lambda^2_{j,k}| = \left(\frac{2}{\pi}\right)^2 \left|\alpha^2_{2,j} - \left(\frac{k\pi}{2}\right)^2\right| \geq \left(\frac{2}{\pi}\right)^2 \left(\frac{\pi}{8}\right) \alpha_{2,j} \geq \frac{1}{2\pi}\alpha_{2,1},$$

for all $j \geq 1$, $k \geq 0$, where $\alpha_{2,1}$ denote the first positive zero of $J_0(x)$, and hence 0 is not an eigenvalue in this case. When n is an arbitrary even integer, we have the same result as in the case $n = 2$.

LEMMA 2. *Suppose that $n > 2$, $T = 2\pi$, and $2a = \pi$. Then the eigenvalues $\lambda^n_{j,k}$, $(j \geq 1,\ k \geq 0)$ have finite multiplicity.*

PROOF. It is known from classical arguments [12] that for large j we have the asymptotic expansion

$$x_{\nu,j} \simeq b_{\nu,j} - \frac{(4\nu^2 - 1)}{8b_{\nu,j}} - \frac{(4\nu^2 - 1)(28\nu^2 - 31)}{384 b^3_{\nu,j}} - \cdots,$$

where $b_{\nu,j} = (j + \frac{1}{2}\nu - \frac{1}{4})\pi$, and $x_{\nu,j}$ is the j^{th} positive zero of $J_\nu(x)$ with $\nu = (n-2)/2$. If we put $\epsilon_{\nu,j} = (2/\pi)(b_{\nu,j} - x_{\nu,j})$, then, $\epsilon_{\nu,j} \to 0$ when $j \to +\infty$ (see [11]). Let j_0 be so large that $|\epsilon_{\nu,j}| < \delta$ for $j \geq j_0$ with $\delta \in]0, 1/2[$. Hence

$$2j + \nu - \tfrac{1}{2} - \delta < \frac{2x_{\nu,j}}{\pi} < 2j + \nu - \tfrac{1}{2} + \delta \tag{9}$$

and consequently, $|2x_{\nu,j}/\pi - k|$ is minimum for $k = 2j + \nu - 1$ or for $k = 2j + \nu$ (which are integers since n is even) and its minimum value is larger or equal to $(1/2 - \delta) > 0$. Thus if $\lambda = (2x_{\nu,j}/\pi - k)(2x_{\nu,j}/\pi + k)$, we deduce that $|\lambda| > (1/2 - \delta)|2x_{\nu,j}/2\pi + k|$ and there exist only finitely many pairs (j, k), $(j \geq j_0,\ k \geq 0)$, which satisfy the last inequality.

Now if $j < j_0$ and λ is an eigenvalue, we easily deduce that $k^2 = (2x_{\nu,j}/\pi)^2 - \lambda$ and there exist a finite number of k, which satisfy this equality since there is a finite number of zeros of $J_\nu(x)$ for $j < j_0$. □

Let us finally notice that when n is odd, there may be eigenvalues with infinite multiplicity. In particular, for $n = 1$ and $\nu = -1/2$ the positive zeros of

$$J_{-1/2}(x) = x^{-1/2} \cos x$$

are given by $\alpha_{1,j} = (2j - 1)\pi/2$, $j \in \mathbb{N}^*$. For $n = 3$ and $\nu = 1/2$ the positive zeros of

$$J_{1/2}(x) = x^{-1/2} \sin x$$

are given by $\alpha_{3,j} = j\pi$, $j \in \mathbb{N}^*$. Thus, for $T = 2\pi$ and $2a = \pi$ we have

$$\lambda^1_{j,k} = (2j - 1)^2 - k^2 \qquad \text{and} \qquad \lambda^3_{j,k} = 4j^2 - k^2,$$

which shows that, in both cases, 0 is an eigenvalue of infinite multiplicity.

3. Preliminary lemmas for the linear problem

Let $T = 2\pi$, $2a = \pi$, and the eigenfunctions in this case

$$\phi^n_{j,k}(t,r) = \begin{Bmatrix} \cos(kt) \\ \sin(kt) \end{Bmatrix} r^{(2-n)/2} J_{(n-2)/2}(2\alpha_{n,j}/\pi),$$

for $k \in \mathbb{N}$, $j \in \mathbb{N}^*$. Each $u \in H$ can be writen as the Fourier series

$$u \simeq \sum_{k\in\mathbb{N},\, j\in\mathbb{N}^*} u_{j,k}\phi^n_{j,k}(t,r),$$

where $u_{j,k} = (u, \phi^n_{j,k})$. We define the abstract realization L in H of the radial symmetric wave operator with the periodic-Dirichlet conditions on J as follows. Let

$$\operatorname{dom} L = \{u \in H;\ \sum_{k\in\mathbb{N},\, j\in\mathbb{N}^*} [(2\alpha_{n,j}/\pi)^2 - k^2]^2 \, |u_{j,k}|^2 < \infty\},$$

and

$$L : \operatorname{dom} L \to H, \quad u \to Lu = \sum_{k\in\mathbb{N},\, j\in\mathbb{N}^*} [(2\alpha_{n,j}/\pi)^2 - k^2] u_{j,k}\phi_{j,k}.$$

Then L is a self-adjoint operator in H, with spectrum

$$\sigma(L) = \{\lambda^n_{j,k} = (2\alpha_{n,j}/\pi)^2 - k^2,\ k \in \mathbb{N},\ j \in \mathbb{N}^*\}$$

made of isolated eigenvalues. We denote by K its right inverse.

Let H_1 (resp H_2) be the space spanned by the eigenfunctions of L associated with the eigenvalues smaller or equal to λ (resp. larger or equal to μ). For each $u \in H$, define the subsets $J_\pm$ of J by

$$J_+(u) = \{(t,r) \in J;\ u(t,r) > 0\},$$

$$J_-(u) = \{(t,r) \in J;\ u(t,r) < 0\},$$

and denote by $\chi_{J_\pm}$ the corresponding characteristic functions. If $p_+ \in L^\infty(J)$ and $p_- \in L^\infty(J)$, define the operator $A_p : H \to L^\infty(J)$ by

$$A_p(u) = p_+\chi_{J_+(u)} + p_-\chi_{J_-(u)} \tag{10}$$

and the operator $B_p : H \to H$ by

$$B_p(u)(t,r) = (A_p(u))(t,r)u(t,r) = [A_p(u)u](t,r). \tag{11}$$

If $u_1 \in H_1$, $u_2 \in H_2$, we have for all $u \in \operatorname{dom} L$

$$\begin{aligned}(Lu - B_p(u), u_2 - u_1) &= (L(u_2 + u_1) - A_p(u)(u_2 + u_1), u_2 - u_1) \\ &= (Lu_2 - A_p(u)u_2, u_2) - (Lu_1 - A_p(u)u_1, u_1)\end{aligned} \tag{12}$$

and for every $\lambda \geq 0$, $B_p(\lambda u) = \lambda B_p(u)$, so that B_p is positive homogeneous. Moreover, if S is a vector space of $\ker L$, and P_S the corresponding orthogonal projector, we shall denote by N_S the mapping defined on $H_S = S \oplus \operatorname{Im} L$ by

$N_S = Q_S N$, with $Q_S = P_S + Q$. The first lemma for linear problem goes as follows.

LEMMA 3. *Let $\alpha_\pm$ and $\beta_\pm$ be elements of $L^\infty(J)$ such that*

$$\lambda \le \alpha_+(t,r) \le \beta_+(t,r) \le \mu, \tag{13}$$

$$\lambda \le \alpha_-(t,r) \le \beta_-(t,r) \le \mu, \tag{14}$$

a.e. on J, and such that

$$\int_J [(\alpha_+ - \lambda)(v^+)^2 + (\alpha_- - \lambda)(v^-)^2] > 0, \qquad \forall v \in \ker(L - \lambda I)\backslash\{0\}, \tag{15}$$

and

$$\int_J [(\mu - \beta_+)(w^+)^2 + (\mu - \beta_-)(w^-)^2] > 0, \qquad \forall w \in \ker(L - \mu I)\backslash\{0\}. \tag{16}$$

Then, for each subspace $S \subset \ker L$, there exists $\epsilon > 0$, and $\delta > 0$ such that for all real measurable functions p_+ and p_- on J with

$$\alpha_+(t,r) - \epsilon \le p_+(t,r) \le \beta_+(t,r) + \epsilon,$$

$$\alpha_-(t,r) - \epsilon \le p_-(t,r) \le \beta_-(t,r) + \epsilon,$$

a.e. on J and for all $u \in \operatorname{dom} L \cap H_S$, one has $|Lu - B_{p,S}(u)| \ge \delta |u|$.

PROOF. If it is not the case, we can find $S \subset \ker L$, a sequence (u^m) in $\operatorname{dom} L \cap H_S$ with $|u^m| = 1$ and sequences (p_+^m) (p_-^m) in $L^\infty(J)$ with

$$\alpha_+(t,r) - 1/m \le p_+^m(t,r) \le \beta_+(t,r) + 1/m, \tag{17}$$

$$\alpha_-(t,r) - 1/m \le p_-^m(t,r) \le \beta_-(t,r) + 1/m, \tag{18}$$

a.e. on J and

$$|Lu^m - B_{p^m,S}(u^m)| \le 1/m \qquad (m = 1, 2, \dots), \tag{19}$$

that is,

$$Lu^m - B_{p^m,S}(u^m) = f^m \tag{20}$$

with $|f^m| \le 1/m$ and $|u^m| = 1$, $(m = 1, 2, \dots)$.

Writing $u^m = u_1^m + u_2^m$ with $u_1^m \in H_1$, $u_2^m \in H_2$, we have $u_1^m \in \operatorname{dom} L \cap H_1 \cap H_S$ and $u_2^m \in \operatorname{dom} L \cap H_2 \cap H_S$ for $m = 1, 2, \dots$, and, taking inner product with (20), we have

$$(Lu^m - B_{p^m,S}(u^m), u_2^m - u_1^m) = (f^m, u_2^m - u_1^m),$$

i.e. as Q_S is self-adjoint and $u_2^m - u_1^m \in H_S$,

$$(Lu^m - B_{p^m}(u^m), u_2^m - u_1^m) = (f^m, u_2^m - u_1^m),$$

or by (12),

$$(21)\quad (Lu_2^m - A_p^m(u^m)u_2^m, u_2^m) - (Lu_1^m - A_p^m(u^m)u_1^m, u_1^m) = (f^m, u_2^m - u_1^m).$$

Now let $\lambda' < \lambda < \mu < \mu'$ (with λ', λ and μ, μ' two pairs of consecutive eigenvalues) and $H_i = \tilde{H}_i \oplus \bar{H}_i$ $(i = 1, 2)$, where $\tilde{H}_1$ (resp. $\tilde{H}_2$) is spanned by the eigenfunctions assciated with eigenvalues smaller or equal to λ' (resp. larger or equal to μ') and $\bar{H}_1$ (resp. $\bar{H}_2$) is the finite dimensional space spanned by the eigenfunctions assciated with the eigenvalue λ (resp. μ). We will write,

$$u_i^m = \bar{u}_i^m + \tilde{u}_i^m, \quad \bar{u}_i^m \in \bar{H}_i,\ \tilde{u}_i^m \in \tilde{H}_i \qquad (i = 1, 2).$$

Using the fact that

$$\lambda - 1/m \le A_{p^m}(u^m)(t, r) \le \mu + 1/m \qquad (m = 1, 2, \dots),$$

we have

$$\begin{aligned}
(Lu_2^m &- A_{p^m}(u^m)u_2^m, u_2^m) - (Lu_1^m - A_{p^m}(u^m){u_1}^m, u_1^m) \\
&\ge (Lu_2^m - (\mu + 1/m)u_2^m, u_2^m) - (Lu_1^m - (\lambda - 1/m)u_1^m, u_1^m) \\
&= (L\tilde{u}_2^m - (\mu + 1/m)\tilde{u}_2^m, \tilde{u}_2^m) + (L\bar{u}_2^m - (\mu + 1/m)\bar{u}_2^m, \bar{u}_2^m) \\
&\qquad - (L\tilde{u}_1^m - (\lambda - 1/m)\tilde{u}_1^m, \tilde{u}_1^m) - (L\bar{u}_2^m - (\lambda - 1/m)\bar{u}_2^m, \bar{u}_2^m) \\
&= (L\tilde{u}_2^m - \mu\tilde{u}_2^m, \tilde{u}_2^m) - (L\tilde{u}_2^m - \lambda\tilde{u}_1^m, \tilde{u}_1^m) - 1/m \\
&= \sum_{\lambda_{j,k}^n \ge \mu'} [\lambda_{j,k}^n |u_{j,k}^m|^2 - \mu |u_{j,k}^m|^2] - \sum_{\lambda_{j,k}^n \le \lambda'} [\lambda_{j,k}^n |u_{j,k}^m|^2 - \lambda |u_{j,k}^m|^2] - 1/m \\
&\ge (\mu' - \mu)|\tilde{u}_2^m|^2 + (\lambda - \lambda')|\tilde{u}_1^m|^2 - 1/m.
\end{aligned}$$

From this relation and (21) we get

$$0 \le (\mu' - \mu)|\tilde{u}_2^m|^2 + (\lambda - \lambda')|\tilde{u}_1^m|^2 \le (1/m) + |f_m||u_2^m - u_1^m| \le 2/m, \quad (m = 1, 2, \dots)$$

and hence

$$(22)\qquad \tilde{u}_1^m \to 0, \quad \tilde{u}_2^m \to 0, \qquad \text{if } m \to \infty.$$

On other hand, as $|\bar{u}_1^m| \le 1$, $|\bar{u}_2^m| \le 1$, and $\bar{H}_1$ and $\bar{H}_2$ are finite dimensional, we can assume, going if necessary to a subsequence, that there exists $\bar{u}_1 \in \bar{H}_1$ and $\bar{u}_2 \in \bar{H}_2$ such that

$$(23)\qquad \bar{u}_1^m \to \bar{u}_1, \quad \bar{u}_2^m \to \bar{u}_2,$$

when $m \to \infty$. Thus $u_1^m \to \bar{u}_1$, $u_2^m \to \bar{u}_2$ for $m \to \infty$, so that

$$(24)\qquad |\bar{u}_1|^2 + |\bar{u}_2|^2 = 1.$$

On other hand,

$$
\begin{aligned}
(25)\qquad 1/m &\geq (f_m, u_2^m - u_1^m)\\
&= (Lu^m - B_{p^m}(u^m), u_2^m - u_1^m)\\
&= (Lu_2^m - A_{p^m}(u^m)u_2^m, u_2^m) - (Lu_1^m - A_{p^m}(u^m)u_1^m, u_1^m)\\
&= (L\tilde{u}_2^m - A_{p^m}(u^m)\tilde{u}_2^m, \tilde{u}_2^m) + (\mu - A_{p^m}(u^m)\bar{u}_2^m, \bar{u}_2^m)\\
&\qquad - 2(A_{p^m}(u^m)\bar{u}_2^m, \tilde{u}_2^m) - (L\tilde{u}_1^m - A_{p^m}(u^m)\tilde{u}_1^m, \tilde{u}_1^m)\\
&\qquad + ((A_{p^m}(u^m) - \lambda)\bar{u}_1^m, \bar{u}_1^m) + 2(A_{p^m}(u^m)\bar{u}_1^m, \tilde{u}_1^m),
\end{aligned}
$$

where $m = 1, 2, \ldots$. By (19) it follows that

$$|A_{p^m}(u^m)\bar{u}_i^m| \leq C, \qquad |A_{p^m}(u^m)\tilde{u}_i^m| \leq C, \qquad |L\tilde{u}_i^m| \leq C,$$

for some $C \geq 0$, $i = 1, 2$ and all $m = 1, 2, \ldots$.

Going if necessary to a subsequence, we can assume that there exists p_+ and p_- such that

$$\alpha_+(t,r) \leq p_+(t,r) \leq \beta_+(t,r), \quad \alpha_-(t,r) \leq p_-(t,r) \leq \beta_-(t,r)$$

and

$$p_+^m \rightharpoonup p_+, \quad p_-^m \rightharpoonup p_-, \tag{26}$$

as $m \to \infty$, and that

$$u^m \to \bar{u} = \bar{u}_1 + \bar{u}_2 \tag{27}$$

a.e. on J as $m \to \infty$. Consequently

$$\chi_{J_\pm(u^m)} \to \chi_{J_\pm(\bar{u})} \tag{28}$$

a.e. on J as $m \to \infty$. Moreover, we have

(29)

$$
\begin{aligned}
&\left|\int_J [p_\pm^m \chi_{J_\pm(u^m)}(\bar{u}_i^m)^2 - p_\pm \chi_{J_\pm(\bar{u})}(\bar{u}_i)^2]\right|\\
&\quad \leq \left|\int_J (p_\pm^m - p_\pm)\chi_{J_\pm(\bar{u})}(\bar{u}_i)^2\right| + \left|\int_J p_\pm^m(\chi_{J_\pm(u^m)}(\bar{u}_i^m)^2 - \chi_{J_\pm(\bar{u})}(\bar{u}_i^2))\right|\\
&\quad \leq \left|\int_J (p_\pm^m - p_\pm)\chi_{J_\pm(\bar{u})}(\bar{u}_i)^2\right| + C\int_J |\chi_{J_\pm(u^m)} - \chi_{J_\pm(\bar{u})}|\,(\bar{u}_i)^2\\
&\qquad\qquad + C\int_J |(\bar{u}_i^m)^2 - (\bar{u}_i)^2|
\end{aligned}
$$

for $i = 1, 2$ and all $k = 1, 2 \ldots$.

Consequently, letting $m \to \infty$ in (25) and using (26) to (29) we obtain

$$0 \geq \int_J (\mu - p_+\chi_{J_+(\bar{u})} - p_-\chi_{J_-(\bar{u})})(\bar{u}_2)^2 + \int_J (p_+\chi_{J_+(\bar{u})} + p_-\chi_{J_-(\bar{u})} - \lambda)(\bar{u}_1)^2 \geq 0.$$

Each integral being nonegative, this gives

$$\int_J (\mu - p_+\chi_{J_+(\bar{u})} - p_-\chi_{J_-(\bar{u})})(\bar{u}_2)^2 = 0, \tag{30}$$

$$\int_J (p_+\chi_{J_+(\bar{u})} + p_-\chi_{J_-(\bar{u})} - \lambda)(\bar{u}_1)^2 = 0. \tag{31}$$

Therefore, if $J_i = \{(t,r) \in J,\ \bar{u}_i(t,r) \neq 0\}$ $(i = 1,2)$, we have

$$\mu = p_+\chi_{J_+(\bar{u})} + p_-\chi_{J_-(\bar{u})} \qquad \text{a.e. on } J_2, \tag{32}$$

$$\lambda = p_+\chi_{J_+(\bar{u})} + p_-\chi_{J_-(\bar{u})} \qquad \text{a.e. on } J_1, \tag{33}$$

and hence $J_1 \cap J_2 = \emptyset$. If $J_1 = \emptyset$, then $\bar{u} = \bar{u}_2$ and (44) becomes

$$\int_J (\mu - p_+)(\bar{u}_2^+)^2 + (\mu - p_-)(\bar{u}_2^-)^2 = 0,$$

which implies

$$0 \geq \int_J (\mu - \beta_+)(\bar{u}_2^+)^2 + (\mu - \beta_-)(\bar{u}_2^-)^2,$$

and hence $\bar{u}_2 = 0$ by assumption (16), so that $\bar{u} = 0$, a contradiction with (24). If we now assume $J_1 \neq \emptyset$, then on J_1 we have (32) (33) and hence by (30) and the fact that the integrand functions in (30) and (31) are nonegative, we get

$$\int_{J_1} (\mu - \lambda)(\bar{u}_2)^2 = 0,$$

so that $\bar{u}_2 = 0$ a.e. on J_1. Consequently, by (31),

$$\begin{aligned} 0 &= \int_{J_1} (p_+\chi_{J_+(\bar{u})} + p_-\chi_{J_-(\bar{u})} - \lambda)(\bar{u}_1)^2 = \int_{J_1} (p_+\chi_{J_+(\bar{u}_1)} + p_-\chi_{J_-(\bar{u}_1)} - \lambda)(\bar{u}_1)^2 \\ &= \int_{J_1} (p_+ - \lambda)(\bar{u}_1^+)^2 + (p_- - \lambda)(\bar{u}_1^-)^2 = \int_J (p_+ - \lambda)(\bar{u}_1^+)^2 + (p_- - \lambda)(\bar{u}_1^-)^2 \\ &\geq \int_J (\alpha_+ - \lambda)(\bar{u}_1^+)^2 + (\alpha_- - \lambda)(\bar{u}_1^-)^2, \end{aligned}$$

so that $\bar{u}_1 = 0$ by assumption (15), a contradiction with $J_1 \neq \emptyset$. □

LEMMA 4. *Under the assumptions of Lemma 3, one has*

$$|D_{L_S}(L_S - B_{p,S}, B(\gamma))| = 1,$$

for every finite dimensional vector subspace $S \subset \ker L$, every open ball $B(\gamma)$ in H_S and every p_+ and p_- satisfying the conditions of Lemma 3.

PROOF. It is immediate that the mapping $B_{p,S}$ is L_S-completely continuous in H_S. It then follows from Lemma 3 and the homotopy invariance of degree that for each $\delta > 0$ one has

$$D_{L_S}(L_S - B_{p,S}, B(\gamma)) = D_{L_S}(L_S - B_{\beta,S}, B(\gamma)).$$

Let $b \in]\lambda, \mu[$; then, for every $s \in [0,1]$ and a.e. $(t,r) \in J$, one has

$$\lambda \leq (1-s)b + s\beta_{\pm}(t,r) \leq \mu,$$

and for every $v \in \ker(L - \lambda I)\backslash\{0\}$ we get

$$\int_J \{[(1-s)b + s\beta_+ - \lambda](v^+)^2 + [(1-s)b + s\beta_- - \lambda](v^-)^2\}$$
$$= (1-s)(b-\lambda)\int_J v^2 + s\int_J (\beta_+ - \lambda)(v^+)^2 + (\beta_- - \lambda)(v^-)^2 > 0.$$

Similary, we have

$$\int_J \{[\mu - (1-s)b - s\beta_+](w^+)^2 + [\mu - (1-s)b + s\beta_-](w^-)^2\} > 0,$$

for every $s \in [0,1]$ and every $w \in \ker\,(L - \mu I)\backslash\{0\}$.

Hence, by Lemma 3 applied with the common values $(1-s)b + s\beta_{\pm}$ for $\alpha_{\pm}, \beta_{\pm}$ and $p_{\pm}$, we see that equation

$$Lu - [(1-s)bu + sB_{\beta,S}(u)] = 0$$

has in $\operatorname{dom} L \cap H_S$ only the zero solution, which implies that for each $\delta > 0$ one has

$$D_{L_S}(L_S - B_{\beta,S}, B(\gamma)) = D_{L_S}(L_S - bI_S, B(\gamma)) = \pm 1.$$

□

4. The nonlinear problem in the case of even dimension

Now let $J = [0, 2\pi] \times [0, \pi/2]$ and let $g : J \times \mathbb{R} \to \mathbb{R}$ be a function such that $g(\cdot, \cdot, u)$ is measurable on J for each $u \in \mathbb{R}$, $g(t, r, \cdot)$ is continuous on $\mathbb{R}$ for a.e. $(t,r) \in J$. Moreover, assume that for each $\rho > 0$, there exists $h_\rho \in H$ such that

$$|g(t,r,u)| \leq h_\rho(t,r) \tag{34}$$

when $(t,r) \in J$ and $|u| \leq \rho$. We shall say that g satisfies the Caratheodory conditions for H.

Consider weak radially symmetric solutions of the semilinear wave problem

$$\begin{aligned} &u_{tt} - u_{rr} - \tfrac{n-1}{2}u_r - g(t,r,u) = 0, && (t,r) \in]0, 2\pi[\times]0, \pi/2[, \\ &u(t, \pi/2) = 0, && t \in]0, 2\pi[, \\ &u(0,r) - u(2\pi, r) = u_t(0,r) - u_t(2\pi, r) = 0, && r \in]0, \pi/2[. \end{aligned} \tag{35}$$

The function $u \in H$ is a weak solution of this problem, provided

$$\int_0^{2\pi}\int_0^{\pi/2} [u(\phi_{tt} - \phi_{rr} - \frac{n-1}{r}\phi_r) + g(t,r,u)\phi]r^{n-1}dr\,dt = 0,$$

for every $\phi \in D$. Then we have the following existence result.

THEOREM 1. *Let n be an even integer, $T = 2\pi$, $2a = \pi$, and let $\lambda < \mu$ be two consecutive eigenvalues of (2). Assume that g satisfies (34) and that the inequalities*

$$(36) \qquad \alpha_+(t,r) \le \liminf_{u\to+\infty} u^{-1}g(t,r,u) \le \limsup_{u\to+\infty} u^{-1}g(t,r,u) \le \beta_+(t,r),$$

$$(37) \qquad \alpha_-(t,r) \le \liminf_{u\to-\infty} u^{-1}g(t,r,u) \le \limsup_{u\to-\infty} u^{-1}g(t,r,u) \le \beta_-(t,r)$$

hold uniformly a.e. in $(t,r) \in J$, where $\alpha_\pm$ and $\beta_\pm$ are functions in $L^\infty(J)$ such that

$$(38) \qquad \lambda \le \alpha_+(t,r) \le \beta_+(t,r) \le \mu,$$

$$(39) \qquad \lambda \le \alpha_-(t,r) \le \beta_-(t,r) \le \mu.$$

Moreover, assume that

$$(40) \quad \int_J [(\alpha_+ - \lambda)(v^+)^2 + (\alpha_- - \lambda)(v^-)^2] > 0, \qquad \forall v \in \ker(L - \lambda I)\backslash\{0\},$$

and

$$(41) \quad \int_J [(\mu - \beta_+)(w^+)^2 + (\mu - \beta_-)(w^-)^2] > 0, \qquad \forall w \in \ker(L - \mu I)\backslash\{0\}.$$

Then problem (35) *has at least one weak solution.*

REMARK 2. Conditions of the form (36) to (39) (with $\alpha_+ = \alpha_- = \alpha$ and $\beta_+ = \beta_- = \beta$) were first introduced for elliptic Dirichlet problems in [10], and for semilinear wave equations in one-dimensional space variable in [9]. An abstract treatment is given in [5].

REMARK 3. If $\alpha_+ = \alpha_- = \alpha$ and $\beta_+ = \beta_- = \beta$, then (40) and (41) respectively become

$$\int_J (\alpha - \lambda)v^2 > 0, \qquad \text{for all } v \in \ker(L - \lambda I)\backslash\{0\}$$

and

$$\int_J (\mu - \beta)w^2 > 0, \qquad \text{for all } w \in \ker(L - \mu I)\backslash\{0\},$$

which is equivalent to $\alpha(t,x) > \lambda$ (resp. $\beta(t,x) < \mu$) on a subset of J of positive measure.

PROOF OF THEOREM 1. We now return to the periodic-Dirichlet problem on J for the semilinear radial symmetric wave problem (35) and following Mawhin-Ward [8], proceed to the proof of Theorem 1 stated above. Let $S = \ker L$, which is finite dimensional from Lemma 2 and $\delta > 0$, $\epsilon > 0$ be given by Lemma 3. By

(36) (37) we can find $\rho > 0$ such that, for a.e. $(t, r) \in J$ and all u with $|u| \geq \rho$, we have

$$\begin{aligned} \alpha_+(t,r) - \epsilon \leq u^{-1} g(t,r,u) \leq \beta_+(t,r) + \epsilon, \qquad & \text{if } u \geq \rho, \\ \alpha_-(t,r) - \epsilon \leq u^{-1} g(t,r,u) \leq \beta_-(t,r) + \epsilon, \qquad & \text{if } u \leq -\rho. \end{aligned}$$

This implies by (34) that

$$|g(t,r,u)| \leq (c+\epsilon)|u| + h_\rho(t,r),$$

for a.e $(t, r) \in J$ and all $u \in \mathbb{R}$, with $c = \mu$ if $\lambda \geq 0$ and $|\lambda|$ if $\mu \leq 0$. Consequently the mapping N defined on H by

$$N(u)(t,r) = g(t,r,u(t,r))$$

will map H continuously into itself and take bounded sets into bounded sets. Moreover, the weak radial symmetric solutions of the periodic-Dirichlet problem on J will be the solutions in dom L of the abstract equation in H

$$Lu - Nu = 0. \tag{42}$$

Clearly, from Lemma 2, L is a Fredholm operator of index zero and N is L-compact.

Define the nonlinear operator $B_\alpha : H \to H$ by

$$B_\alpha(u) = \alpha_+ u^+ - \alpha_- u^-.$$

It follows from Lemma 4 that

$$|D_L(L - B_\alpha, B(\gamma))| = 1$$

for every $\gamma > 0$. According to the Theorem A in the Appendix, equation (42) will have a solution if the set of possible solutions of the family of equations

$$Lu - (1-s)B_\alpha(u) - sN(u) = 0, \qquad s \in [0,1], \tag{43}$$

is a priori bounded independently of s.

Define f_+ on $J \times \mathbb{R}_+$ and f_- on $J \times \mathbb{R}_-$ by

$$f_+(t,r,u) = \begin{cases} u^{-1} g(t,r,u) & \text{if } u \geq \rho; \\ (1 - \frac{u}{\rho})\rho^{-1} g(t,r,\rho) + \frac{u}{\rho} u^{-1} g(t,r,u) & \text{if } 0 \leq u < \rho, \end{cases}$$

$$f_-(t,r,u) = \begin{cases} u^{-1} g(t,r,u) & \text{if } u \leq -\rho; \\ (1 + \frac{u}{\rho})(-\rho^{-1}) g(t,r,\rho) - \frac{u}{\rho} u^{-1} g(t,r,u) & \text{if } 0 \geq u > -\rho, \end{cases}$$

and e on $J \times \mathbb{R}$ by

$$\begin{aligned} e(t,r,0) &= g(t,r,0), \\ e(t,r,u) &= g(t,r,u) - f_+(t,r,u)u \qquad \text{if } u > 0, \\ e(t,r,u) &= g(t,r,u) - f_-(t,r,u)u \qquad \text{if } u < 0. \end{aligned}$$

Then e satisfies the Carathéodory conditions for H and

$$|e(t,r,u)| \leq 2h_\rho(t,r) \tag{44}$$

for a.e. $(t,r) \in J$ and all $u \in \mathbb{R}$. Notice that

$$\alpha_+(t,r) - \epsilon \leq f_+(t,r,u) \leq \beta_+(t,r) + \epsilon, \qquad \text{on } J \times \mathbb{R}_+,$$

and

$$\alpha_-(t,r) - \epsilon \leq f_-(t,r,u) \leq \beta_-(t,r) + \epsilon, \qquad \text{on } J \times \mathbb{R}_-,.$$

Moreover, for every $u \in H$,

$$g(t,r,u(t,r)) = f_+(t,r,u^+(t,r))u^+(t,r) - f_-(t,r,u^-(t,r))u^-(t,r) + e(t,r,u(t,r)).$$

Define $C_f : H \to L^\infty(J)$ by

$$C_f(u)(t,r) = f_+(t,r,u^+(t,r))\chi_{J_+(u)} + f_-(t,r,u^-(t,r))\chi_{J_-(u)},$$

and $D_f : H \to H$ by

$$D_f(u)(t,r) = [C_f(u)(t,r)]u(t,r) = (C_f(u)u)(t,r),$$

so that

$$D_f(u)(t,r) = f_+(t,r,u^+(t,r))u^+(t,r) - f_-(t,r,u^-(t,r))u^-(t,r),$$

and hence

$$Nu = D_f(u) + E(u) = C_f(u)u + E(u), \qquad \text{where } \begin{cases} E : H \to H, \\ u \to e(\cdot\,,\cdot\,,u(\cdot\,,\cdot)). \end{cases}$$

Thus equation (41) can be written as

$$Lu - (1-s)B_\alpha(u) + sD_f(u) - sE(u) = 0$$

or

$$Lu - [(1-s)A_\alpha(u) + sC_f(u)]u - sE(u) = 0, \qquad s \in [0,1]. \tag{45}$$

Let u be a possible solution of (43) (and then of (45)) for some $s \in [0,1]$. We have

$$\begin{aligned} (1-s)A_\alpha(u)(t,r) + sC_f(u)(t,r) = {} & [(1-s)\alpha_+(t,r) + sf_+(t,r,u^+(t,r)]\chi_{J_+(u)} \\ & + [(1-s)\alpha_-(t,r) + sf_-(t,r,u^-(t,r)]\chi_{J_-(u)} \end{aligned}$$

with

$$\alpha_+(t,r) - \epsilon \leq (1-s)\alpha_+(t,r) + sf_+(t,r,u^+(t,r)) \leq \beta_+(t,r) + \epsilon$$

$$\alpha_-(t,r) - \epsilon \leq (1-s)\alpha_-(t,r) + sf_-(t,r,u^-(t,r)) \leq \beta_-(t,r) + \epsilon.$$

Consequently, Lemma 3 and

$$p_\pm(t,r) = (1-s)\alpha_\pm(t,r) + sf_\pm(t,r,u^\pm(t,r))$$

imply, together with (44) and (45), that

$$\delta|u| \leq |Lu - [(1-s)A_\alpha(u) + sC_f(u)]u| = |sE(u)| \leq 2|h_\rho|,$$

i.e. $|u| \leq 2\delta^{-1}|h_\rho|$. □

REMARK 4. Since all properties on L are satisfied for $(-L)$, Theorem 1 holds true for the radially symmetric periodic-Dirichlet problem

$$\begin{aligned} &u_{tt} - u_{rr} - \tfrac{1}{2}(n-1)u_r + g(t,r,u) = 0, && (t,r) \in]0,2\pi[\times]0,\pi/2[\,, \\ &u(t,\pi/2) = 0, && t \in]0,2\pi[\,, \\ &u(0,r) - u(2\pi,r) = u_t(0,r) - u_t(2\pi,r) = 0, && r \in]0,\pi/2[\,, \end{aligned}$$

if $\lambda < \mu$ denote now the eigenvalues of the negative of the radial symmetric wave operator with the periodic-Dirichlet boundary conditions on $]0,2\pi[\times]0,\pi/2[$.

COROLLARY 1. *Let $g : J \times \mathbb{R} \to \mathbb{R}$ satisfy the Carathéodory conditions for H and be such that*

$$\alpha(t,r) \leq \frac{g(t,r,u) - g(t,r,v)}{u-v} \leq \beta(t,r) \tag{46}$$

for a.e. $(t,r) \in J$ and all $u \neq v \in \mathbb{R}$, with $\alpha_- = \alpha_+ = \alpha$ and $\beta_- = \beta_+ = \beta$ as in Theorem 1. Then the periodic-Dirichlet problem (35) *has a unique weak radially symmetric solution .*

PROOF. It follows from (46) that conditions (36) (37) hold. Thus the existence follows from Theorem 1. If, now u and v are solutions, then letting $w = u - v$, w will be a weak solution of the radially symmetric periodic-Dirichlet problem for equation

$$w_{tt} - w_{rr} - \tfrac{1}{r}(n-1)w_r - [g(t,r,v+w) - g(t,r,v)] = 0. \tag{47}$$

Setting

$$f(t,r,w) = \begin{cases} w^{-1}[g(t,r,v+w) - g(t,r,v)] & \text{if } w \neq 0; \\ \alpha(t,r) & \text{if } w = 0, \end{cases}$$

we see that (47) can be written as

$$w_{tt} - w_{rr} - \tfrac{1}{r}(n-1)w_r - f(t,r,w)w = 0 \tag{48}$$

with

$$\alpha(t,r) \le f(t,r,w) \le \beta(t,r)$$

for a.e. $(t,r) \in J$ and all $w \in \mathbb{R}$. Consequently, by Lemma 3, we easily see from (48) that $w = 0$, i.e. $u = v$. □

5. The nonlinear problem in the case of odd dimension

With the notations of section 4, we shall now study problem (35) when the space dimension n is odd. Then the zero eigenvalue of the linear part may have infinite multiplicity and a Leray-Schauder type approach is excluded. Using instead an existence theorem of [7] we shall be able to prove in the case $n = 1$ or $n = 3$ (for which the nonzero eigenvalues all have finite multiplicity) an existence theorem when g satisfies nonuniform nonresonance conditions.

THEOREM 2. *Let $n = 1$ or $n = 3, T = 2\pi, 2a = \pi$, and let $\lambda < \mu$ be two consecutive nonzero eigenvalues of* (2).

Assume that g satisfies (34), $\operatorname{sign}\lambda \cdot g(t,r,\cdot)$ *is nondecreasing and that the inequalities*

$$\alpha_-(t,r) \le \liminf_{u\to-\infty} u^{-1}g(t,r,u) \le \limsup_{u\to-\infty} u^{-1}g(t,r,u) \le \beta_-(t,r),$$
$$\alpha_+(t,r) \le \liminf_{u\to+\infty} u^{-1}g(t,r,u) \le \limsup_{u\to+\infty} u^{-1}g(t,r,u) \le \beta_+(t,r)$$

hold uniformly a.e. in $(t,r) \in J$ where $\alpha_\pm$ and $\beta_\pm$ are functions in $L^\infty(J)$ satisfying

$$\lambda \le \alpha_-(t,r) \le \beta_-(t,r) \le \mu$$
$$\lambda \le \alpha_+(t,r) \le \beta_+(t,r) \le \mu.$$

Moreover, assume that

$$\int_J [(\alpha_+ - \lambda)(v^+)^2 + (\alpha_- - \lambda)(v^-)^2] > 0, \qquad \textit{for all } v \in \ker(L-\lambda I)\setminus\{0\},$$

and

$$\int_J [(\mu - \beta_+)(w^+)^2 + (\mu - \beta_-)(w^-)^2] > 0, \qquad \textit{for all } w \in \ker(L-\mu I)\setminus\{0\}.$$

Then the radial symmetric periodic-Dirichlet problem (35) *has at least one weak solution.*

PROOF. The proof is also based on the two Lemmas 3 and 4. Let $\delta > 0$ and $\epsilon > 0$ be given by Lemma 3. We can find $\rho > 0$ such that for a.e. $(t,r) \in J$,

$$\alpha_+(t,r) - \epsilon \le u^{-1}g(t,r,u) \le \beta_+(t,r) + \epsilon, \qquad \text{if } u \ge \rho,$$
$$\alpha_-(t,r) - \epsilon \le u^{-1}g(t,r,u) \le \beta_-(t,r) + \epsilon, \qquad \text{if } u \le -\rho.$$

This implies by (34) that

$$|g(t,r,u)| \leq (C+\epsilon)|u| + h_\rho(t,r)$$

for a.e. $(t,r) \in J$ and all $u \in \mathbb{R}$, with $C = \mu$ if $\lambda > 0$ and $|\lambda|$ if $\mu < 0$. Consequently, the mapping N defined on H by

$$(Nu)(t,r) = g(t,r,u(t,r))$$

will map H continuously into itself and takes bounded sets into bounded sets. Moreover, the weak solutions of the radially symmetric periodic-Dirichlet problem (35) will be the solutions in dom L of the abstract equation in H

$$Lu - Nu = 0. \tag{49}$$

Without loss of generality, we can assume from now that $\lambda > 0$. Our assumption on g implies that N is monotone in H. As the right inverse K of L is compact, we see that KQN is compact on bounded sets on H. The (nonlinear) operator B_α defined by

$$B_\alpha(u) = \alpha_+ u^+ - \alpha_- u^-$$

is continuous, takes bounded sets into bounded sets and is such that

$$(B_\alpha(u) - B_\alpha(v), u-v) \geq \lambda |u-v|^2,$$

for all $u, v \in H$ as is easily checked. Thus KQB_α is compact on bounded sets and B_α is strongly monotone. It follows from Lemma 4, that

$$|D_{L_S}(L_S - B_{\alpha,S}, B(\gamma))| = 1$$

for every $\gamma > 0$ and every finite dimensional vector subspace S of ker L. According to the Theorem B in the Appendix, equation (49) will have a solution if the set of possible solutions of the family of equations

$$Lu - (1-s)B_\alpha(u) - sN(u) = 0, \qquad s \in [0,1],$$

is a priori bounded independently of s, which can be obtained as in Theorem 1. $\square$

REMARK 5. If $\lambda < \mu$ denote now the eigenvalues of the negative of the radial symmetric operator with the periodic-Dirichlet boundary conditions on J, then Theorem 2 holds true for the problem in Remark 3. In particular, with the notation of Smiley [11] and $g(t,r,u)$ of the form $g(u) - h(t,r)$, the conditions

$$|g(u) - g(v)| \leq \beta_1 |u-v|, \quad (g(u)-g(v))(u-v) \geq \beta_0 |u-v|^2,$$

with $0 < \beta_0 \leq \beta_1$, and $\beta_1^2 < 3\beta_0$ if $n = 1$ and $\beta_1^2 < 5\beta_0$ if $n = 3$, are replaced, for the existence, by g continuous and

$$0 < \beta_0 \leq \liminf_{|u|\to\infty} u^{-1} g(u) \leq \limsup_{|u|\to\infty} u^{-1} g(u) \leq \beta_1,$$

with $\beta_1 < 3$ if $n = 1$ and $\beta_1 < 5$ if $n = 3$.

COROLLARY 2. *Let $g : J \times \mathbb{R} \to \mathbb{R}$ satisfy the Carathéodory conditions for H and be such that*

$$\alpha(t,r) \le \frac{g(t,r,u) - g(t,r,v)}{u - v} \le \beta(t,r) \tag{50}$$

for a.e. $(t,r) \in J$ and all $u \neq v \in J$, with $\alpha_- = \alpha_+ = \alpha$ and $\beta_- = \beta_+ = \beta$ as in Theorem 2. Then the problem (35) has a unique solution.

PROOF. It follows from (50), that conditions (36) (37) holds with $\alpha_- = \alpha_+ = \alpha$ and $\beta_- = \beta_+ = \beta$, and that sign $\lambda \cdot g(t,r,\cdot\,)$ is nondecreasing for a.e. $(t,r) \in J$. Thus the existence follows from Theorem 2. Following the proof of Corollary 1 and using Lemma 3, we easily obtain the uniqueness. □

REMARK 6. Corollary 2 also holds for the problem in Remark 3, and in particular, with Smiley's notations [11] and $g(t,r,u)$ of the form $g(u) - h(t,r)$, uniqueness will hold when

$$0 < \beta_0 \le \frac{g(u) - g(v)}{u - v} \le \beta_1$$

with $\beta_1 < 3$ if $n = 1$ and $\beta_1 < 5$ if $n = 3$. In contrast to Smiley's one, our existence and uniqueness conditions are sharp.

We consider finally the problem (35), with a/T a rational number, and $n \ge 5$ and odd. Let β_0 and β_1 be two real constants, and g satisfy

$$\lambda < \beta_0 \le \frac{g(t,r,u) - g(t,r,u)}{u - v} \le \beta_1 < \mu, \tag{51}$$

for all $u \neq v \in \mathbb{R}$ and a.e. $(t,r) \in J$. We can suppose that $\lambda \ge 0$ (see Remark 7 below). It is clear, that a solution of the abstract equation in H

$$Lu - Nu = 0 \tag{52}$$

will be a solution of problem (35).

Let $L_\lambda = -L + \lambda I$ and $N_\lambda = N - \lambda I$. Then, equation (52) is equivalent to

$$L_\lambda u + N_\lambda u = 0. \tag{53}$$

From (51) we have

$$\beta_0(L_\lambda)|u - v|^2 \le (N_\lambda u - N_\lambda v, u - v) \le \beta_1(L_\lambda)|u - v|^2,$$

with

$$\beta_0(L_\lambda) = \beta_0 - \lambda > 0,$$
$$\beta_1(L_\lambda) = \beta_1 - \lambda < \mu - \lambda.$$

On the other hand, if we remark that

$$d_0^-(L_\lambda) = \operatorname{dist}(0, \sigma(L_\lambda) \cap \mathbb{R}_0^-) = \mu - \lambda$$

and

$$\beta_1(L_\lambda) = \beta_1 - \lambda < \mu - \lambda,$$

we can conclude, from Theorem C in the Appendix, that there exists a unique solution of equation (53) (hence of problem (35)), under the sole hypothesis (51), which improves the results of Smiley [11].

REMARK 7. If the hypothesis (51) holds with $\mu \leq 0$, we consider the equation

$$L_\mu u + N_\mu u = 0 \tag{54}$$

with $L_\mu = L - \mu I$ and $N_\mu = -N - \mu I$ and proceed as in the previous case.

REMARK 8. Under the hypothesis (51), the result holds true for the problem in Remark 3. It suffices to consider the equation (53) with $L_\lambda = L + \lambda I$ and $N_\lambda = N - \lambda I$ if $(\lambda \geq 0)$ and the equation (54) with $L_\mu = -L - \mu I$ and $N_\mu = -N + \mu I$ if $(\mu \leq 0)$.

6. The periodic problem on spheres

Now, let $M = S^1 \times S^n$ and let $g : M \times \mathbb{R} \to \mathbb{R}$ be a function satisfying the Carathéodory conditions for $L^2(M, \mathbb{R})$.

We consider the weak solutions of the semilinear spherical wave equation

$$u_{tt} - \Delta_n u = g(t, x, u), \qquad t \in S^1, x \in S^n. \tag{55}$$

Then $u \in L^2(S^1 \times S^n)$ is a weak solution of (55) provided that

$$\int_{S^1 \times S^n} u(\phi_{tt} - \Delta_n \phi) - g(t, x, u)\phi = 0,$$

for every $\phi \in C^2(S^1 \times S^n)$.

In the study of problem (55), we need to know the spectrum of the spherical wave operator. Let us reproduce some known results in [4] or [2].

It is well known, that the spherical wave operator $(\partial^2/\partial t^2 - \Delta_n)$ is symmetric, with domain $C^2(M)$ and such that, if

$$u(t, x) \simeq \sum_{j,l,m} a_{j,l,m} Y_{l,m}(x) e^{ijt}, \qquad i^2 = -1,$$

then,

$$(\partial^2/\partial t^2 - \Delta_n) u(t, x) = \sum_{j,l,m} [l(l + n - 1) - j^2] a_{j,l,m} Y_{l,m} e^{ijt},$$

where $Y_{l,m}(x)$ are spherical harmonic functions of degree l, $l = 0, 1, 2, \dots,$ $m = 1, 2, \dots, h_l$, $h_l = C^n_{n+l} - C^n_{n+l-2}$.

Then $(\partial^2/\partial t^2 - \Delta_n)$ can be extended to be a self adjoint operator A with domain

$$\operatorname{dom} A = \{u \in L^2(M, \mathbb{R}) : \sum_{j,l,m} \{1 + [l(l + n - 1) - j^2]^2\} |a_{j,l,m}|^2 < +\infty\},$$

and spectrum

$$\sigma(A) = \{l(l+n-1) - j^2,\ (l,j) \in \mathbb{N} \times \mathbb{Z}\}.$$

We will use the following result proved in [4].

LEMMA 5. *For each $b \in \mathbb{R}$,*

$$\dim \ker(A + bI) < +\infty,$$

except if n is odd and $b = (\frac{1}{2}(n-1))^2$.

PROOF. We have to prove the finiteness of the number of solutions pairs (l, j) of the Diophantine equation

$$\begin{aligned} 0 =& l(l+n-1) - j^2 + b \\ =& (l + \tfrac{1}{2}(n-1) - |j|)\,(l + \tfrac{1}{2}(n-1) + |j|) - (\tfrac{1}{2}(n-1))^2 + b, \end{aligned} \tag{56}$$

but this follows from the fact that when $l + \frac{1}{2}(n-1) - |j| \neq 0$, the right hand side of (56) tends to $\pm\infty$ as $l \to +\infty$ or $|j| \to +\infty$. □

Consequently, if n is even, $\sigma(A)$ is made of isolated eigenvalues with finite multiplicity and hence A has compact resolvent, although if n is odd, $-(\frac{1}{2}(n-1))^2$ is an eigenvalue of A having infinite multiplicity and $\sigma(A)\backslash\{-(\frac{1}{2}(n-1))^2\}$ is made of isolated eigenvalues with finite multiplicity.

Let us finally notice that from Lemma 5, analogues of the Lemmas 3 and 4 hold true for L defined by $L = A + bI$, where b is given by

$$b = \begin{cases} 0, & \text{for even } n; \\ (\frac{1}{2}(n-1))^2, & \text{for odd } n. \end{cases}$$

So we can give the two following existence theorems for the spherical wave equation (55).

THEOREM 3. *Let n be an even integer, and let $\lambda < \mu$ be two consecutive eigenvalues of A. Assume that g satisfies (34) and that the inequalities*

$$\alpha_+(t,x) \le \liminf_{u\to+\infty} u^{-1} g(t,x,u) \le \limsup_{u\to+\infty} u^{-1} g(t,x,u) \le \beta_+(t,x),$$
$$\alpha_-(t,x) \le \liminf_{u\to-\infty} u^{-1} g(t,x,u) \le \limsup_{u\to-\infty} u^{-1} g(t,x,u) \le \beta_-(t,x)$$

hold uniformly a.e. in $(t,x) \in M$, where $\alpha_\pm$ and $\beta_\pm$ are functions in $L^\infty(M)$ such that

$$\lambda \le \alpha_+(t,x) \le \beta_+(t,x) \le \mu$$
$$\lambda \le \alpha_-(t,x) \le \beta_-(t,x) \le \mu.$$

Moreover, assume that

$$\int_M [(\alpha_+ - \lambda)(v^+)^2 + (\alpha_- - \lambda)(v^-)^2] > 0, \qquad \textit{for all } v \in \ker(L - \lambda I)\backslash\{0\},$$

and

$$\int_M [(\mu - \beta_+)(w^+)^2 + (\mu - \beta_-)(w^-)^2] > 0, \qquad \textit{for all } w \in \ker(L - \mu I)\backslash\{0\}.$$

Then the equation (55) *has at least one weak solution.*

THEOREM 4. *Let n an odd integer and $\lambda < \mu$ be two consecutive eigenvalues of A different of $-(\frac{1}{2}(n-1))^2$. Assume that g satisfies (34),*

$$\operatorname{sign}(\lambda + (\tfrac{1}{2}(n-1))^2) \cdot (g(t,x,\cdot\,) + (\tfrac{1}{2}(n-1))^2 I)$$

is nondecreasing and that the inequalities

$$\alpha_-(t,x) \le \liminf_{u\to-\infty} u^{-1} g(t,x,u) \le \limsup_{u\to-\infty} u^{-1} g(t,x,u) \le \beta_-(t,x),$$
$$\alpha_+(t,x) \le \liminf_{u\to+\infty} u^{-1} g(t,x,u) \le \limsup_{u\to+\infty} u^{-1} g(t,x,u) \le \beta_+(t,x),$$

hold uniformly a.e. in $(t,x) \in M$ where $\alpha_\pm$ and $\beta_\pm$ are functions in $L^\infty(M)$ satisfying

$$\lambda \le \alpha_-(t,x) \le \beta_-(t,x) \le \mu$$
$$\lambda \le \alpha_+(t,x) \le \beta_+(t,x) \le \mu.$$

Moreover, assume that

$$\int_M [(\alpha_+ - \lambda)(v^+)^2 + (\alpha_- - \lambda)(v^-)^2] > 0, \qquad \textit{for all } v \in \ker(L - \lambda I)\backslash\{0\}$$

and

$$\int_M [(\mu - \beta_+)(w^+)^2 + (\mu - \beta_-)(w^-)^2] > 0, \qquad \textit{for all } w \in \ker(L - \mu I)\backslash\{0\}.$$

Then the equation (55) *has at least one weak solution.*

As in the previous problem, results about the uniqueness of the solution, which are similar to Corollaries 1 and 2 can be obtained in both cases.

7. Appendix

We state here for the reader's convenience, the three abstract existence theorems used in the paper. This requires some definitions and notations of [5] [6].

Let X,Z be real vectors normed spaces, L : dom $L \subset X \to Z$ a linear Fredholm operator of index zero, $B(\gamma) \subset X$, an open ball of center 0 and radius γ, and $N : \bar{B}(\gamma) \to Z$ a (possibly) nonlinear L-compact operator. The first two results are continuation theorems of the Leray-Schauder type.

THEOREM A. *Assume that there exist a number* $\gamma > 0$ *and a* L*-compact operator* $M : X \to Z$ *such that*

(i) *for every* $(u, s) \in (\operatorname{dom} L \cap \partial B(\gamma)) \cap]0, 1[$

$$Lu - (1 - s)Mu - sNu \neq 0,$$

(ii) $0 \notin (L - M)(\operatorname{dom} L \cap \partial B(\gamma))$,

(iii) $D_L(L - M, B(\gamma)) \neq 0$.

Then the equation

$$Lu - Nu = 0$$

has at least one solution in $\operatorname{dom} L \cap \bar{B}(\gamma)$.

Now, let H be a real Hilbert space, with inner product $(\cdot\,, \cdot)$ and corresponding norm $|\cdot|$. Let $L : \operatorname{dom} L \subset H \to H$ be a self-adjoint operator with closed range. If $\mathbb{R}_0^-$ (resp. $\mathbb{R}_0^+$) denotes the set of negative (resp. positive) real numbers, we shall set

$$d_0^- = \operatorname{dist}(0, \sigma(L) \cap \mathbb{R}_0^-),$$

with the convention $d_0^- = +\infty$ if $\sigma(L)\backslash\{0\} \subset \mathbb{R}_0^+$.

Denote by K the right inverse of L defined by

$$K = [L|\operatorname{dom} L \cap \operatorname{Im} L]^{-1} : \operatorname{Im} L \to \operatorname{Im} L,$$

and by Q the orthogonal projection onto Im L.

Recall that if $N : H \to H$ is a nonlinear operator, then N is said to be monotone (resp. strongly monotone) on H, if, for all u, v in H, one has

$$(Nu - Nv, u - v) \geq 0 \qquad \text{(resp. } (Nu - Nv, u - v) \geq c|u - v|^2,\ c > 0),$$

and demicontinuous on H if $u_k \to u \Rightarrow Nu_k \rightharpoonup Nu$, where $\rightharpoonup$ denotes the weak convergence in H.

THEOREM B. *Let* L *a self adjoint operator (with right inverse* K*) and* $N : H \to H$ *be a monotone demicontinuous operator. Assume that there exist a demicontinuous strongly monotone operator* $M : H \to H$ *and a number* $\gamma > 0$ *such that the following conditions are satisfied :*

(i) KQN *and* KQM *are compact on the closed ball* $\bar{B}(\gamma)$ *of center* 0 *and radius* γ *in* H.

(ii) $M(\bar{B}(\gamma))$ *is bounded.*

(iii) $(\forall\, s \in [0, 1[\,)\ (\forall u \in \operatorname{dom} L \cap \partial B(\gamma))$:

$$Lu - (1 - s)Mu - sNu \neq 0.$$

(iv) *For each finite dimensional subspace* S *of* $\ker L$ *such that*

$$0 \notin (L_S - M_S)(\operatorname{dom} L \cap \partial B(\gamma) \cap H_S),$$

the coincidence degree of $L_S - M_S$,

$$D_{L_S}(L_S - M_S, B(\gamma) \cap H_S)$$

is nonzero.

Then the equation $Lu - Nu = 0$ *has at least one solution* $u \in \operatorname{dom} L \cap \bar{B}(\gamma)$.

The last theorem is based upon fixed point theory for non-expansive or contractive mappings.

THEOREM C. *Assume that* $N : H \to H$ *is a gradient, that* $0 < d_0^- < \infty$, *and that there exist positive constants* $\beta_0, \beta_1, \gamma_1, \delta_1$ *such that the assumptions*

(i) $\beta_0|u - v|^2 \leq (Nu - Nv, u - v) \leq \beta_1|u - v|^2$,

(ii) $|Nu - (d_0^-/2)u| \leq \gamma_1|u| + \delta_1$,

are satisfied for all $u, v \in H$.

If d_0^- *is finite and the following conditions*

(iii) $\beta_1 \leq d_0^-$,

(iv) $\gamma_1 < d_0^-/2$,

hold, then the equation

$$Lu + Nu = f$$

has at least one solution for each $f \in H$.

If condition (i) *holds together with the inequality*

$$\beta_1 < d_0^-,$$

then the above equation has, for each $f \in H$, *a unique solution.*

See [5], [6] for Theorem A, [5], [7], Theorem B and [3], for Theorem C.

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