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A HOMOTOPICAL PROPERTY OF ATTRACTORS

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Abstract. We construct a 2-dimensional torus $\mathcal{T} \subseteq \mathbb{R}^3$ having the property that it cannot be an attractor for any homeomorphism of $\mathbb{R}^3$. To this end we show that the fundamental group of the complement of an attractor has certain finite generation property that the complement of $\mathcal{T}$ does not have.

1. Introduction

Given a manifold M and a dynamical system defined on it, we say that a compact set $K \subseteq M$ is an attractor if it is invariant, Lyapunov stable and there is a neighbourhood $U = U(K)$ such that all orbits starting at U converge to the set K. This definition leads to the following question: what compact sets can be realized as attractors of some dynamical system? In the last thirty years several authors have dealt with this question and the known results depend critically on the type of dynamical system and the dimension of the ambient space. For continuous flows we refer to [5], [8], [10], [11], [12], [16], [18] and to [6], [7], [9], [13], [17] for the discrete case.

In the present paper we assume that the ambient space is $M = \mathbb{R}^3$ and the system is discrete, produced by a homeomorphism $f \colon \mathbb{R}^3 \to \mathbb{R}^3$. The general (unsolved) problem is to describe the class of compact sets $K \subseteq \mathbb{R}^3$ which are

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attractors for some h. Our more modest goal will be to construct a curious example of a set that cannot be realized as an attractor. In the process we will find an abstract homotopical obstruction that can be of independent interest. To describe our result let us consider one of the most natural attractors in the Euclidean space, the torus of revolution $T \subseteq \mathbb{R}^3$. We aim at constructing a set $\mathcal{T} \subseteq \mathbb{R}^3$ that is homeomorphic to T but cannot be an attractor of any h. At first sight the existence of $\mathcal{T}$ may seem paradoxical but those readers who are familiar with topology in three dimensions will probably agree that $\mathcal{T}$ is conceivable as long as it is a wild surface. Roughly speaking, a surface $S \subseteq \mathbb{R}^3$ is wild if it contains a point p such that S cannot be flattened within $\mathbb{R}^3$ near p. There is nothing particular about the torus in our construction and similar examples of different genus can be constructed. In particular we refer to [17] for a different construction in the case of the sphere.

At this point it seems convenient to discuss the connections of our result with the existing literature. The question posed earlier about the realization of compact sets as attractors can be interpreted in different ways. In our approach the ambient space is fixed ($M = \mathbb{R}^3$) but other authors have considered the problem in different terms: the set K is a given compact metric space and the unknowns are the ambient manifold M (of arbitrary dimension) and the homeomorphism producing an attractor that is homeomorphic to K. The two problems are different but certainly there are links between them. In particular we refer to the approach taken by Günther in [9]. In this interesting paper the very general case of continuous maps $f\colon M \to M$ is considered to show that certain solenoids cannot be realized as attractors on any manifold M. To prove this result Günther considers the Čech cohomology groups of an attractor K and the induced homomorphism $f^*\colon \check{H}^*(K) \to \check{H}^*(K)$, showing that there must exist a finitely generated subgroup $G \subseteq \check{H}^*(K)$ which acts as a sort of algebraic attractor for h^*. This rather vague statement means exactly that

$$\bigcup_{n=1}^{\infty} (f^*)^{-n}(G) = \check{H}^*(K).$$

Our paper is organized as follows. In Section 2 we adapt the idea of Günther to our setting, proving that it still holds after replacing the Čech cohomology group of the attractor by the first homotopy group of its complement $\mathbb{R}^3 - A$. Our construction of the wild torus $\mathcal{T}$ that cannot be an attractor is based on two sets with very surprising topological properties: the Cantor set of Antoine A and the wild sphere of Antoine $\mathcal{A}$. These sets were discovered (invented?) almost one century ago and they seem to be very well adapted for the needs of dynamics. Section 3 reviews how A is constructed and some of its properties. In Section 4 we introduce a number $\delta(\alpha)$ that somehow quantifies the amount of entanglement of a loop $\alpha \subseteq \mathbb{R}^3 - A$ with the set A. Section 5 starts by reviewing