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GROUND STATE SOLUTIONS FOR A CLASS OF NONLINEAR MAXWELL–DIRAC SYSTEM

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Abstract. This paper is concerned with the following nonlinear Maxwell–Dirac system

$$\begin{cases} -i\displaystyle\sum_{k=1}^{3}\alpha_k\partial_k u + a\beta u + \omega u - \phi u = F_u(x,u), \\ -\Delta\phi = 4\pi|u|^2, \end{cases}$$

for $x \in \mathbb{R}^3$. The Dirac operator is unbounded from below and above, so the associated energy functional is strongly indefinite. We use the linking and concentration compactness arguments to establish the existence of ground state solutions for this system with asymptotically quadratic nonlinearity.

1. Introduction and main results

We study the following nonlinear Maxwell–Dirac system

$$\begin{cases} -i\displaystyle\sum_{k=1}^{3}\alpha_k\partial_k u + a\beta u + \omega u - \phi u = F_u(x,u), \\ -\Delta\phi = 4\pi|u|^2, \end{cases} \tag{1.1}$$

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where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $u \in \mathbb{C}^4$, $\partial_k = \partial/\partial x_k$, $a > 0$, α_1, α_2, α_3 and β are the 4×4 complex matrices:

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad k = 1, 2, 3,$$

with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

ϕ is the electron field. In this paper, we are interested in the existence of ground state solutions of system (1.1) with asymptotically quadratic nonlinearity, that is, solutions corresponding to the least energy for the energy functional of system (1.1).

The Maxwell–Dirac system, which describes the interaction of a particle with its self-generated electromagnetic field, plays an important role in quantum electrodynamics. Also it has been used as effective theories in atomic, nuclear and gravitational physics (see [39]). In the past decade, system (1.1) has been studied for a long time and many results are available concerning the Cauchy problem, see for instance, [8], [9], [27], [29], [32], [30], [38] and the references therein. As we known, the existence of stationary solutions of the Maxwell-Dirac system has been an open problem for a long time, see [31, p. 235]. As far as the existence of stationary solutions of system (1.1) is concerned by using variational methods, there is a poineering work by Esteban et al. [23] in which a multiplicity result is studied. After that, Abenda [1] obtained the existence result of solitary wave solutions. And a strong localization result was obtained in [36]. On the other hand, in [28], Garrett Lisi gave numerical evidence of the existence of bounded states by using an axially symmetric ansatz. For more detailed descriptions for equations and systems related to Dirac equations, we refer to the recent review [24] and the references therein.

We emphasize that the works mentioned above mainly concerned with the autonomous system with null self-coupling ($F \equiv 0$). In [12], Chen and Zheng studied system (1.1) with nonlinear self-coupling model ($F \neq 0$), and the existence of least energy stationary solutions of system (1.1) was obtained for the special superquadratic power nonlinearity $F_u(x, u) = a(x)|u|^{p-2}u$ with $2 < p < 3$. Zhang et al. [46] considered the general superquadratic nonlinearity. Besides, for other related topics including the superquadratic singular perturbation problem and concentration phenomenon of semi-classical states, see, for instance [20]–[22] and the references therein.

Inspired by the above works, the purpose of this paper is to consider system (1.1) with non-autonomous asymptotically quadratic nonlinearity. To the best of our knowledge, there has been no work concerning on this case up to now.