Abstract. In this paper, we present a calculus of paraconsistent logic. We propose an axiomatisation and a semantics for the calculus, and prove several important meta-theorems. The calculus, denoted as $\text{CB}^1$, is an extension of systems $\text{PI}$, $\text{C}_{\text{min}}$ and $\text{B}^1$, and a proper subsystem of Sette’s calculus $\text{P}^1$. We also investigate the generalization of $\text{CB}^1$ to the hierarchy of related calculi.

Keywords: paraconsistent logic; paraconsistency; hierarchy of the paraconsistent calculi

1. Introduction

One of the most commonly quoted definitions of paraconsistent logic runs as follows: a logic $\langle \mathcal{L}, \vdash \rangle$ is said to be paraconsistent if $\{\alpha, \neg\alpha\} \not\vdash \beta$, for some formulas $\alpha, \beta$. The definition is very general and covers a broad range of logics. Therefore, some authors have suggested that additional criteria should be taken into account when introducing a calculus of paraconsistent logic. It is worth mentioning three of them here: (a) the law of non-contradiction must not be a valid schema in any paraconsistent calculus; (b) a paraconsistent calculus should be rich enough to enable practical inference; and last but not least, (c) a paraconsistent calculus should have an intuitive justification.\footnote{Criterion (a) was introduced by Newton C. A. da Costa [10, p. 498]. Criteria (b) and (c) were formulated by Stanisław Jaśkowski [12, the second English translation, p. 38]. More complex and up-to-date definitions of paraconsistency are proposed in [see esp. Chapter 2 of 2; and Chapter 1 of 4].} Unfortunately, the latter two are rather vague and imprecise. They suffer from a lack of accuracy.
and open a wide field for speculation and conjecture. On the other hand, there are a few significant examples of paraconsistent calculi in which the law of non-contradiction has not been abandoned. Jaśkowski’s discursive calculus and Asenjo–Tamburino’s logic of antinomies may serve as good examples of this kind of calculi [see 9, pp. 52, 71–72].

The aim of this paper is to propose a calculus of paraconsistent logic which is intended to satisfy at least some of the requirements. The calculus, denoted as CB$$^1$$, arises as a result of the extension of the system C$$_{\text{min}}$$ with the principle of weak explosion \( \alpha \rightarrow (\neg\alpha \rightarrow (\neg\neg\alpha \rightarrow \beta)) \) or the calculus B$$^1$$ with the law of double negation \( \neg\neg\alpha \rightarrow \alpha \). It can also be viewed as an extension of the propositional logic PI, or as a proper subsystem of the calculus P$$^1$$ [16]. All of them form together a lattice of paraconsistent calculi. In addition, we will investigate the generalization of CB$$^1$$ to a hierarchy of related calculi.

2. Basic notation

Let Var denote a denumerable set of propositional variables: \( p, q, p_1, p_2, \ldots \). The set For of formulas is standardly defined using variables from Var and the symbols \( \neg, \lor, \land \) and \( \rightarrow \) for negation, disjunction, conjunction and implication, respectively. The connective of equivalence, \( \alpha \leftrightarrow \beta \), is treated as an abbreviation for \( (\alpha \rightarrow \beta) \land (\beta \rightarrow \alpha) \).

In For, we will consider axiomatic propositional calculi in a Hilbert-style formalization with (MP) as the only rule of interference: \( \alpha \rightarrow \beta \), \( \alpha \vdash \beta \). Such a calculus \( C \) is determined by its set of axioms Ax$$^C$$ which is included in For. For \( C \), any \( \alpha \in \text{For} \) and any \( \Gamma \subseteq \text{For} \), we say that \( \alpha \) is provable from \( \Gamma \) within \( C \) (in symbols: \( \Gamma \vdash^C \alpha \)) iff there is a finite sequence of formulas, \( \beta_1, \beta_2, \ldots, \beta_n \) such that \( \beta_n = \alpha \) and for each \( i \leq n \),

\footnote{See [5, p. 31], [6, sections 1–3] and [9, pp. 80–84]. Note that C$$_{\text{min}}$$ has been also considered in [17, pp. 18–19], under the abbreviation \( \langle 13 \rangle \). A modern discussion on C$$_{\text{min}}$$ can be found in [see 2, Section 7.4]. The calculus B$$^1$$ originally appeared in [7] as mbC$$^1$$. Unfortunately, the chosen abbreviation and narrative in the paper are a bit misleading, e.g. “mbC$$^1$$ […] essentially coincides with mbC by Carnielli, Coniglio and Marcos” [17, p. 174]. As a result, one could conclude that mbC$$^1$$/B$$^1$$ was equivalent to mbC. Obviously, it is not the case and such a claim should be rejected [see 9, p. 140].}

\footnote{For details, see [3]. Nowadays, the system PI is perhaps better known under the abbreviation CLuN. The calculi C$$_{\text{min}}$$ and B$$^1$$ are not the only extensions of PI. Some other (not necessarily paraconsistent) extensions of PI, are, e.g., presented in [2, 11, 15, 17, 18].}

\footnote{2 See [5, p. 31], [6, sections 1–3] and [9, pp. 80–84]. Note that C$$_{\text{min}}$$ has been also considered in [17, pp. 18–19], under the abbreviation \( \langle 13 \rangle \). A modern discussion on C$$_{\text{min}}$$ can be found in [see 2, Section 7.4]. The calculus B$$^1$$ originally appeared in [7] as mbC$$^1$$. Unfortunately, the chosen abbreviation and narrative in the paper are a bit misleading, e.g. “mbC$$^1$$ […] essentially coincides with mbC by Carnielli, Coniglio and Marcos” [17, p. 174]. As a result, one could conclude that mbC$$^1$$/B$$^1$$ was equivalent to mbC. Obviously, it is not the case and such a claim should be rejected [see 9, p. 140].}
either \( \beta_i \in \Gamma \), or \( \beta_i \in \text{Ax}_C \), or for some \( j, k \leq i \), we have \( \beta_k = \beta_j \to \beta_i \).

A formula \( \alpha \) is a thesis of \( C \) iff \( \alpha \) is provable from \( \emptyset \) within \( C \). Let \( \text{Th}(C) \) be the set of all theses of \( C \). Observe that \( C \) can be identified with the triple \( \langle \text{For}, \text{Ax}_C, \vdash \rangle \), but \( C \) is determined by \( \text{Ax}_C \). Also, it can be easily seen that \( \vdash \) is a finitary consequence relation satisfying Tarskian properties (reflexivity, monotonicity, transitivity).

**Lemma 2.1.** For every \( \Gamma, \Delta \subseteq \text{For} \) and \( \alpha, \beta \in \text{For} \):

1. \( \Gamma \vdash_C \alpha \) iff for some finite \( \Delta \subseteq \Gamma \), \( \Delta \vdash_C \alpha \).
2. If \( \alpha \in \Gamma \), then \( \Gamma \vdash_C \alpha \).
3. If \( \Gamma \subseteq \Delta \) and \( \Gamma \vdash_C \alpha \), then \( \Delta \vdash_C \alpha \).
4. If \( \Delta \vdash_C \alpha \) and, for every \( \beta \in \Delta \) such that \( \Gamma \vdash_C \beta \), then \( \Gamma \vdash_C \alpha \).
5. If \( \Gamma \cup \{ \alpha \} \vdash_C \beta \) and \( \Delta \vdash_C \alpha \), then \( \Gamma \cup \Delta \vdash_C \beta \); in particular, if \( \Gamma \cup \{ \alpha \} \vdash_C \beta \) and \( \alpha \) is a thesis of \( C \), then \( \Gamma \vdash C \beta \).

Each calculus considered in this work, except for the calculi discussed in the last section, is expected to contain all axiom schemas of the positive fragment of Classical Propositional Calculus (CPC\(^+\) for short), i.e., all instances of the following schemas:

\[
\begin{align*}
&\alpha \to (\beta \to \alpha) \quad (A1) \\
&(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma)) \quad (A2) \\
&((\alpha \to \beta) \to \alpha) \to \alpha \quad (A3) \\
&(\alpha \land \beta) \to \alpha \quad (A4) \\
&(\alpha \land \beta) \to \beta \quad (A5) \\
&\alpha \to (\beta \to (\alpha \land \beta)) \quad (A6) \\
&\alpha \to (\alpha \lor \beta) \quad (A7) \\
&\beta \to (\alpha \lor \beta) \quad (A8) \\
&(\alpha \to \gamma) \to ((\beta \to \gamma) \to (\alpha \lor \beta \to \gamma)) \quad (A9)
\end{align*}
\]

Notice that if \((A1), (A2)\) are theses of \( C \) and (MP) is the sole rule of inference, then the deduction theorem holds for \( C \), that is, for any \( \Gamma \subseteq \text{For} \) and \( \alpha, \beta \in \text{For} \), we have:

\[
\Gamma \cup \{ \alpha \} \vdash_C \beta \iff \Gamma \vdash_C \alpha \to \beta. \quad (\text{DT})
\]

If \((A9)\) is a thesis of \( C \) then, for any \( \Gamma, \Delta \subseteq \text{For} \) and \( \alpha, \beta, \gamma \in \text{For} \), the following holds:

if \( \Gamma \cup \{ \alpha \} \vdash_C \gamma \) and \( \Gamma \cup \{ \beta \} \vdash_C \gamma \), then \( \Gamma \cup \{ \alpha \lor \beta \} \vdash_C \gamma \). \quad (\text{Dis})
From Lemma 2.1(2) and (DT), it follows that
\[ \alpha \rightarrow \alpha \]  
(R)
is a thesis of \( C \).

From (DT), it also immediately follows that
\[ (\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow (\beta \rightarrow (\alpha \rightarrow \gamma)) \]  
(PoC)
\[ (\alpha \rightarrow \beta) \rightarrow ((\beta \rightarrow \gamma) \rightarrow (\alpha \rightarrow \gamma)) \]  
(HS)
\[ (\alpha \rightarrow (\alpha \rightarrow \beta)) \rightarrow (\alpha \rightarrow \beta) \]  
(C)
are theses of \( C \).

For any calculi \( C_1 \) and \( C_2 \) (in For), we say that \( C_1 \) is an extension of \( C_2 \) iff \( \text{Th}(C_2) \subseteq \text{Th}(C_1) \). We say that \( C_2 \) is a proper subsystem of \( C_1 \) (in symbols: \( C_2 \sqsubset C_1 \)) iff \( \text{Th}(C_2) \subseteq \text{Th}(C_1) \) and \( \text{Th}(C_1) \not\subseteq \text{Th}(C_2) \).

Finally, let \( \text{INT}^+ \) denote the positive fragment of intuitionistic propositional calculus obtained from \( \text{CPC}^+ \) by dropping \( (A3) \), also known as Peirce’s law.

### 3. The Paraconsistent calculus \( CB^1 \). Syntax

The paraconsistent calculus \( CB^1 \) is defined, in a Hilbert-style formalization, by (MP), as the sole rule of inference, the axiom schemas \( (A1) \)–\( (A9) \) and the following ones involving negation:
\[ \alpha \vee \neg \alpha \]  
(ExM)
\[ \neg \neg \alpha \rightarrow \alpha \]  
(NN)
\[ \alpha \rightarrow (\neg \alpha \rightarrow (\neg \neg \alpha \rightarrow \beta)) \]  
(DS²)

In the succeeding paragraphs, we consider several subsystems of \( CB^1 \), i.e.: \( PI (= \text{CLuN}) \), \( C_{\text{min}} \) and \( B^1 \). They are all defined by (MP), axiom schemas \( (A1) \)–\( (A9) \) and additionally:
- \( PI \) has the axiom \( (\text{ExM}) \),
- \( C_{\text{min}} \) contains the axioms \( (\text{ExM}) \) and \( (\text{NN}) \),
- \( B^1 \) has the axiom schemas \( (\text{ExM}) \) and \( (\text{DS}^2) \).

It is obvious that \( C_{\text{min}} \) and \( B^1 \) are extensions of \( PI \), whereas \( CB^1 \) is an extension of \( C_{\text{min}} \), \( B^1 \) and \( PI \). It is also known that: (i) \( (\text{DS}^2) \) and \( (\text{NN}) \) are not theses of \( PI \); (ii) \( (\text{NN}) \) is not a thesis of \( B^1 \); (iii) \( (\text{DS}^2) \) is not a thesis of \( C_{\text{min}} \). Therefore, we have:

**Fact 3.1.** 1. \( B^1 \not\sqsubset C_{\text{min}} \) and \( C_{\text{min}} \not\sqsubset B^1 \).
2. \( PI \sqsubset B^1 \), \( PI \sqsubset C_{\text{min}} \), \( B^1 \sqsubset CB^1 \) and \( C_{\text{min}} \sqsubset CB^1 \).
Notice that from \((DT), (A9), (R), (ExM)\) and \((MP)\), we obtain:

\[
\begin{align*}
(\neg \alpha \to \alpha) & \to \alpha \quad (CM1) \\
(\alpha \to \neg \alpha) & \to \neg \alpha \quad (CM2)
\end{align*}
\]

which means that \((CM1)\) and \((CM2)\) are theses of \(PI\). Furthermore, from \((DT), (DS^2), (NN), (HS), (PoC), (C), (CM2)\) and \((MP)\), we receive:

\[
(\alpha \to \neg \beta) \to ((\alpha \to \neg \neg \beta) \to \neg \alpha) \quad (NI)
\]

Thus, this formula is a thesis of \(CB^1\).

**Fact 3.2.** The axioms \((NN)\) and \((DS^2)\) can be replaced by a single one:

\[
\neg \alpha \to (\neg \neg \alpha \to \beta). \quad (DSn)
\]

Consequently, the calculus \(CB^1\) may as well be defined by the axioms \((A1)–(A9), (ExM), (DSn)\) and \((MP)\) as the sole rule of inference.

**Proof.** To demonstrate that \((DSn)\) is provable in \(CB^1\), consider the sequence of formulas: \(\neg \alpha, \neg \neg \alpha; \alpha, \) by \((NN), \neg \neg \alpha\) and \((MP)\); \(\beta, \) by \((DS^2)\), \(\alpha, \neg \alpha, \neg \neg \alpha\) and \((MP)\); \(\neg \alpha \to (\neg \neg \alpha \to \beta), \) by \((DT)\).

Observe that \((PoC)\) and \((CM1)\) are provable from axioms of \(CPC^+\), \((ExM), (DSn)\) and \((MP)\). Now, for \((NN)\): Assume that \(\neg \neg \alpha\). Then we have \(\neg \alpha \to \alpha, \) by \((DSn), (PoC), \) the assumption and \((MP)\). Now, apply \((MP)\) to \((CM1)\) and \(\neg \alpha \to \alpha, \) to get \(\alpha\). But this means that, by \((DT)\), we obtain \((NN)\). For \((DS^2)\): It is enough to apply \((MP)\) to \((A1)\) and \((DSn)\).

**Theorem 3.3.** For \(CB^1\), the following weaker variants of the indirect deduction theorem hold, where \(\Gamma \subseteq For\) and \(\alpha, \beta \in For:\)

1. If \(\Gamma, \alpha \vdash_{CB^1} \neg \beta\) and \(\Gamma, \alpha \vdash_{CB^1} \neg \neg \beta\), then \(\Gamma \vdash_{CB^1} \neg \alpha\).
2. If \(\Gamma, \neg \alpha \vdash_{CB^1} \neg \beta\) and \(\Gamma, \neg \alpha \vdash_{CB^1} \neg \neg \beta\), then \(\Gamma \vdash_{CB^1} \alpha\).

**Proof.** \(Ad\ 1.\) Assume that \(\Gamma, \alpha \vdash_{CB^1} \neg \beta\) and \(\Gamma, \alpha \vdash_{CB^1} \neg \neg \beta\). Then, by \((DT)\), we have \(\Gamma \vdash_{CB^1} \alpha \to \neg \beta\) and \(\Gamma \vdash_{CB^1} \alpha \to \neg \neg \beta\). Since \((NI)\) is a thesis of \(CB^1\), then \(\Gamma \vdash_{CB^1} \neg \alpha\).

\(Ad\ 2.\) Assume that \(\Gamma, \neg \alpha \vdash_{CB^1} \neg \beta\) and \(\Gamma, \neg \alpha \vdash_{CB^1} \neg \neg \beta\). Then, by 1, \(\Gamma \vdash_{CB^1} \neg \alpha\). Since \((NN)\) is an axiom of \(CB^1\), we also have \(\Gamma \vdash_{CB^1} \alpha\).

There is an important point which has not been discussed yet, namely, whether \(CB^1\) is a paraconsistent calculus. The fact below shows that this is indeed the case.
**Fact 3.4.** The formulas:
\[
\begin{align*}
    p & \rightarrow (\neg p \rightarrow q) \quad \text{(DS)} \\
    p & \rightarrow (\neg p \rightarrow \neg q) \quad \text{(DS')} \\
    \neg(p \land \neg p) & \quad \text{(NC)} \\
    p & \rightarrow \neg\neg p \quad \text{(NN')} \\
\end{align*}
\]
are not provable in CB\(^1\). Moreover, neither \{\alpha, \neg\alpha\} \vdash_{\text{CB}\(^1\)} \beta, nor \{\alpha, \neg\alpha\} \vdash_{\text{CB}\(^1\)} \neg\beta, nor \{\alpha \rightarrow \beta\} \vdash_{\text{CB}\(^1\)} \neg\beta \rightarrow \neg\alpha, nor \{\neg\alpha \rightarrow \neg\beta\} \vdash_{\text{CB}\(^1\)} \beta \rightarrow \alpha\) hold.

**Proof.** Apply the matrix \(\mathcal{M}^3 = \langle\{1, 2, 0\}, \{1, 2\}, \neg, \land, \lor, \rightarrow\rangle\), where \{1, 2, 0\} and \{1, 2\} are the sets of logical values and designated values, respectively; and \neg, \land, \lor, \rightarrow\) are defined as follows:
\[
\begin{array}{c|ccc}
\rightarrow & 1 & 2 & 0 \\
\hline
1 & 1 & 2 & 0 \\
2 & 1 & 2 & 0 \\
0 & 1 & 1 & 1 \\
\end{array}
\quad
\begin{array}{c|ccc}
\neg & 1 & 1 & 1 \\
\hline
1 & 1 & 2 & 0 \\
2 & 2 & 2 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
\quad
\begin{array}{c|ccc}
\lor & 1 & 2 & 0 \\
\hline
1 & 1 & 1 & 1 \\
2 & 1 & 2 & 2 \\
0 & 1 & 2 & 0 \\
\end{array}
\]

The truth tables for implication, conjunction and disjunction are isomorphic to the ones given by Asenjo and Tamburino [1, p. 18]. The truth table for negation seems to be pretty new. Note that each axiom schema of CB\(^1\) is valid in the matrix \(\mathcal{M}^3\) and (MP) preserves validity. To demonstrate that (DS), (DS'), (NC) and (NN') are not valid in \(\mathcal{M}^3\), assign 1 to \(p\) in the formulas \(\neg(p \land \neg p)\) and \(p \rightarrow \neg p\), respectively; 1 to \(p\) and 0 to \(q\) in \(p \rightarrow (\neg p \rightarrow q)\); and 1 to \(p\) and 2 to \(q\) in (DS'). Next, assign 1 to \(\alpha\) and 0 to \(\beta\) in \{\alpha, \neg\alpha\} \vdash_{\text{CB}\(^1\)} \beta; 1 to \(\alpha\) and 2 to \(\beta\) in \{\alpha, \neg\alpha\} \vdash_{\text{CB}\(^1\)} \neg\beta; 2 to \(\alpha\) and 1 to \(\beta\) in \{\alpha \rightarrow \beta\} \vdash_{\text{CB}\(^1\)} \neg\beta \rightarrow \neg\alpha; and finally, 0 to \(\alpha\) and 1 to \(\beta\) in \{\neg\alpha \rightarrow \neg\beta\} \vdash_{\text{CB}\(^1\)} \beta \rightarrow \alpha.

Now we can prove [for details, see 16 and 8]:

**Fact 3.5.** CB\(^1\) \sqsubseteq P\(^1\), where P\(^1\) is Sette’s calculus.

**Proof.** In [9, pp. 116–120], it is demonstrated that (A1)–(A9) and (ExM) are theses of P\(^1\). In [8, p. 267], we prove that (DSn) is a thesis of P\(^1\).

---

4 The arguments given in Fact 3.4 might be expressed more concisely in a more advanced terminology. For instance, one could perceive that (a) the connective of \neg\) is not explosive in CB\(^1\) and \{p, \neg p\} \not\vdash_{\text{CB}\(^1\)} \neg q; then CB\(^1\) is strongly pre-\neg\-paraconsistent; (b) \neg\) is left-involutive (but not right-involutive); (c) \neg\) is not contrapositive; etc. [for details see 2, Chapter 2]. For the purpose of this paper, however, we have decided to use a simpler set of terms.
Notice that (MP) is the sole rule of inference of both calculi. Thus, all theses of CB¹ are provable in P¹.

Now we show that the following axiom of Sette’s calculus P¹ is not provable in CB¹:

\[(p \rightarrow q) \rightarrow \neg
\neg(p \rightarrow q)\]  

($)$

We apply the matrix \(M^3\) = \(\langle\{1, 2, 0\}, \{1, 2\}, \neg, \land, \lor, \rightarrow\rangle\), where \(\{1, 2, 0\}\) and \(\{1, 2\}\) are the sets of logical values and designated values, respectively; and \(\neg, \land, \lor, \rightarrow\) are defined as follows:

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All axioms of CB¹ are valid in \(M^3\) and (MP) preserves validity. Now, assign the value 1 (or 2) to \(p\) and 2 to \(q\) in ($$) to demonstrate that it is not valid in \(M^3\). Thus, ($$) is a not a thesis of CB¹. ⊣

Let us recall a few well-known facts. For this reason, they will be given without proofs.

**FACT 3.6.** 1. INT⁺ ⊆ CPC⁺ ⊆ PI.
2. PI ⊆ B¹ and PI ⊆ Cmin.
3. P¹ ⊆ CPC, where CPC is the classical propositional calculus.

As a summary of this section, let us note that the calculi can be represented by the lattice structure of Figure 1.

**4. Bivaluational semantics for CB¹**

In this section, we introduce a bivaluational semantics for CB¹. It can be easily obtained from the semantics proposed in [7].

**DEFINITION 4.1.** A CB¹-valuation is any function \(v: \text{For} \rightarrow \{1, 0\}\) satisfying, for any \(\alpha, \beta \in \text{For}\), the following conditions:

\[(\lor) \quad v(\alpha \lor \beta) = 1 \text{ iff } v(\alpha) = 1 \text{ or } v(\beta) = 1,\]

\[(\land) \quad v(\alpha \land \beta) = 1 \text{ iff } v(\alpha) = 1 \text{ and } v(\beta) = 1,\]

\[(\rightarrow) \quad v(\alpha \rightarrow \beta) = 1 \text{ iff } v(\alpha) = 0 \text{ or } v(\beta) = 1,\]

\[(\neg) \quad \text{if } v(\neg \alpha) = 0 \text{ then } v(\alpha) = 1,\]

\[(\neg\neg) \quad \text{if } v(\neg\neg \alpha) = 1 \text{ then } v(\neg \alpha) = 0.\]

A formula \(\alpha\) is a CB¹-tautology iff \(v(\alpha) = 1\), for any CB¹-valuation \(v\).
Figure 1. A lattice of the paraconsistent calculi

Definition 4.2. For all $\alpha \in \text{For}$ and $\Gamma \subseteq \text{For}$, $\alpha$ is a semantic consequence of $\Gamma$ (in symbols: $\Gamma \models_{CB} \alpha$) iff for any $\text{CB}^1$-valuation $v$: if $v(\beta) = 1$ for any $\beta \in \Gamma$, then $v(\alpha) = 1$.

The soundness of $\text{CB}^1$ can be obtained in the standard way, by induction on the length of a derivation in $\text{CB}^1$:

Theorem 4.1. If $\Gamma \vdash_{\text{CB}^1} \alpha$, then $\Gamma \models_{\text{CB}^1} \alpha$.

For the proof of completeness $\vdash_{\text{CB}^1}$, we use the method which is based on the notion of maximal non-trivial sets of formulas. We apply the technique described in [see 4, Section 2.2]. To begin with, let us recall some important definitions and results.

Let $\mathcal{C} = \langle \text{For}, \text{Ax}_\mathcal{C}, \vdash_\mathcal{C} \rangle$ be a calculus (satisfying Tarskian properties) and $\Delta \subseteq \text{For}$. We say that $\Delta$ is a closed theory of $\mathcal{C}$ iff for any $\beta \in \text{For}$: $\Delta \vdash_\mathcal{C} \beta$ iff $\beta \in \Delta$. We say that $\Delta$ is maximal non-trivial with respect to $\alpha \in \text{For}$ in $\mathcal{C}$ iff (a) $\Delta \not\vdash_\mathcal{C} \alpha$ and (b) for any $\beta \in \text{For}$, if $\beta \not\in \Delta$ then $\Delta \cup \{\beta\} \vdash_\mathcal{C} \alpha$.

Lemma 4.2 (4, Lemma 2.2.5). Every maximal non-trivial set with respect to some formula is a closed theory.

Of course, Lemma 4.2 holds for $\text{CB}^1$, as well. Moreover, we have:

Lemma 4.3. For any maximal non-trivial set $\Delta$ with respect to $\alpha$ in $\text{CB}^1$ the mapping $v$: $\text{For} \rightarrow \{1, 0\}$ defined, for any $\delta \in \text{For}$, as $(\star)$: $v(\delta) = 1$ iff $\delta \in \Delta$, is a $\text{CB}^1$-valuation.
Proof. We only show that the conditions for (¬) and (¬¬) are fulfilled. The rest of the proof is analogous to that of Theorem 2.2.7 in [4].

Assume, for a contradiction, that \( v(\neg \beta) = 0 \) and \( v(\beta) = 0 \). Thus we have \( \neg \beta \notin \Delta \) and \( \beta \notin \Delta \), by (⋆). Since \( \Delta \) is a maximal non-trivial set with respect to \( \alpha \), then \( \Delta \cup \{\beta\} \vdash_{CB^1} \alpha \) and \( \Delta \cup \{\neg \beta\} \vdash_{CB^1} \alpha \). Consequently, \( \Delta \cup \{\beta \lor \neg \beta\} \vdash_{CB^1} \alpha \), by (Dis). Hence \( \Delta \vdash_{CB^1} \alpha \), by (ExM) and Lemma 2.1(5). Observe that \( \Delta \) is a closed theory (see Lemma 4.2), so \( \alpha \in \Delta \). But \( \alpha \notin \Delta \) (by the main assumption). This yields a contradiction.

Assume, for a contradiction, that \( v(\neg \neg \beta) = 1 \) and \( v(\neg \beta) = 1 \). Then \( \neg \neg \beta \in \Delta \) and \( \neg \beta \in \Delta \), by (⋆). Hence \( \Delta \vdash_{CB^1} \neg \neg \beta \) and \( \Delta \vdash_{CB^1} \neg \beta \), by Lemma 2.1(2). Notice that (DSn) is a thesis of \( CB^1 \), so we have \( \Delta \vdash_{CB^1} \alpha \). Since \( \Delta \) is a closed theory, hence \( \alpha \in \Delta \). But \( \alpha \notin \Delta \), by the assumption. This results in a contradiction.

Note that the so-called Lindenbaum–Łoś theorem holds, for any finitary calculus \( C = \langle \text{For}, \text{Ax}_C, \vdash_C \rangle \):

Lemma 4.4 (4, Theorem 2.2.6; 14, Theorem 3.31). For any \( \Gamma \subseteq \text{For} \) and \( \alpha \in \text{For} \) such that \( \Gamma \not\vdash_C \alpha \), there is a maximal non-trivial set \( \Delta \) with respect to \( \alpha \) in \( C \) such that \( \Gamma \subseteq \Delta \).

Thus, the completeness of \( CB^1 \) follows:

Theorem 4.5. For all \( \Gamma \subseteq \text{For} \) and \( \alpha \in \text{For} \): if \( \Gamma \vdash_{CB^1} \alpha \), then \( \Gamma \vdash_{CB^1} \alpha \).

Proof. Assume that \( \Gamma \not\vdash_{CB^1} \alpha \) and \( \Delta \) is a maximal non-trivial set with respect to \( \alpha \) in \( CB^1 \) such that \( \Gamma \subseteq \Delta \) (see Lemma 4.4). Then \( \alpha \notin \Delta \). Therefore, by Lemma 4.3, there is a valuation \( v \) such that \( v(\alpha) = 0 \) and \( v(\beta) = 1 \), for any \( \beta \in \Delta \). Hence \( \Gamma \not\vdash_{CB^1} \alpha \).

5. Hierarchy of \( CB^m \)-calculi

Let \( m \in \mathbb{N} \) and \( \neg^m \alpha \) be an abbreviation for \( \neg \neg \ldots \neg \alpha \). The hierarchy is obtained by replacing (DSn) with the following schema, for any \( m \in \mathbb{N} \):

\[-^m \alpha \rightarrow (\neg^{m+1} \alpha \rightarrow \beta)\]  \hspace{1cm} (DSn\(^m\))

More precisely, for any \( m \in \mathbb{N} \), let \( CB^m \) result from \( CPC^+ \) by adding to it (ExM) and (DSn\(^m\)). For \( m = 0 \), the logic \( CB^m \) collapses into the classical propositional calculus.
As a semantics for CB\(^m\), for each \(m \in \mathbb{N}\), we use CB\(^m\)-valuations which are obtained from CB\(^1\)-valuations by replacing the condition \((\neg \neg)\) with a more general one, that is,

\((\neg^{m+1})\) if \(v(\neg^{m+1} \alpha) = 1\) then \(v(\neg^m \alpha) = 0\).

The semantic clauses for (\(\lor\)), (\(\land\)), (\(\rightarrow\)) and for (\(\neg\)) remain unchanged.

The soundness of CB\(^m\) is obtained analogously to that of CB\(^1\). For the completeness of CB\(^m\), we need to modify the proof of Lemma 4.3, i.e.:

**Lemma 5.1.** For any maximal non-trivial set \(\Delta\) with respect to \(\alpha\) in CB\(^m\) the mapping \(v:\) For \(\rightarrow\) \(\{1,0\}\) defined, for any \(\delta \in \text{For}\), as (\(\ast\)): \(v(\delta) = 1\) iff \(\delta \in \Delta\), is a CB\(^m\)-valuation.

**Proof.** Assume, for a contradiction, that \(v(\neg^{m+1} \beta) = 1\) and \(v(\neg^m \beta) = 1\). Then \(\neg^{m+1} \beta \in \Delta\) and \(\neg^m \beta \in \Delta\), by (\(\ast\)). Hence \(\Delta \vdash_{\text{CB}^1} \neg^{m+1} \beta\) and \(\Delta \vdash_{\text{CB}^1} \neg^m \beta\). The formula \((\text{DSn}^m)\) is a thesis of CB\(^m\), so it follows that \(\Delta \vdash_{\text{CB}^m} \alpha\). Since \(\Delta\) is a closed theory (see Lemma 4.2), then \(\alpha \in \Delta\). But \(\alpha \notin \Delta\) (by the main assumption). This yields a contradiction. \(\dashv\)

Thus, we receive:

**Theorem 5.2.** For all \(m \in \mathbb{N}\), \(\Gamma \subseteq \text{For}, \alpha \in \text{For}: \) \(\Gamma \vdash_{\text{CB}^m} \alpha\) iff \(\Gamma \models_{\text{CB}^m} \alpha\).

Notice that each calculus in the hierarchy is essentially weaker than the preceding one(s), viz., \(\text{CB}^1 \supseteq \text{CB}^2 \supseteq \text{CB}^3 \supseteq \cdots \supseteq \text{CB}^m \supseteq \cdots\). The proof that \(\text{Th}(\text{CB}^k) \subseteq \text{Th}(\text{CB}^m)\), for \(k > m\), basically reduces to the observation that every instance of \((\text{DSn}^m)\) is also an instance of \((\text{DSn}^m)\). Consequently, it suffices to show that the following holds: if \(k > m\), then

\(\neg^m p \rightarrow (\neg^{m+1} p \rightarrow q)\)

is not a thesis of CB\(^k\). But this fact can be easily proved with the help of the completeness theorem for CB\(^k\). There is a CB\(^k\)-valuation such that \(v(\neg^m p) = 1 = v(\neg^{m+1} p)\) and \(v(q) = 0\). This entails that:

**Fact 5.3.** For any \(k, m \in \mathbb{N}\) such that \(k > m\), we have \(\text{CB}^k \supseteq \text{CB}^m\).

Moreover, we obtain the following result:

**Fact 5.4.** Let \(m, k \in \mathbb{N}\) and \(m \geq k\), then the formula \(\neg^{k-1} p \rightarrow \neg^{k+1} p\) is not provable in any CB\(^m\)-calculus.
On the system CB\textsuperscript{1} and a lattice ...

Proof. We have already noticed that \( p \rightarrow \neg\neg p \) (i.e., \( \neg^0 p \rightarrow \neg^2 p \)) is not provable in CB\textsuperscript{1} (and neither is in any CB\textsuperscript{m}-calculus weaker than CB\textsuperscript{1}). For the other cases, it suffices to apply the completeness theorem for CB\textsuperscript{m}.

At the end of this section, we state two simple facts about CB\textsuperscript{m}.

Fact 5.5. For any \( m \in \mathbb{N} \), enriching the set of axiom schemas of CB\textsuperscript{m} with the formula \( \alpha \rightarrow \neg\neg\alpha \), results in the axiom system of CPC.

In other words, for any \( m > 0 \), we need to prove that the axiom schemas of CPC\textsuperscript{+}, (ExM), (DS\textsuperscript{m}), (NN\textsuperscript{*}) and (MP), as the sole rule of inference, constitute an axiomatization of CPC.

Proof. It immediately follows due to the fact that the schemas (DS\textsuperscript{m}) and (NN\textsuperscript{*}) are equivalent to (DS) in CPC; to put it more precisely, let CPC be defined by CPC\textsuperscript{+}, (ExM), (DS) and (MP). Then (NN\textsuperscript{*}) follows from the deduction theorem, (DS), (CM2) and (MP); (DS\textsuperscript{m}) is an instance of (DS). Now, for CPC being defined by CPC\textsuperscript{+}, (ExM), (DS\textsuperscript{m}), (NN\textsuperscript{*}) and (MP), assume that \( \alpha \) and \( \neg\alpha \). Let \( m \) be even (the proof for \( m \) being odd is similar). Then, by \( \alpha \), (NN\textsuperscript{*}) (applied \( \frac{m}{2} \) times) and (MP), we receive \( \neg^m\alpha \). Likewise, by \( \neg\alpha \), (NN\textsuperscript{*}) and (MP), we get \( \neg^{m+1}\alpha \), and finally \( \beta \) by (DS\textsuperscript{m}) and (MP). Hence, by (DT), we obtain: \( \alpha \rightarrow (\neg\alpha \rightarrow \beta) \). \( \dashv \)

The proof of the following fact is analogous to the proof of Fact 5.5.

Fact 5.6. For any \( m > 1 \), enriching the set of axiom schemas of any CB\textsuperscript{m}-calculus with the formula \( \neg\alpha \rightarrow \neg\neg\neg\alpha \), results in obtaining CB\textsuperscript{1}.

Proof. By (DT), (DS\textsuperscript{n}), (CM2) and (MP), we find that \( \neg\alpha \rightarrow \neg\neg\neg\alpha \) is a thesis of CB\textsuperscript{1}. But it is not a thesis of any CB\textsuperscript{m}-calculus that is weaker than CB\textsuperscript{1} (see Fact 5.4). Now it suffices to show that \( \neg^m\alpha \rightarrow (\neg^{m+1}\alpha \rightarrow \beta) \), where \( m > 1 \), and \( \neg\alpha \rightarrow \neg\neg\neg\alpha \) are equivalent to (DS\textsuperscript{2}). \( \dashv \)

6. Final remarks

So far, every calculus that has been discussed here contains CPC\textsuperscript{+} as its positive base. In this section, we will weaken the base to the positive fragment of intuitionistic propositional calculus. As a result, we will be able to enrich our discussion with a few interesting calculi among which Newton da Costa’s calculus \( C_\omega \) seems to be the most remarkable [see 6, 10, 13 and 9, Section 2.6].
We define, in a Hilbert-style formalization, the following calculi:

1. \( \text{INTuN} := \text{INT}^+ + (\text{ExM}) \),
2. \( C_\omega := \text{INTuN} + (\text{NN}) \),
3. \( A^1 := \text{CPC}^+ + (\text{DS}^2) \).

The calculus \( C_\omega \) is well-known in the literature. The calculi \( \text{INTuN} \) and \( A^1 \) are extremely weak and far less known than \( C_\omega \). Let us state a few facts about the calculi.

**Fact 6.1.**
1. \( \text{INT}^+ \sqsubseteq \text{INTuN} \sqsubseteq C_\omega \sqsubseteq C_{\text{min}} \).
2. \( \text{INTuN} \sqsubseteq \text{PI} \sqsubseteq B^1 \).
3. \( \text{CPC}^+ \sqsubseteq A^1 \sqsubseteq B^1 \).
4. It is not the case that
   (a) \( C_\omega \sqsubseteq \text{PI} \) or \( \text{PI} \sqsubseteq C_\omega \),
   (b) \( C_\omega \sqsubseteq A^1 \) or \( A^1 \sqsubseteq C_\omega \),
   (c) \( A^1 \sqsubseteq \text{PI} \) or \( \text{PI} \sqsubseteq A^1 \).

**Proof.** *Ad 1.* It is obvious that both \( \text{INT}^+ \sqsubseteq \text{INTuN} \) and \( C_\omega \sqsubseteq \text{Cmin} \). To prove that \( \text{INTuN} \sqsubseteq C_\omega \), we should slightly modify the matrix \( M^3 \) (see Fact 3.4), i.e., replace the truth table for negation with the three-valued table for the so-called cyclic (or rotary) negation:

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and assign 0 to \( p \) in \( \neg
\neg p \rightarrow p \) of the form \( \text{NN} \).

*Ad 2.* We have already noticed that \( \text{PI} \sqsubseteq B^1 \) (ct. Fact 3.6). Since \( ((\alpha \rightarrow \beta) \rightarrow \alpha) \rightarrow \alpha \) is unprovable in \( C_\omega \) [see 10, Theorem 15, p. 501] and \( \text{INTuN} \sqsubseteq C_\omega \), then \( ((p \rightarrow q) \rightarrow p) \rightarrow p \) of the form \( \text{A3} \) is not provable in \( \text{INTuN} \), either.

*Ad 3.* The case \( \text{CPC}^+ \sqsubseteq A^1 \) is trivial. To show that \( A^1 \sqsubseteq B^1 \), apply the classical truth tables for implication, conjunction, and disjunction. Next, define negation as follows:

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and assign 0 to \( p \) in \( p \lor \neg p \rightarrow p \) of the form \( \text{ExM} \).

*Ad 4a.* Observe that \( ((p \rightarrow q) \rightarrow p) \rightarrow p \) of the form \( \text{A3} \) is not a thesis of \( C_\omega \) (see above). To demonstrate that \( \neg \neg p \rightarrow p \) of the form \( \text{NN} \)
is not a thesis of PI, it suffices to apply the completeness theorem and semantics for the calculus.

*Ad 4b.* We need to show that either \( \neg \neg p \rightarrow p \) of the form \((\text{NN})\) or \( p \lor \neg p \) of the form \((\text{ExM})\) is not a theses of \( A^1 \) on the one hand, and \( p \rightarrow (\neg p \rightarrow (\neg \neg p \rightarrow q)) \) of the form \((\text{DS}^2)\) is not a thesis of \( C_\omega \) on the other. The former is obvious (cf. the item 3). The latter can be easily proved with the help of the completeness theorem for \( C_\omega \).

*Ad 4c.* Notice that \( p \rightarrow (\neg p \rightarrow (\neg \neg p \rightarrow q)) \) of the form \((\text{DS}^2)\) is not a theorem of PI (by the completeness theorem and semantics for PI). To prove that \( p \lor \neg p \) of the form \((\text{ExM})\) is not a thesis of \( A^1 \), it is enough to recall the two-valued matrix given in the item 3 and assign 0 to \( p \) in \( p \lor \neg p \).

As a final remark, let us emphasise that all calculi presented in this paper fulfil the requirements for being considered as paraconsistent (at least in a broad sense of the term), and they form a more complex structure than that of Figure 1; see Figure 2.

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