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Defining Measures in a Mereological Space
(an exploratory paper)

Abstract. We explore the notion of a measure in a mereological structure and we deal with the difficulties arising. We show that measure theory on connection spaces is closely related to measure theory on the class of ortholattices and we present an approach akin to Dempster's and Shafer's. Finally, the paper contains some suggestions for further research.

Keywords: connection structures; measures; mereological space; mereology; region-based theories of space

1. Introduction and scope

Point-free geometry (or *geometry of solids*) is predicated on the idea that in geometry it is not necessary to assume points as primitive. Indeed, as an alternative, it is possible to start from the notion of a "region"(or "solid") and, successively, to define points in terms of regions. The rationale for point-free geometry is usually ontological in nature, and this is because many researchers believe that the existence of regions is more convincing than that of points.

Research on point-free geometry originated in the first half of the twentieth century as a consequence of two facts: the first being the natural evolution of Leśniewski's mereology [see 14] and the second being the publication of three books by the famous philosopher and mathematician A. N. Whitehead [see 21, 22, 23].

Leśniewski's mereology, coming from the famous Polish school of logic, provides a formal basis for point-free geometry. Indeed, the first

rigorous treatment, which became one of the most important papers on this subject, is due to A. Tarski, the most famous student of this school. We quote the initial part of Tarski's paper [20]:

> Some years ago Leśniewski suggested the problem of establishing the foundation of "geometry of solids", understanding by this term a system of geometry destitute of such geometrical figures as points, lines, and surfaces, and admitting as figures only solids.

Whitehead's books are philosophical in nature. Their aim is not to define a formal system of axioms but rather to analyze four-dimensional space with the idea that the events (intended as four-dimensional entities) are the foundation of this space. Whitehead's books and Tarski's paper spawned an extensive and interesting literature on point-free geometry.

The theory of measure for region-based theories is in statu nascendi and few contributions have been made so far, we mention the notable contributions by T. Lando and D. Scott [13], F. Arntzenius [1, 2], J. Russell [18] and others.[1]

In exploring the notion of measure in a mereological structure (where the main role is played by the inclusion relation), our paper is located in the tradition of the Polish school.[2] In our forthcoming paper we will tackle the problem of measurement in the more structured environment of point-free geometry.

The paper is organized as follows. In Section 2 we recall the definition of closed regular subset of a topological space since this notion is usually used in literature to represent the the notion of a region. In Section 3 we deal with the difficulties arising from the required additivity of the measures. In Section 4 we present some proposals for solutions to these difficulties. In Section 5 we show that — by accepting the connection spaces defined by B. Clarke — measure theory on connection spaces is closely related to measure theory on the class of ortholattices. In Section 6 we explore the possibility of defining measures in a mereological space by an approach akin to Dempster's [7, 8] and Shafer's [19]. The last section is devoted to some final considerations and suggestions for further research.

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[1] Unfortunately, the questions addressed by these authors are not well-known by those working on point-free geometry.

[2] For a deeper understanding of the theory of mereological structures we recommend reading, for instance, [15, 16, 10, 11].

Let us emphasize that this paper and the forthcoming are merely exploratory. Their purpose is to be a stimulus for a greater presence of the problem of measuring areas and volumes in the community of point-free geometry.

2. Regular closed subsets for an analytic representation of the notion of region

We denote by $\mathbb{R}^n$ the n-dimensional Euclidean space and by i and c the interior and closure operators, respectively.

DEFINITION 2.1. Define $r_c \colon \mathcal{P}(\mathbb{R}^n) \to \mathcal{P}(\mathbb{R}^n)$ by setting $r_c(X) = c(i(X))$. We say that a subset X of $\mathbb{R}^n$ is a *regular closed subset* whenever X is a fixed point of r_c. Analogously, define r_o by $r_o(X) := i(c(X))$, then we say that X is a *regular open subset* whenever X is a fixed point of r_o.

In point-free geometry regular closed sets (alternatively, regular open sets) of the Euclidean space are almost always assumed as a reference model for the idea of a region. This is so since since all the figures, usually considered in Euclidean geometry (triangles, rectangles, circles), are regular closed subsets while sets of points whose dimension is less than n (points, lines, sets of rational numbers) are not. However, this choice is due to the fact that regular closed subsets define an elegant algebraic structure as the following theorem says.

THEOREM 2.1. *Let $n \in \mathbb{N}$ and denote by* RC *the class of regular closed subsets of the Euclidean space $\mathbb{R}^n$. Then $(\mathrm{RC}, \subseteq)$ is an atomless complete Boolean algebra where the empty set is the zero element and $\mathbb{R}^n$ is the unity. Equivalently, $(\mathrm{RC}, \subseteq)$ is a complete mereological space with zero element.*

We omit the proof and observe only that the Boolean operations in RC are defined by regularizing the corresponding set-theoretical operations, i.e.

$$X \vee Y := r_c(X \cup Y), \quad X \wedge Y := r_c(X \cap Y), \qquad -_c X := r_c(-X).$$

As a matter of fact, we have

$$X \vee Y = X \cup Y, \quad -_c X = c(-X).$$

An analogous version of this theorem is satisfied by regular open subsets.

THEOREM 2.2. *Let* RO *be the class of regular open subsets of the Euclidean space* $\mathbb{R}^n$*, then* $(\mathrm{RO}, \subseteq)$ *is an atomless complete Boolean algebra where the empty set is the bottom element and* $\mathbb{R}^n$ *is the top element.*

On the other hand the following proposition shows that the structures $(\mathrm{RC}, \subseteq)$ and $(\mathrm{RO}, \subseteq)$ are isomorphic.

PROPOSITION 2.3. *The Boolean algebras* $(\mathrm{RC}, \subseteq)$ *and* $(\mathrm{RO}, \subseteq)$ *are isomorphic. Namely,* $i \colon \mathrm{RC} \to \mathrm{RO}$ *is an isomorphism from* $(\mathrm{RC}, \subseteq)$ *onto* $(\mathrm{RO}, \subseteq)$ *and* $c \colon \mathrm{RO} \to \mathrm{RC}$ *is its inverse.*

In this paper we refer to regular closed sets.

PROPOSITION 2.4. *Let* X *be a regular closed subset of Euclidean space, then there is no nonempty regular closed subset of the boundary of* X*.*

PROOF. If $R \in \mathrm{RC}$ and $R \subseteq c(X) \cap c(-X) = X \cap c(-X)$, then $i(R) \subseteq X$ and $i(R) \subseteq i(c(-X))$. On the other hand $i(c(-X)) = i(-i(X)) = -c(i(X)) = -X$. This proves that $i(R) = \emptyset$. Therefore $R = c(i(R)) = c(\emptyset) = \emptyset$.   ⊣

3. Countable additivity and finite additivity: some difficulties

As is known, every Boolean algebra carries finitely additive measures [see 12, p. 470].[3] In particular, that is true for the Boolean algebra of regular closed subsets of $\mathbb{R}^n$. However, the following questions arise:

- What is the situation in the case of σ-additivity?
- Is there a "natural" measure in $(\mathrm{RO}, \subseteq)$ which is geometrical in nature?
- Is $(\mathrm{RO}, \subseteq)$ adequate to represent the notion of a region?

To face these questions we consider the following two theorems [see 2, p. 142].[4]

THEOREM 3.1. *There is no* σ-additive measure μ *in the mereological space of regular closed subsets of* $\mathbb{R}$ *satisfying* $\mu([a, b]) = b - a$*. In particular, the Lebesgue measure is not* σ-additive over this space.*

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[3] The results in this and in the successive section are related to possible prototypical models of Roeper's theory [see 17]. Nevertheless, they are very general.

[4] We thank the anonymous referee for reporting them to us.

PROOF. According to the terminology in [2], we call *island* a closed interval contained in $[0, 1]$ and *archipelago* a union of pairwise disjoint islands. We define a sequence $(S_n)_{n \in \mathbb{N}}$ of archipelagos step-by-step as follows:
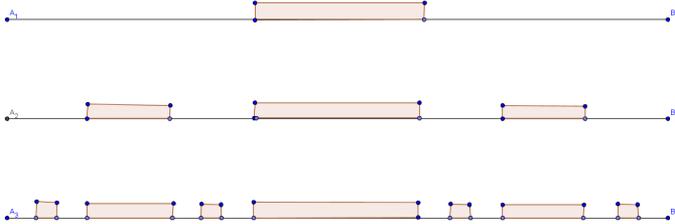
Step 1. At the centre of the segment $[0, 1]$ we slot the island S_1 whose measure is one fourth the length of $[0, 1]$, i.e. S_1 is the interval $[3/8, 5/8]$. Obviously, one has $\mu(S_1) = (1/2)^2$ and, for every $P \in [0, 1]$, $d(P, i(S_1)) \leqslant 1/2$, where — as usual — given a point P and a set X, we put $d(P, X) := \inf\{d(P, x) : x \in X\}$.

Step 2. Repeat the previous procedure in each of the two connected components of the complement of S_1. Namely, slot in the centre of each component an island whose measure is $1/4$ the measure of S_1 and therefore is $1/16$. We obtain the islands $[7/32, 9/32]$ and $[23/32, 25/32]$ and we denote by S_2 the union of these islands. Then S_2 is disjoint from S_1 and $\mu(S_2) = (1/2)^3$. Moreover, if P is an element of $[0, 1]$ such that $P \notin i(S_1)$, then $d(P, i(S_1 \cup S_2)) \leqslant (1/2)^2$.

Step 3. Repeat this procedure in each of the 4 connected components of the complement of $S_1 \cup S_2$. Namely, slot in the centre of each component an island whose measure is $1/4$ the measure of the islands of the archipelago S_2. Let S_3 be the union of these islands. Then S_3 is disjoint from $S_2 \cup S_1$ and $\mu(S_3) = (1/2)^4$. Moreover, if P is an element of $[0, 1]$ such that $P \notin i(S_1) \cup i(S_2)$, then $d(P, i(S_1) \cup i(S_2) \cup i(S_3)) \leqslant (1/2)^3$.

Continuing in this way we obtain a sequence of regions $(S_n)_{n \in \mathbb{N}}$ such that

(i) the elements of $(S_n)_{n \in \mathbb{N}}$ are pairwise disjoint,
(ii) $\mu(S_n) = (1/2)^{n+1}$,
(iii) if P is an element of $[0, 1]$ such that $P \notin i(S_1) \cup \cdots \cup i(S_{n-1})$, then
$d(P, i(S_1) \cup \cdots \cup i(S_n)) \leq (1/2)^n$.

Set $M := \bigcup_{n \in \mathbb{N}} i(S_n)$ and call it the *Cantor Archipelago*.[5] Since M is open, $c(M)$ is the region $r_c(M)$ generated by M. We claim that $c(M)$

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[5] We have slightly changed the nomenclature in [2] where the name

coincides with $[0,1]$. Indeed, for every $P \notin M$ and $m \in \mathbb{N}$;

$$d(P,M)) \leqslant d(P,i(S_m)) \leqslant (1/2)^m.$$

Thus, $d(P,M) = 0$, and so $c(M) = [0,1]$. As $M \subseteq \bigcup_{n \in \mathbb{N}} S_n$ we have

$$[0,1] = c(M) \subseteq c(\textstyle\bigcup_{n \in \mathbb{N}} S_n) \subseteq [0,1]$$

and therefore $c(M) = c(\bigcup_{n \in \mathbb{N}} S_n) = [0,1]$. Hence the region $[0,1]$ is the least upper bound $\bigvee_{n \in \mathbb{N}} S_n$ of the sequence $(S_n)_{n \in \mathbb{N}}$ of regions in the Boolean algebra of regular closed subsets of $\mathbb{R}$. Therefore $\mu(\bigvee_{n \in \mathbb{N}} S_n) = \mu([0,1]) = 1$. On the other hand, since μ is σ-additive,

$$\mu(\textstyle\bigvee_{n \in \mathbb{N}} S_n) = \sum_{n \in \mathbb{N}} \mu(S_n) = 1/2. \hspace{3em} \dashv$$

Difficulties related with finite additivity are also expressed by the following theorem [see 2, 18].

Theorem 3.2. *The Lebesgue measure is not finitely additive on the Boolean algebra of regular closed subsets.*

Proof. Split the sequence $(S_n)_{n \in \mathbb{N}}$, defined in Theorem 3.1, into the subsequences $(S_{2n-1})_{n \in \mathbb{N}}$ and $(S_{2n})_{n \in \mathbb{N}}$ and put $M_o = \bigcup_{n \in \mathbb{N}} i(S_{2n-1})$ and $M_e = \bigcup_{n \in \mathbb{N}} i(S_{2n})$. Then M_o and M_e are two disjoint open subsets of $[0,1]$ called *odd Cantor archipelago* and *even Cantor archipelago*, respectively.

Claim 1. Let P be a point which is not in $\bigcup_{n \in \mathbb{N}} i(S_n)$, then P lies both on the boundary of M_o and on the boundary of M_e.

Proof of Claim 1. Given $m \in \mathbb{N}$, by iii) in the proof of Theorem 3.1 $d(P,i(S_{2m-1})) \leqslant (1/2)^{2m-1}$. Then for every $m \in \mathbb{N}$ we have

$$d(P,M_o) \leqslant d(P,i(S_{2m-1})) \leq (1/2)^{2m-1}$$

and therefore $d(P,M_o) = 0$. Hence P lies on the boundary of M_o. In a similar way one proves that P lies on the boundary of M_e.

Claim 2. The region $c(M_o)$ is the complement of the region $c(M_e)$ in the Boolean algebra of regular closed subsets of $[0,1]$.

Cantor Archipelago denotes $r_c(\bigcup_{n \in \mathbb{N}} S_n))$ and not $\bigcup_{n \in \mathbb{N}} i(S_n)$. We do the same for the definitions of *odd Cantor Archipelago* and *even Cantor Archipelago* in the proof of Theorem 3.2.

Proof of Claim 2. By Claim 1 we get $c(M_o \vee M_e) = [0, 1]$. To prove that $c(M_o) \wedge c(M_e) = \emptyset$ we observe that $c(M_o) \wedge c(M_e) = r_c(c(M_o) \cap c(M_e))$ and $c(M_o) \cap c(M_e)$ is the boundary of both the regions $c(M_o)$ and $c(M_e)$. By Proposition 2.4 there is no nonempty regular closed set contained in it. Denoting by μ the Lebesgue measure, since μ is σ-additive we have

$$\mu(c(M_o)) = \mu(M_o) = \mu\left(\bigcup_{n \in \mathbb{N}} i(S_{2n-1})\right) = \sum_{n=2}^{\infty} (1/2)^{2n-1} = 1/6$$

and therefore $\mu(c(M_e)) = 1 - \mu(-M_o) = 1 - 1/6 = 5/6$. Analogously, $\mu(c(M_e)) = \sum_{n=1}^{\infty} (1/2)^{2n} = 1/3$, therefore $\mu(c(M_o)) = 2/3$. Thus $5/6 + 2/3 \neq 1$ and finite additivity fails. ⊣

4. Some possible answers

There are several attempts to give an answer to the difficulties emerging from the previous two theorems. For example, Arntzenius proposes to consider the *Lebesgue measure algebra B_n* instead of the Boolean algebra of regular closed subsets. This algebra is defined by introducing in the σ-algebra $\mathrm{Borel}(\mathbb{R}^n)$ of the Borel subsets of $\mathbb{R}^n$ the congruence $\equiv$ defined by setting $A \equiv B$ provided that the Lebesgue measure of the symmetric difference of A and B is zero. Then B_n is the quotient of $\mathrm{Borel}(\mathbb{R}^n)$ modulo $\equiv$. This structure is an atomless complete Boolean algebra. Putting $\underline{\mu}([x]) = \mu(x)$, where μ is the Lebesgue measure, then $\underline{\mu}: B_n \to [0, \infty]$ is a countably additive measure.

Moreover, two further proposals were made by Land and Scott in [13]. The first one is defined by the class $\mathrm{RCN}(\mathbb{R}^n)$ of the regular closed subsets of $\mathbb{R}^n$ whose boundaries have Lebesgue-measure zero. This class is a subalgebra of $\mathrm{RC}(\mathbb{R}^n)$ which is not complete and on which Lebesgue measure is finitely additive. The latter proposal is based on the notion of a clopen element of B_n. Recall that $a \in B_n$ is *clopen* if there is an open subset A and a closed subset B of $\mathbb{R}^n$ such that $a = [A] = [B]$. For example an open circle and its closure are equivalent and therefore the related class is clopen. The collection of clopen elements of B_n forms a subalgebra of B_n denoted by $\mathrm{CLOP}(B_n)$. One can prove that $\mathrm{CLOP}(B_n)$ is an atomless Boolean subalgebra of B_n and Lebesgue measure is finitely additive over it.

However, the question of an adequate representation of the notion of region is crucial and open. It is also a hard question as Sections 5–7 show. In any case, we have to be aware that, in the spirit of point-free

geometry, there is the idea of a direct formalization of spatial intuition. This requires us to assume as a primitive notion the region of the three-dimensional place that a perceptible object can occupy. This excludes the claim that strange sets of points obtained by infinitary methods have the right to be called a "region".

5. Measures in connection-based mereological spaces

An approach to point-free geometry cannot be purely mereological in nature. For example, Whitehead [23] considers the notion of *connection*, which is necessary for defining points as classes of regions. A system of axioms for this notion was successively proposed by Clarke in his influential papers [5] and [6]. We are introducing a modification of Clarke's approach that introduces an empty region in a connection space.

DEFINITION 5.1. Let Re be at least a two-element set, C a binary relation on Re and $\mathsf{o} \in Re$. A *connection space with an empty region* is a structure $(Re, \mathsf{C}, \mathsf{o})$ satisfying:

(C1) $x \mathrel{\mathsf{C}} x$ for every $x \neq \mathsf{o}$ and there is no region connected with o;
(C2) $x \mathrel{\mathsf{C}} y$ implies $y \mathrel{\mathsf{C}} x$;
(C3) $\mathsf{C}(x) = \mathsf{C}(y)$ implies $x = y$;[6]
(C4) for any subset X of Re there is a $z \in Re$ such that $\mathsf{C}(z) = \bigcup\{\mathsf{C}(x) : x \in X\}$.

As usual we call *regions* the elements of Re, o is called the *empty region* and we say that *x is connected with y* whenever $x \mathrel{\mathsf{C}} y$. For arbitrary regions x and y we say that *x is part* of y (notation: $x \leq y$) if $\mathsf{C}(x) \subseteq \mathsf{C}(y)$. One proves that $\leq$ is an order relation. Since $\mathsf{C}(\mathsf{o}) = \emptyset$, for any $x \in Re$ we have $\mathsf{C}(\mathsf{o}) \subseteq \mathsf{C}(x)$ and therefore o is the least element (zero) in $(Re, \leq)$. Moreover, when there is a non-empty region z such that $z \leq x$ and $z \leq y$ we say that x and y *overlap* and we write $x \mathrel{\mathsf{O}} y$.

Remark 5.1. 1. Clarke [5, 6] also used four axioms, two of which have been modified by us. The first axiom said that the relation C is reflexive, the fourth were applicable to non-empty sets of regions only.

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[6] For any binary symmetric relation R we put $R(r) := \{z \in Re : z \mathrel{R} r\}$.

In Clarke's theory $x \, \mathsf{O} \, y$ iff there is a region z such that $z \leq x$ and $z \leq y$. Namely, apart from the trivial case, i.e. when Re is a singleton, there is no least element (zero).[7]

2. Let $Re_+ := Re \setminus \{0\}$ and C_+ be the restriction of C to Re_+. If $(Re, \mathsf{C}, 0)$ satisfies (C1)–(C4), then (Re_+, C_+) satisfies Clarke's axioms. ⊣

Remark 5.2. In any complete mereological space, the overlapping relation satisfies Clarke's axioms ($\mathsf{C} := \mathsf{O}$). Moreover, if in any non-trivial complete Boolean algebra we define $x \, \mathsf{O} \, y$ iff $x \wedge y \neq 0$, then the overlapping relation satisfies conditions (C1)–(C4). ⊣

Like Clarke, we say that a region z is the C-fusion of a set X of regions if $\mathsf{C}(z) = \bigcup \{\mathsf{C}(x) : x \in X\}$. In virtue of (C3), in this case we can write $z = \bigsqcup X$, i.e., $z = \bigsqcup X$ iff for any $y \in Re$: $y \, \mathsf{C} \, z$ iff there is an $x \in X$ such that $x \, \mathsf{C} \, y$. We have $0 = \bigsqcup \emptyset$. We can put $1 := \bigsqcup Re$; 1 is the greatest element (unity) in $(Re, \leq)$. So $1 \neq 0$.

Moreover, we define the following three operations on Re by setting:

$$-x := \begin{cases} \bigsqcup \{z : \neg \, z \, \mathsf{C} \, x\} & \text{if } 0 \neq x \neq 1, \\ 0 & \text{if } x = 1, \\ 1 & \text{if } x = 0, \end{cases}$$

$$x \vee y := \bigsqcup \{z : z \leq x \text{ or } z \leq y\},$$
$$x \wedge y := \bigsqcup \{z : z \leq x \text{ and } z \leq y\}.$$

In this section we will show that measure theory in connection spaces is equivalent with measure theory in the class of ortholattices. To do this, we make use of a result by Biacino and Gerla for constructing a bridge between free-point geometry and ortholattices [see 4, Theorem 3.9]. We recall the definition of an ortholattice.

In any lattice $(L, \wedge, \vee)$ we use the order relation $\leq$ by putting: $x \leq y$ iff $x \wedge y = x$ (iff $x \vee y = y$). An *ortholattice* is a structure $(L, \wedge, \vee, -, 0, 1)$ such that $(L, \wedge, \vee, 0, 1)$ is a bounded lattice and $- : L \to L$ is an *ortho-complementation*, i.e. an operation satisfying

(L1) $-(-x) = x$,
(L2) $x \wedge -x = 0$,
(L3) $x \leq y$ implies $-y \leq -x$.

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[7] Assume that $0 \in Re$ is the least element in $(Re, \leq)$. Then for any $x \in Re$, we have $0 \leq x$, i.e., $\mathsf{C}(0) \subseteq \mathsf{C}(x)$. Hence, by the reflexivity and symmetry of C, we have $x \in \mathsf{C}(0)$. From the arbitrariness of x we derive $\mathsf{C}(0) = Re$; and so $\mathsf{C}(x) = \mathsf{C}(0)$. Hence, by the third axiom, $x = 0$; i.e. Re is a singleton.

Notice that from each complete ortholattice, we can create a connection space with the empty region o.

THEOREM 5.1. *Let* $(L, \wedge, \vee, -, o, 1)$ *be a complete ortholattice and let* C *be a binary relation on* L *defined by*:

$$x \, C \, y \iff x \not\leq -y.$$

Then (L, C, o) *is a connection space with the empty region* o.

Conversely, we can create a complete ortholattice from any connection space with zero.

THEOREM 5.2. *Let* (Re, C, o) *be a connection space with the empty region* o. *Then* $(Re, \wedge, \vee, -, o, 1)$ *is a complete ortholattice such that* $\sup X = \bigsqcup X$ *for any subset* X *of* Re *and for all* $x, y \in Re$ *we have*:

$$x \, C \, y \iff x \not\leq -y.$$

Remark 5.3. An ortholattice $(L, \wedge, \vee, -, o, 1)$ is *orthomodular* if it satisfies the following *orthomodular law*:

(L4) $x \leq y$ implies $x \vee (-x \wedge y) = y$.

The above in connection structures becomes:

(C5)   if x is not connected with y, then $y = -x \wedge (y \vee x)$.

Hence, in virtue of Theorem 5.1, from each complete orthomodular lattice, we can create a connection space satisfying (C5) and vice versa, by Theorem 5.2. ⊣

In literature there is a definition of a measure in an orthomodular lattice. We extend this definition to any ortholattice.

DEFINITION 5.2. Let $(L, \wedge, \vee, -, o, 1)$ be an ortholattice. A *measure* is a function $\mu \colon L \to [0, +\infty]$ such that $\mu(o) = 0$ and $\mu(x \vee y) = \mu(x) + \mu(y)$ whenever $x \leq -y$.

So, in the light of Theorem 5.2, a *measure* on a connection structure (Re, C, o) is a function $\mu \colon Re \to [0, +\infty]$ such that $\mu(o) = 0$ and $\mu(x \vee y) = \mu(x) + \mu(y)$ whenever x and y are not connected.

In the theory of non-commutative measure theory, the archetypal example of orthomodular lattices is the lattice of closed subspaces in a Hilbert space that can be seen as an event structure of a quantum experiment.

We recall an important result by Gleason:

Theorem 5.3. *Let $\mathcal{H}$ be a separable Hilbert space of dimension at least three, let $L(\mathcal{H})$ be the quantum logic of all closed subspaces of $\mathcal{H}$. Then there is a one-to-one correspondence between the class of σ-additive measures on $L(\mathcal{H})$ and the class of von Neumann operators. This correspondence is given by $\mu(A) = \mathrm{trace}(P_A)$, for $A \in L(H)$, where P_A is the orthogonal projector from $\mathcal{H}$ onto A.*

A *Hermitian operator* on a finite-dimensional complex vector space $\mathcal{H}$ with inner product $\langle \cdot, \cdot \rangle$ is a linear map T (from $\mathcal{H}$ to itself) that is its own adjoint, i.e. $\langle Tv, w \rangle = \langle v, Tw \rangle$, for all vectors v and w. Moreover, if T is represented by a square matrix with real or complex entries and $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of T (listed according to their algebraic multiplicities), then $\mathrm{trace}(T) = \sum_i^n \lambda_i$. Finally, a σ-additive measure is a map such that $\mu(0) = 0$ and $\mu(B) = \sum \mu(A_i)$ where A_i are mutually orthogonal elements and B is the closed linear span of this collection A_i.

If $\dim \mathcal{H}$ is finite, then on $L(\mathcal{H})$ there is a unique discrete σ-additive measure, namely $\mu(A) = \frac{\dim A}{\dim \mathcal{H}}$, for any $A \in L(\mathcal{H})$.

Example 5.1. Let $(L, \wedge, \vee, -, 0, 1)$ be the ortholattice of linear subspaces of three-dimensional linear space $\mathbb{R}^3$, where for any subspace x, let $-x$ be the subspace orthogonal to x. We define the measure μ by

$$\mu(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1/3 & \text{if } x \text{ is a one-dimensional subspace,} \\ 2/3 & \text{if } x \text{ is a two dimensional subspace,} \\ 1 & \text{if } x \text{ is the universe.} \end{cases}$$

In $\mathcal{H}$ we put:

$x \mathsf{C} y$ iff x is not contained in the orthogonal complement of y.

Then we obtain a connection space $(L, \mathsf{C}, 0)$ with the measure μ. Moreover, the function μ' defined by $\mu'(x) = 3\mu(x)$ (i.e. μ' maps each subspace to its dimension) is a measure, too. $\dashv$

We conclude by observing that the measures obtained in this section do not seem adequate for point-free geometry.

6. Rough and approximate measures

In the sequel, by a mereological space, we understand a (not necessarily complete) Boolean lattice, that is, a relationally characterized Boolean

algebra. Thus, in every mereological space fusion is just the supremum operation, and for any x and y, $x \vee y$ always exists. A large class of "rough measures" in a mereological space is suggested by Dempster-Shafer theory (where the word "measure" must be understood in a broad sense as assignment of a number to a region).[8] The notion of mass plays a crucial role.

DEFINITION 6.1. Given a mereological space $(\mathrm{Re}, \leq, \mathrm{o})$, a *mass* is a function $m \colon \mathrm{Re} \to [0, +\infty]$ such that $m(\mathrm{o}) = 0$. We say that z is a *focal* region for m if $m(z) \neq 0$, and we denote by $\mathrm{Fo}(m)$ the set of focal regions.

In Dempster-Shafer theory one assumes that $\sum_{x \in \mathrm{Re}} m(x) = 1$, too. We do not make this assumption since it is related to probabilistic valuations and not to geometrical measures.

DEFINITION 6.2. We say that m is *dense* if every region overlaps at least one focal region. Moreover, for every region x, we put

$$\mathrm{Co}(x) := \{z \in \mathrm{Fo}(m) : z \leq x\},$$
$$\mathrm{Ov}(x) := \{z \in \mathrm{Fo}(m) : z \, \mathbf{O} \, x\},$$
$$\mathrm{Bv}(x) := \mathrm{Ov}(x) - \mathrm{Co}(x) = \{z \in \mathrm{Fo}(m) : z \, \mathbf{O} \, x \text{ and } z \nleq x\}.$$

It is evident that $\mathrm{Co}(x) \subseteq \mathrm{Ov}(x)$, $\mathrm{Co}(\mathrm{o}) = \emptyset$ and Co and Ov are monotonic. The fusions of the classes $\mathrm{Co}(x)$, $\mathrm{Ov}(x)$ and $\mathrm{Bo}(x)$ are named *lower m-approximation, upper m-approximation* and *m-border* of x, respectively.

PROPOSITION 6.1. *For all* $x, y \in Re$, $\mathrm{Co}(x) \cup \mathrm{Co}(y) \subseteq \mathrm{Co}(x \vee y)$, *while* $\mathrm{Co}(x) \cup \mathrm{Co}(y) \neq \mathrm{Co}(x \vee y)$ *in general. Moreover, if x and y are not connected, then* $\mathrm{Co}(x) \cap \mathrm{Co}(y) = \emptyset$.

PROOF. The first inclusion is an immediate consequence of the inclusions $x \leq x \vee y$ and $y \leq x \vee y$ and of the monotonicity of Co. We will prove that $\mathrm{Co}(x) \cup \mathrm{Co}(y) \neq \mathrm{Co}(x \vee y)$. Assume that there exists a join-prime focal element z. Then there are x, y such that $z \leq x \vee y$, $z \nleq x$ and $z \nleq y$. The remaining part of the proposition is straightforward.   ⊣

PROPOSITION 6.2. *For all* $x, y \in Re$, $\mathrm{Ov}(x) \cup \mathrm{Ov}(y) = \mathrm{Ov}(x \vee y)$. *Moreover, $\mathrm{Ov}(x) \cap \mathrm{Ov}(y)$ may be different from the empty set, also if x and y are not connected.*

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[8] This theory, which is based on the notion of belief function, has a vast, interesting and increasing literature [see 7, 8, 19].

PROOF. The first part is an immediate consequence of the fact that $x \vee y$ is defined as the fusion of $\{x, y\}$. Moreover, assume that there is a focal element z which is not an atom. Then there exists a region $x \neq 0$ such that $x < z$. So there exists a region $y \neq o$ such that $y \leq z$ and x and y are not connected. It is immediately evident that z is an element of $\mathrm{Ov}(x) \cap \mathrm{Ov}(y)$. ⊣

We are now able to give the following definitions.

DEFINITION 6.3. Given a mass m, we put $\mathrm{int}(x) := \sum_{z \leq x} m(z)$, $\mathrm{ext}(x)$ $:= \sum_{zOx} m(z)$ and $\mathrm{err}(x) := \sum_{zOx, z \not\leq x} m(z)$. Moreover, we say that int and ext are the *lower approximation* and the *upper approximation* measures, respectively.[9]

Both the functions int and ext are monotonic. Moreover, $\mathrm{int}(x) = 0$ iff $\mathrm{Co}(x) = \emptyset$ iff there is no focal region z contained in x. We get $\mathrm{ext}(x) = 0$ iff $\mathrm{Ov}(x) = 0$, so $x = o$.

DEFINITION 6.4. Denote by $I([0, \infty])$ the class of all closed intervals contained in $[0, \infty]$. We call the *interval approximate measure* the function $\mu \colon Re \to I([0, \infty])$ defined by setting $\mu(x) = [\mathrm{int}(x), \mathrm{ext}(x)]$. We say that x is *measurable* if $\mathrm{int}(x) = \mathrm{ext}(x)$ and in this case we write $\mu(x) = \mathrm{int}(x)$ instead of $\mu(x) = \{\mathrm{int}(x)\}$.

THEOREM 6.3. *The function* int *is superadditive but not necessarily additive. The function* ext *is subadditive but not necessarily additive. As a consequence μ is additive on the class of measurable regions.*

PROOF. We will prove the superadditivity of int. Assume x and y are not connected. Since $\mathrm{Co}(x) \cap \mathrm{Co}(y) = \emptyset$ and $\mathrm{Co}(x) \cup \mathrm{Co}(y) \subseteq \mathrm{Co}(x \vee y)$, we have:

$$\mathrm{int}(x) + \mathrm{int}(y) = \sum_{z \leq x} m(z) + \sum_{z \leq y} m(z) \leq \sum_{z \leq x \vee y} m(z) = \mathrm{int}(x \vee y).$$

We will prove that *int* is not necessarily additive. Assume that there is a focal element z which is not join-prime. Then there are x and y such that $z \leq x \vee y$ but $z \not\leq x$ and $z \not\leq y$. Consequently, $int(x \vee y) - (\mathrm{int}(x) + \mathrm{int}(y)) > 0$ and therefore *int* is not additive.

To prove that *ext* is subadditive, we observe that $Ov(x \vee y) = (Ov(x) \cup Ov(y))$ and therefore:

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[9] In Dempster-Shafer theory *int* and *ext* are named *belief* and *plausibility* function, respectively.

$$ext(x \vee y) = \sum_{z\mathcal{O}(x\vee y)} m(z) =$$

$$\sum_{z\mathcal{O}x} m(z) + \sum_{z\mathcal{O}y} m(z) - \sum_{z\mathcal{O}x \wedge z\mathcal{O}y} m(z) \leq ext(x) + ext(y).$$

Finally, the map ext may be not additive, since $\mathrm{Ov}(x) \cap \mathrm{Ov}(y)$ may be different from $\emptyset$.   ⊣

The proof of the following proposition is straightforward.

PROPOSITION 6.4. *The following statements hold*:

   (i) *A region x is measurable if and only if every focal region overlapping x is contained in x.*

  (ii) *A region x is measurable if and only if the measure of the m-border is zero.*

 (iii) *Every nonempty region strictly contained in a focal region z is not measurable.*

THEOREM 6.5. *Assume that every focal region is measurable, then the following items hold*:

   (i) *The measure of a focal region z is $m(z)$.*

  (ii) *The focal regions are pairwise disjoint.*

 (iii) *If x is the fusion of a set F of focal regions, then it is measurable and $\mu(x) = \sum\{m(z) : z \in F\}$.*

PROOF. (i) By Proposition 6.4(iii) we have $\mathrm{Co}(z) = \{z\}$, therefore $\mu(z) = \mathrm{int}(z) = m(z)$.

(ii) Assume for a contradiction that for two focal regions z and z' we have $z \, \mathrm{O} \, z'$. Then by Proposition 6.4(i) we have $z \leq z'$ and $z \geq z'$, and therefore we obtain a contradiction: $z = z'$.

(iii) Let x be the fusion of a class F of focal regions and recall that this fusion is the least upper bound of F. Thanks to the equalities $\mathrm{Co}(z) = \{z\}$ and $\mathrm{Co}(z) = \mathrm{Ov}(z)$, we have $\mathrm{O}(x) = \bigcup\{\mathrm{O}(z) : z \in F\} = \bigcup\{\mathrm{Co}(z) : z \in F\} = \bigcup\{\{z\} : z \in F\} = F$.

On the other hand, since x is the fusion of F, $\mathrm{Co}(x) = \{z \in \mathrm{Fo}(m) : z \leq x\} \supseteq \{z \in F : z \leq x\} = F$. The inverse inequality follows from $\mathrm{Co}(x) \subseteq \mathrm{O}(x) = F$. From the equalities $\mathrm{O}(x) = F$ and $\mathrm{Co}(x) = F$ we can derive the measurability of x and $\mu(x) = \sum\{m(z) : z \in F\}$.   ⊣

THEOREM 6.6. *Assume that the mereological space is complete and m is dense, then x is measurable if and only if it is the fusion of a set F of*

focal regions. In this case $\mu(x) = \sum\{m(z) : z \in F\}$. *Moreover, the set of focal regions is a partition of the universe.*[10]

PROOF. Assume that x is measurable and let f be the fusion of $\mathrm{Co}(x)$, then $x \leq f$. If $x < f$, then there exists a region z such that $z \leq f$ and z and x are not connected. Since m is dense, there exists a focal region z' overlapping z and therefore overlapping f. We get $\mathrm{O}(f) = \bigcup\{\mathrm{O}(z) : z \in \mathrm{Co}(x)\} \supseteq \bigcup\{\mathrm{O}(z) \cap \mathrm{Fo}(m) : z \in \mathrm{Co}(x)\} = \bigcup\{\mathrm{Ov}(z) : z \in \mathrm{Co}(x)\} = \bigcup\{\mathrm{Co}(z) : z \in \mathrm{Co}(x)\} = \mathrm{Co}(x)$, whence the thesis comes.

For proving that the set $\mathrm{Fo}(m)$ of focal elements is a partition of the universe, it is sufficient to prove that the fusion f of $\mathrm{Fo}(m)$ is 1. We get $\mathrm{O}(f) = \bigcup\{\mathrm{O}(x) : x \in \mathrm{Fo}(m)\} = \bigcup\{\mathrm{O}(x) : x \in \mathrm{Re}\}$ since m is dense. Therefore $\mathrm{O}(f) = \mathrm{O}(1)$, whence $f = 1$ by the uniqueness of the fusion. ⊣

The theorems proved above suggest that we should assume that the class of focal regions is a partition of the universe.

DEFINITION 6.5. We say that an approximate measure in a complete mereological space is a *partition-based measure* if it is obtained by a mass $m\colon \mathrm{Re} \to [0, \infty]$ whose class of focal elements is a partition of the universe.

As an example, consider the figure below, where the partition is formed by the squares whose mass is 1. Then the approximate measure of the circle is the interval $[12, 32]$ and therefore the (considerable) error is 20.

_____
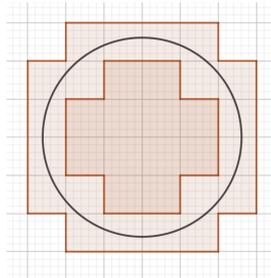
[10] We say that a class $\mathcal{Z}$ of regions is a *partition* of a non-empty region x if the elements of $\mathcal{Z}$ are mutually disjoint and x is a fusion of $\mathcal{Z}$.

7. Conclusions and future avenues of enquiry

By and large, research on providing an adequate measure theory in point-free geometry is only in its infancy, despite the contributions in [1, 2, 13, 18].

In Section 6 we explored the possibility of defining partition-based measures as a rough approach to Peano-Jordan measure theory. This approach has yielded correct information on the extension of a region x. Unfortunately, this information is significant inasmuch partition elements are very small with respect to the regions to be measured. On the other hand, once a mass is fixed, there is no way of improving this information by using the proposed method.

A workaround is suggested by the figure in the previous section: Together with the partition defined by major squares, there are also further partitions defined by smaller squares mirroring the pattern of graph paper. We will elaborate on this solution in a forthcoming paper [see 3] in which a basic role is assigned to a sequence of partitions $(\Pi_n)_{n \in \mathbb{N}}$ such that:

- Π_{n+1} is finer than Π_n,
- the elements of Π_n are pairwise congruent squares,
- $\lim_{n \to \infty} \mu(\Pi_n) = 0$, where $\mu(\Pi_n)$ is a measure of the squares in Π_n.

To do this, we need a system of axioms to define a good notion of congruence and, more generally, we need suitable definitions of several geometrical notions. This will be done starting from the system of axioms for point-free geometry proposed in [9].

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