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Kilwardby's 55th Lesson

Abstract. In "Lectio 55" of his *Notule libri Priorum*, Robert Kilwardby discussed various objections that had been raised against Aristotle's Theses. The first thesis, AT1, says that no proposition q is implied both by a proposition p and by its negation, $\sim p$. AT2 says that no proposition p is implied by its own negation. In *Prior Analytics*, Aristotle had shown that AT2 entails AT1, and he argued that the assumption of a proposition p such that $(\sim p \rightarrow p)$ would be "absurd".

The unrestricted validity of AT1, AT2, however, is at odds with other principles which were widely accepted by medieval logicians, namely the law *Ex Impossibili Quodlibet*, EIQ, and the rules of disjunction introduction. Since, according to EIQ, the impossible proposition $(p \wedge \sim p)$ implies every proposition, it also implies $\sim(p \wedge \sim p)$, in contradiction to AT2. Furthermore, by way of disjunction introduction, the proposition $(p \vee \sim p)$ is implied both by p and by $\sim p$, in contradiction to AT1.

Kilwardby tried to defend AT1, AT2 against these objections by claiming that EIQ holds only for accidental but not for natural implications. The second argument, however, cannot be refuted in this way because Kilwardby had to admit that every disjunction $(p \vee q)$ is naturally implied by its disjuncts. He therefore introduced the further requirement that, in order to constitute a genuine counterexample to AT1, $(p \rightarrow q)$ and $(\sim p \rightarrow q)$ have to hold "by virtue of the same thing".

In a recently published paper, Spencer Johnston accepted this futile defence of AT1 and developed a formal semantics that would fit Kilwardby's presumably connexive implication. This procedure, however, is misguided because the remaining considerations of Lesson 55 — which were entirely ignored by Johnston — show that Kilwardby eventually recognized that AT2 is bound to fail. After all he concluded: "So it should be granted that from the impossible its opposite follows, and that the necessary follows from its opposite".

Keywords: connexive logic; Kilwardby; Aristotle's Thesis

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1. Introduction

In a paper recently published in this journal, Spencer Johnston developed a formal semantics for "the logic" — or, at least, for some features of the logic — of the 13$^{\text{th}}$ century logician Robert Kilwardby. In particular, within Lesson 55 of his impressive work *Notule libri Priorum*, Kilwardby distinguishes between two kinds of implication: *natural* implication as opposed to *accidental* implication. According to Johnston, the former relation can be characterized by three conditions:

> 1. The natural implication relationship validates Aristotle's Thesis [...].
> 2. Disjunction introduction is a natural inference.
> 3. The natural implication relationship does not validate *ex impossibile quodlibet*. [2, p. 472]

These features, however, appear hardly reconcilable. On the one hand, if an implication operator '$\rightarrow$' validates Aristotle's Theses, then it is *connexive*, which means that *no* proposition q is implied both by a proposition p and by p's negation:

AT1 $$\sim((p \rightarrow q) \wedge (\sim p \rightarrow q)).$$

Alternatively, the connexivity of '$\rightarrow$' may be characterized by the requirement that no proposition p is ever implied by its own negation:[1]

AT2 $$\sim(\sim p \rightarrow p).$$

As should be evident from these formulas, '$\sim$' and '$\rightarrow$' are here used as symbols for *negation* and (any kind of) *implication*, respectively. As a default assumption, '$\rightarrow$' may be viewed as a strict implication, while *material* implication would be symbolized by '$\supset$'. Furthermore, '$\wedge$' and

[1] Cf. [13]: "One basic idea [of connexive logic] is that no formula implies or is implied by its own negation. This conception may be expressed by requiring [...] that certain schematic formulas [... such as AT1, AT2] are theorems." The "passive" versions of the connexive principles as formulated by Aristotle (by means of the expression 'is implied by') are basically equivalent to the "active versions" saying that no proposition p implies its own negation, or that no proposition p implies both a proposition q and $\sim q$. Some related criteria have been put forward in [9, p. 569]: "(1) No proposition implies its own negation. (2) No proposition implies each of two contradictory propositions. (3) No proposition implies every proposition. (4) No proposition is implied by every proposition".

'$\vee$' abbreviate *conjunction* and *disjunction*; and '$\Diamond$', '$\Box$' symbolize the modal operators 'it is *possible* that' and 'it is *necessary* that'.

Now if a relation '$\rightarrow$' fulfils Johnston's condition 1, it has to fulfil condition 3, too, i.e. '$\rightarrow$' cannot satisfy *ex impossibili quodlibet*:

EIQ If p is impossible, then, for every q, $(p \rightarrow q)$.

For if '$\rightarrow$' were to satisfy EIQ, then the impossible proposition $(p \wedge \sim p)$ would imply *every* proposition q; hence it would also imply its own negation, in contradiction to AT1:

ANTICONN1 $(p \wedge \sim p) \rightarrow \sim(p \wedge \sim p)$.

Similarly, if '$\rightarrow$' is connexive, it cannot satisfy the related principle *necessarium ex quodlibet*:

NEQ If q is necessary, then, for every p, $(p \rightarrow q)$.

Otherwise the necessary proposition $(p \vee \sim p)$ would be implied by *any* proposition, hence, again in contradiction to AT1, it would be implied by its own negation, too:

ANTICONN2 $\sim(p \vee \sim p) \rightarrow (p \vee \sim p)$.

However, if an implication operator '$\rightarrow$' satisfies either *conjunction elimination*:

CONJ1 $(p \wedge q) \rightarrow p$,

CONJ2 $(p \wedge q) \rightarrow q$,

or *disjunction introduction*:

DISJ1 $p \rightarrow (p \vee q)$,

DISJ2 $q \rightarrow (p \vee q)$,

then ANTICONN1, ANTICONN2 become provable, provided that '$\rightarrow$' satisfies *transitivity* and *contraposition*:

TRANS If $(p \rightarrow q) \wedge (q \rightarrow r)$, then $(p \rightarrow r)$,

CONTRA If $(p \rightarrow q)$, then $(\sim q \rightarrow \sim p)$.

In [5] it was already shown that ANTICONN1 follows from CONJ1, CONJ2. Let us now prove that ANTICONN2 follows similarly from DISJ1, DISJ2:[2]

(i)	$p \to (p \vee \sim p)$	DISJ1
(ii)	$\sim p \to (p \vee \sim p)$	DISJ2
(iii)	$\sim(p \vee \sim p) \to \sim p$	CONTRA (i)
(iv)	$\sim(p \vee \sim p) \to (p \vee \sim p)$	TRANS (iii), (ii).

Hence if Kilwardby's natural implication really satisfies Johnston's condition 1, it would be connexive. But if it satisfies condition 3 (plus TRANS and CONTRA), it would *not* be connexive. The task of this note is to resolve this contradiction by examining:

- whether Kilwardby's $(p \to q)$ satisfies principles TRANS and CONTRA;
- whether Kilwardby's $(p \to q)$ satisfies the full — or perhaps only a restricted — version of the connexivist principles AT1 and AT2; and
- whether Kilwardby's $(p \to q)$ satisfies the full — or perhaps only a restricted — version of the anti-connexivist principles EIQ, NEQ or their corollaries ANTICONN1 and ANTICONN2.

2. Some facts about connexive implication

One rather trivial point which may not always have been observed with sufficient clarity should be stressed from the very beginning:

THEOREM 1. *Neither Aristotle's Thesis nor any other characteristic axiom of connexivity holds for material implication.*

Loosely speaking, "fifty percent" of all propositions materially imply their own negations, and "fifty percent" of all propositions are materially implied by their own negation. For whenever p is *true*, then $(\sim p \supset p)$ is *true*, and whenever p is false, then $(p \supset \sim p)$ is true. Hence, with respect to issues of connexivity, material implication is "out of the game" from the very beginning, and we can suppose in what follows that '$\to$' always stands for a *strict* implication (or some other implication stronger than '$\supset$').

[2] Johnston does not explicitly state that Kilwardby's natural implication satisfies CONJ1 and CONJ2, but he says: "Inferences like disjunction introduction (and also, as we shall see, conjunction elimination) preserve this notion of meaning containment" [2, p. 472]. According to Johnston, the condition of meaning containment is a characteristic feature of natural implication.

However, for "almost all" systems of *strict implication*, the following *restrictions* of the characteristic axioms of connexive implication turn out to be *provable*. Aristotle's Theses (and related "passive" principles formulated by means of the expression 'is implied by') just have to be restricted to *non-necessary consequents*:

AT1$_{\text{rest}}$ If $\sim\Box q$, then $\sim((p \to q) \wedge (\sim p \to q))$,

AT2$_{\text{rest}}$ If $\sim\Box p$, then $\sim(\sim p \to p)$.

The "active" counterparts (formulated by means of 'implies')[3] instead have to be restricted to *self-consistent antecedents*, e.g.:

AT3$_{\text{rest}}$ If $\Diamond p$, then $\sim((p \to q) \wedge (p \to \sim q))$,

AT4$_{\text{rest}}$ If $\Diamond p$, then $\sim(p \to \sim p)$.

The validity of these principles in almost all systems of modal logic can be recognized as follows. Suppose that '$\to$' is *reflexive* and satisfies the principle of *conjunction-introduction*:

REFL $(p \to p)$,

CONJ3 If $(p \to q) \wedge (p \to r)$, then $(p \to (q \wedge r))$.

Suppose furthermore that '$\to$' *preserves self-consistency*, i.e., if p implies q and if p is possible, then q must be possible, too:

POSS If $(p \to q)$ and $\Diamond p$, then $\Diamond q$.

Suppose, finally, that, within the respective system of modal logic, the contradiction $(p \wedge \sim p)$ is (provably) *impossible*:

IMPOSS $\sim\Diamond(p \wedge \sim p)$.

Then, e.g., condition AT2$_{\text{rest}}$ may be proven as follows:

(i)	$\sim p \to p$	Assumption
(ii)	$\sim p \to \sim p$	REFL
(iii)	$\sim p \to (p \wedge \sim p)$	(ii), (i), CONJ3
(iv)	If $\sim p \to (p \wedge \sim p) \wedge \Diamond\sim p$, then $\Diamond(p \wedge \sim p)$	POSS
(v)	If $\Diamond\sim p$, then $\Diamond(p \wedge \sim p)$	(iii), (iv)

[3] The "active" principles A3 $\sim((p \to q) \wedge (p \to \sim q))$, A4 $\sim(p \to \sim p)$ are sometimes referred to as 'Boethius' Thesis' and 'Abelard's Thesis', respectively.

(vi) $\sim\Diamond(p \wedge \sim p)$ IMPOSS
(vii) $\sim\Diamond\sim p$ (v), (vi)

Hence the assumption that proposition p is (strictly) implied its own negation entails that $\sim p$ is impossible, i.e. that p is *necessary*.

More generally, *restricted connexivity*, or "*humble*" connexivity, as some logicians prefer to call it [cf., e.g., 3, section 3], is not an *extra* requirement which strict implication might either fulfill or not; rather:

THEOREM 2. *Strict implication is* (*almost always*)[4] *"humbly" connexive.*

Let us therefore turn to "unhumble" connexivity! Since, e.g., according to $\mathrm{AT4}_{\mathrm{rest}}$, no *self-consistent* proposition entails its own negation *anyway*, the "hardcore" version boils down to the claim that (even) if p is self-*inconsistent*, p does not entail its own negation:

HC1 If $\sim\Diamond p$, then $\sim(p \to \sim p)$.

Given that $(p \wedge \sim p)$ is the paradigm of an impossible proposition, HC1 entails:

HC2 $\sim[(p \wedge \sim p) \to \sim(p \wedge \sim p)]$.

Hence, any "hardcore connexivist" has to uphold that the *contradictory* proposition $(p \wedge \sim p)$ does *not* entail the *tautology* $\sim(p \wedge \sim p)$.

3. Aristotle and "Aristotle's Theses"

In *Prior Analytics* II, 4, 57b3–14 Aristotle dealt with the issue that a true conclusion may well follow from false premises but that such inferences cannot be *necessary*:

> But it is impossible that the same thing should be necessitated by the being and by the not-being of the same thing. I mean, for example, that it is impossible that B should necessarily be great if A is white and that B should necessarily be great if A is not white. For whenever if this, A, is white[,] it is necessary that that, B, should be great, and if B is great that C should not be white, then it is necessary if A is white that C should not be white. And whenever it is necessary, if one of two things is, that the other should be, it is necessary, if the latter is not, that the former should not be. If then B is not great[,] A cannot be

[4] 'Almost always' means that the underlying system of strict implication satisfies at least REFL, CONJ3, POSS and IMPOSS.

> white. But if, if A is not white, it is necessary that B should be great, it necessarily results that if B is not great, B itself is great. But this is impossible. [1, p. 91]

About 1,500 years later, in his extensive commentary on Aristotle's work, Kilwardby paraphrased this argument as follows:

> Next he proves [...] that a truth does not follow from falsehoods of necessity, like this. Something's being so does not follow of necessity from the same thing's being so and not being so. [...] He expounds it, saying that when it follows of necessity 'If A is white, B is great', it does not follow of necessity 'If A is not white, B is great'. [...] Second he proves it as follows. If from A's being white it follows of necessity that B is great, then from the denial of the consequent, if B is not great, A is not white. But ex hypothesi it follows 'If A is not white, B is great'. So, from the first to the last, it follows 'If B is not great, B is great'. But this is impossible. [10, p. 1139]

Another 800 years later, Storrs McCall summarized and simplified this so:

> What Aristotle is trying to show here is that two implications of the form 'If p then q' and 'If not-p then q' cannot both be true. The first yields, by *contraposition*, 'If not-q then not-p', and this together with the second gives 'If not-q then q' by *transitivity*. But, Aristotle says, this is impossible: a proposition cannot be implied by its own negation. [8, p. 415; my emphasis]

Three indubitable points can be extracted from these passages:

1. Aristotle proved that if principle AT1 would not hold, i.e. if there would exist propositions p, q such that $(p \to q) \wedge ({\sim}p \to q)$, then there would exist a proposition q such that $({\sim}q \to q)$, i.e. principle AT2 would not hold either.
2. Aristotle considered both principles AT1 and AT2 as *valid*. So it is certainly justified to refer to them as 'Aristotle's Theses'.
3. But Aristotle also considered the principle of the transitivity of '$\to$' and the principle of contraposition as *valid*.

Thus the Aristotelian conception of implication, '$p \to q$' faces a similar dilemma as Kilwardby's natural implication mentioned above. On the one hand, '$p \to q$' would have to be *connexive* because it is assumed to satisfy AT1 and AT2. On the other hand, provided that '$p \to q$' satisfies the principles of *conjunction elimination* and/or *disjunction introduc-*

tion, it would have to be *anti-connexive*.[5] Evidently there are only three possible ways out of this dilemma:

(I) To argue that Aristotle was a "hardcore connexivist" who somehow rejected TRANS and/or CONTRA.
(II) To argue that Aristotle was a "hardcore connexivist" who somehow rejected CONJ1, CONJ2, and DISJ1, DISJ2.
(III) To argue that Aristotle was just a "humble connexivist" who would never have accepted the *unrestricted* principles HC1 or HC2.

Strategy (III) has been explained and defended by me in a recently published paper[6]. There it was admitted that Aristotle's writings contain no *direct* evidence for his rejection of "hardcore connexivism". In particular, he nowhere subscribed to principles like *ex impossibili quodlibet* or *necessarium ex quodlibet* (or their corollaries ANTICONN1, ANTICONN2). In *Prior Analytics* II, 15 he only examined which syllogistic inferences may be drawn from a *pair* of *contradictory* premises[7], but he never considered the crucial issue which logical inferences might be drawn from a *self-contradictory* proposition. Nevertheless in the light of the available evidence it seems extremely likely that Aristotle would have accepted the proof of ANTICONN1 presented in Section 1 above.

Let us now turn to Kilwardby's treatment of this topic which — at least partially — follows strategy (II). Strategy (I), by the way, does not seem to have been taken into consideration by anybody so far.

4. Kilwardby's defence of AT1

Having summarized Aristotle's derivation of AT1 from AT2 (as quoted in the previous section), Kilwardby points out that "there is a doubt about the major premise [i.e., about AT1] in his [i.e., Aristotle's] argument". As a matter of fact, this doubt consists of *three* objections:

> [1] For it seems that one and the same thing does follow from the same thing's being so and not being so, because if you are sitting then God

[5] Cf. the derivation of ANTICONN1 and ANTICONN2 by means of CONTRA and TRANS in Section 1 above.

[6] Cf. [6] and the underlying considerations in [5].

[7] E.g., according to mood Camestres (of the "Second Figure") the contrary premises 'Every science is good' and 'No science is good' entail the absurd conclusion 'No science is a science'.

> exists and if you are not sitting then God exists, because the necessary follows from anything.
> [2] Further, if you are sitting then one of the following is true: that you are sitting, that you are not sitting. And if you are not sitting, one of them is true.
> [3] Further, a disjunctive follows from either of its parts, and in a natural inference. Hence it follows 'If you are sitting you are sitting or you are not sitting' and 'If you are not sitting, you are sitting or you are not sitting'. And thus one and the same thing follows in a natural inference, and thus of necessity, from the same thing's being so and not being so.[8]

Objection [1] is based on the theological doctrine that God's existence can be *proved* and thus is *necessarily* true. In accordance with the principle *necessarium sequitur ex quodlibet*, the proposition $q =$ 'God exists' hence "follows" from any other proposition p, e.g. from the contingent proposition that a certain person P (addressed by Kilwardby as 'you') is sitting, but also from the contrary proposition that P is not sitting, $\sim p$. The pair of implications $\langle p \to q, \sim p \to q\rangle$ thus constitutes a first counterexample to AT1! However, Kilwardby replies:

> To the first objection it should be said that the inferences are of two types, viz. essential or natural (as when the consequent is naturally understood in the antecedent), and incidental inferences. Now it is inferences of the latter type according to which we say that the necessary follows from anything; and Aristotle's remarks are not to be understood as being about these. [10, p. 1141]

Kilwardby here resorts to the distinction between two kinds of an implication that was mentioned already in Section 1: *natural* implication as opposed to *accidental* implication. Without further ado, Kilwardby simply maintains that the full principle *necessarium ex quodlibet* holds only for accidental implication, while natural implication presumably satisfies NEQ only in a restricted form.

In objections [2] and [3], the *theologically* necessary proposition 'God exists' is replaced by the *logically* necessary proposition '$p \vee \sim p$'. Thus the pair $\langle (p \to (p \vee \sim p)), (\sim p \to (p \vee \sim p))\rangle$ forms another counterexample to AT1. The main difference between objections [2] and [3] consists in the underlying *justification*. In [2], the validity of the two implications seems to be based again on principle *necessarium ex quodlibet*, while

[8] Cf. [10, p. 1141]; in Thom and Scott's excellent edition of Kilwardby's text, the numbering of the objections is added in the margin.

[3] explicitly refers to the principles of *disjunction introduction*. The latter argument is thus stronger than the former one because it escapes Kilwardby's reply to objection [1]. As the virtual objector stresses, the inferences $(p \to (p \vee \sim p))$ and $(\sim p \to (p \vee \sim p))$ can no longer be rejected as merely *accidental* implications; $(p \vee \sim p)$ rather follows from p and from $\sim p$ — as Kilwardby had to admit — "in a *natural* inference, and thus of *necessity*".

Yet Kilwardby replies to these objections as follows:

> To the second objection it should be said that the same thing can follow in two ways, viz. either by virtue of the same thing in it (and in this way one and the same thing cannot follow from the same thing affirmed and denied, and this is what Aristotle means), or by virtue of different things in it (and in this way one and the same thing can follow from the same thing affirmed and denied, and this is not what Aristotle means; but the objection proceeded on this basis).
>
> The third is also solved by this means, because a disjunctive follows from both of its parts by virtue of different things in it, not by virtue of the same thing. [10, pp. 1141–1143]

Since the second counterexample can no longer be refuted by simply requiring that the respective implications have to be *natural* — both p and $\sim p$ *do* naturally entail $(p \vee \sim p)$ — Kilwardby hastens to add another condition: The inferences have to obtain "by virtue of the same thing in it". Without any attempt to *clarify* or to *justify* this *ad hoc* requirement, Kilwardby concludes the discussion of Doubt 1 by presenting the following "Solution":

> So Aristotle understands that something does not follow of necessity from the same thing's being so and not being so, in a natural inference, and by virtue of the same thing. [10, p. 1143]

It should be clear that this attempt to save principle AT1 does not truthfully reflect *Aristotle's* opinion of the matter. The Stagirite nowhere even considered the *notion* of a natural inference proceeding "by virtue of the same thing". The quoted "solution" at best expresses *Kilwardby's* view of the problem, but even *this* view remains far from clear. How, in particular, shall we understand the extra requirement that, given two implications $(p_1 \to q)$ and $(p_2 \to q)$, the consequent q follows "by virtue of the same thing" from the antecedents p_1, p_2? Since Kilwardby himself provided no answer, Johnston tried to explain this as follows:

> Kilwardby's claim that "when a proposition follows in natural consequences from each of two contradictory antecedents, this is in virtue of two different things" is best understood in the following way: Say we start a deduction by assuming A and B. For ease of the example, assume we have one rule for or-introduction, namely from 'ϕ' infer 'ϕ or ψ' and another rule that allows us to infer 'ψ or ϕ' from 'ϕ or ψ'. Then we can infer 'A or B' from 'A' and we can infer 'B or A' from 'B', from which it follows that 'A or B'. Kilwardby's point here is that this inference is naturally valid, but the grounds or the basis for inferring the disjunction is different in each case. In one case it is based on the content of 'A' and in the other case it is based on the content of 'B'. This again is further evidence for thinking of natural consequences as preserving the content of the antecedent from which a particular consequent is inferred. The idea here is that natural consequences are sensitive to which propositions are used to ground or justify the inference that follows. [2, p. 462]

The rules of disjunction introduction are usually taken to be perfectly *symmetric*. As DISJ1, DISJ2 show, one and the same disjunction $(p \vee q)$ follows both from the "left" disjunct p and from the "right" disjunct q. Spencer, however, tries to break this symmetry by assuming that, e.g., only DISJ1 functions as a *basic* rule or axiom while DISJ2 becomes a *theorem* derivable from DISJ1 in conjunction with the additional basic principle

DISJ3 $$(p \vee q) \to (q \vee p).$$

In my opinion, this move is just as *ad hoc* as Kilwardby's introduction of the extra requirement that two natural implications must proceed "by virtue of the same thing".[9] Moreover, this move alone does not suffice to save principle AT1 because, as was pointed out earlier, anti-connexive results can be obtained not only by means of *disjunction* introduction but also by means of *conjunction* elimination. And it would seem very hard to deny that, given two natural implications $((p \wedge q) \to p)$ and

[9] The "ad-hoc" reproach does not equally apply to Paul Thom's brand-new (2019) attempt to explain and defend Kilwardby's idea of naturally implying "by virtue of the same thing". One particular feature of Thom's reconstruction is that "primary essential consequences" are *not transitive*, a feature which is said to be shared also by the "Von Wright-Geach-Smiley notion of entailment" [11, p. 177]. Unfortunately, this approach cannot be discussed here. I am grateful to an anonymous referee for having drawn my attention to Thom's important work.

$((p \wedge q) \to q)$, the respective consequents p and q are implied by the antecedent $(p \wedge q)$ "in virtue of the same thing"!

What is even more important, however, is that the quoted "Solution" does not represent Kilwardby's final judgment about the validity of Aristotle's Theses at all. In addition to Doubt1 directed against AT1, Kilwardby's Lesson 55 contains two further doubts directed against AT2, and this additional material, *which was entirely ignored by Johnston*, throws an important new light on the central issue whether Kilwardby was a "hardcore connexivist" or not.

5. Two doubts concerning AT2

> [Doubt 2] After that there is a doubt about his [Aristotle's] concluding that 'If B is not great, B is great' is unacceptable. For this does not seem to be unacceptable, since one opposite may well follow from another as in 'If you are an ass, you are not an ass' because anything follows from the impossible and the necessary follows from anything. [10, p. 1143]

For readers who are not so familiar with the history of logic it may be helpful to point out that 'You are an ass' was never meant as a personal insult. Rather, it's a medieval standard example of an "impossible" assertion because the addressee of any assertion is a human being. But 'No human being is an ass' is an analytic truth. Hence, for any person P, the proposition $p =$ 'P is an ass' is (analytically) "impossible", while $\sim p =$ 'P is not an ass' is (analytically) "necessary". Thus, in accordance with either EIQ or NEQ one obtains $(p \to \sim p)$ as a counterexample to AT2.

However, Kilwardby replies:

> [Solution] And it should be said that inferences are of two types, viz. positing or incidental inferences (and here there is nothing unacceptable in one opposite's following from another, as has been shown), and ones that are natural or essential (and here one opposite does not follow from another, and this is what Aristotle means). [10, p. 1143]

Here Kilwardby once again resorts to the distinction between accidental[10] and natural inferences. He admits that the negation of AT2, i.e.

[10] Kilwardby speaks of a 'consequentia positiva sive accidentalis'; Thom and Scott's translation 'positing or incidental' seems a bit less suitable than 'accidental'.

($\sim p \to p$), may well be satisfied for some accidental inferences; but he sticks to the claim that a proposition never follows *naturally* from its opposite. However, this reply is not the end of the story, because another objection is lurking:

> [Doubt 3] However, according to an alternative suggestion, not just anything follows from the impossible, but anything besides its opposite; and the necessary does not follow from just anything, but from anything besides the opposite. About which there is an incidental doubt.
> [1] And it seems that this is true for the following reason. Nothing posits the same thing and destroys it; but the antecedent posits its consequent; so it does not destroy it. But one opposite destroys the other; so it is not an antecedent to it. Hence neither of a pair of opposites follows from the other, and neither is an antecedent to the other.
> [2] Further, the consequent belongs to the actual understanding of the antecedent. But one opposite does not belong to the actual understanding of the other. [10, p. 1143]

The main idea of Doubt 3 — as formulated in the first paragraph of the quote — is to restrict the principles *ex impossibili quodlibet* and *necessarium ex quodlibet* in the following way:

EIQ* If p is impossible, then, for every $q \neq \sim p$: $(p \to q)$,

NEQ* If q is necessary, then, for every $p \neq \sim q$: $(p \to q)$.

This suggestion is then supported by two somewhat obscure arguments. The first one says that, in an inference $(p \to q)$, the antecedent p always "posits" the consequent q and does not "destroy" it. Since, in general, any proposition q is "destroyed" by its own negation, q can never entail or be entailed by $\sim q$. The second argument maintains that the "understanding" or meaning of the consequent is generally contained in the meaning of the antecedent; but the meaning of the opposite of p can never be contained in the meaning of p; hence p can never imply or be implied by $\sim p$.

With reference to Doubt 3, Kilwardby replies:

> To the contrary, as follows. If from the impossible there follows anything that is not inconsistent with it, then since your running is not inconsistent with your being an ass, it follows 'If you are an ass, you are running'. And if the necessary follows from anything that is not inconsistent with it, and your running is not inconsistent with your not being an ass, it follows 'If you are running you are not an ass'. Hence,

> from the first, it follows 'If you are an ass, you are not an ass'. And thus in such inferences one opposite does follow from another. And this is to be granted according to Augustine, who says 'If there is no truth, it follows that there is truth'. [10, p. 1145]

This passage contains some very interesting ideas. First of all, the restricted principles EIQ* and NEQ* are now formulated somewhat differently by requiring that the conclusion q must "*not be inconsistent*" ("non repugnans") with the antecedent p, while according to the earlier formulation, q just had to be *different from the negation* of p:

EIQ** If p is impossible, then,
for every q such that q is compatible with p: $(p \to q)$,

NEQ** If q is necessary, then,
for every p such that p is compatible with q: $(p \to q)$.

Given any "normal" implication $(p \to q)$, the "new" condition of the compatibility of p and q entails the "old" condition $p \neq \sim q$, but *not vice versa*, because even if $p \neq \sim q$, p may well be incompatible with q.[11] However, what is at stake in connection with principles EIQ*/EIQ** and NEQ*/NEQ** is not a "*normal*" implication but one with an *impossible* antecedents and/or a *necessary* consequent, and with respect to such propositions the situation is quite different.

On the one hand, NEQ** turns out to be *equivalent* to NEQ*. NEQ* requires that any necessary consequent q is entailed by every antecedent p except for the case where p is the *negation* of a *necessary* proposition and where p thus is *impossible*. This very condition, however, is also captured by NEQ** which requires that, in order for $(p \to q)$ to hold, p must be *compatible* with the *necessary* consequent q; but being compatible with a necessary proposition just means that p has to be *self-consistent*.

On the other hand, the re-formulated condition EIQ** is not at all equivalent to the earlier condition EIQ* which required that an impossible antecedent p entails *every* proposition q with the only exception of $q = \sim p$. At least according to the *modern* understanding of the relation of (logical) compatibility, two propositions p, q are *incompatible* if and only if their conjunction, $(p \wedge q)$, is impossible. Hence, if p alone is already impossible, then so is $(p \wedge q)$, for any q. In other words, there simply

[11] E.g., when p is *contrary* (and not just *contradictory*) to q!

does not exist any q which would be compatible with an impossible antecedent p. Therefore EIQ** is vacuously satisfied for every impossible antecedent p, no matter whether $(p \to q)$ holds or not.

These difficulties strongly suggest that, when re-formulating principles EIQ* and NEQ* in the way of EIQ** and NEQ**, Kilwardby must have had a *different understanding* of the *compatibility* of two propositions in mind. Anyway, we want to assume that his reply to Doubt 3 (as developed after the introductory words 'To the contrary, as follows') is based on restrictions of *ex impossibili quodlibet* and *necessarium ex quodlibet* in the sense of EIQ*, NEQ* and not in the sense of EIQ**, NEQ**. Kilwardby's argument can then be analyzed as follows.

In Doubt 2 the counterexamples to AT2 (and to its "active" counterpart AT4; cf. footnote 3) had been obtained by an application of *either* EIQ *or* NEQ: An impossible antecedent p entails *every* consequent, hence it entails also $\sim p$. Similarly, a necessary consequent q is entailed by every antecedent, hence also by $\sim q$. In order to inhibit these straightforward refutations, Doubt 3 suggests *restricting* EIQ to EIQ* and NEQ to NEQ*. Clearly, these restricted principles no longer admit the *direct* derivation of $(p \to \sim p)$ or $(\sim q \to q)$. But — and this is the "clou" of the quoted passage — an *indirect* refutation of AT2 (or AT4) still remains possible by *combining* two inferences in accordance with EIQ* and NEQ*! Start from any impossible proposition p like, e.g., 'Person P is an ass'! Next take any contingent proposition q like, e.g., 'P is running'. Since $q \neq \sim p$, EIQ* allows us to conclude that $(p \to q)$. But the *contingent* proposition q is also $\neq$ any *necessary* proposition r, in particular $q \neq \sim p$. Hence NEQ* admits to conclude that $(q \to \sim p)$, so that by means of Trans one gets $(p \to \sim p)$!

Thus in the end Kilwardby is forced to admit that "in such inferences one opposite *does* follow from another." Furthermore, Kilwardby, who was not only a logician but also a member of the Dominican order (and later even became Archbishop of Canterbury), somehow seems to be *satisfied* with this result because it accords with the teaching of the Church Father Augustine, who is supposed to have said: "If there is no truth, it follows that there is truth". In his work *Contra Academicos* Augustine critically examined the doctrine of the sceptics. In particular, he put forward his famous argument '*si enim fallor, sum*' which means that even if he, Augustine, would be mistaken with respect to any proposition p, there would remain one indubitable truth q, namely that he exists. This idea may rightly be considered as an anticipation of Descartes' famous cogito-argument from the *Meditations*.[12] However, Kilwardby's paraphrase of Augustine's anti-scepticism may also be understood as a variant of the *Liar paradox* in the following way.

When a sceptic, S, maintains 'There is no truth', he (or she) maintains that there does not exist any proposition p such that p is true. With the help of the quantifier '$\exists p$' ('there is at least one proposition p'), and the meta-linguistic truth-predicate '$\mathrm{T}(p)$' ("p' is true') this thesis may be formalized as $\sim\exists p\,\mathrm{T}(p)$. Now, whenever someone *maintains* something, q, he implicitly maintains q *to be true*! Hence S's maintaining $\sim\exists p\,\mathrm{T}(p)$ somehow *implies* maintaining $\mathrm{T}(\sim\exists p\,\mathrm{T}(p))$. But from $\mathrm{T}(\sim\exists p\,\mathrm{T}(p))$ one may derive by means of *existential generalization* that there exists as least one truth q, namely $q = \sim\exists p\,\mathrm{T}(p)$. Thus Augustine's refutation of S's universal scepticism may be interpreted as having the structure $\sim\exists p\,\mathrm{T}(p) \to \exists q\,\mathrm{T}(q)$, or $\sim\exists p\,\mathrm{T}(p) \to \exists p\,\mathrm{T}(p)$, i.e. a counterexample to AT2! Anyway, Kilwardby concludes the discussion of Doubt 3 by the following

> [Solution]: So it should be granted that from the impossible its opposite follows, and that the necessary follows from its opposite.
>
> [10, p. 1145]

6. Summary and conclusion

Robert Kilwardby is a highly gifted logician, and his *Notule libri Priorum* contain many interesting observations concerning Aristotle's *Prior Analytics*.[13] In "Lesson 55" Kilwardby tried to defend "Aristotle's Theses",

[12] Cf. [7, p. 1]: " 'Denn wenn ich mich täusche, bin ich'." Mit dieser einprägsamen Formulierung beginnt eine jener Textpassagen, in denen Augustinus dasjenige Argument vorstellt, welches als sein *Cogito* bekannt geworden ist. In einer vorläufigen Interpretation [. . .] konstatiert er hier, dass bei aller möglichen Täuschung des menschlichen Geistes, die Existenz des Subjekts, des Ichs, welches getäuscht wird, nicht geleugnet werden kann. Offensichtlich weist die Augustinische Argumentation eine gewisse sprachliche und gedankliche Nähe zu jenen Gedankengängen auf, die Descartes in seinen *Meditationes de prima philosophia* formuliert und deren Kernelement wiederum als dessen *Cogito* in die Philosophiegeschichte eingegangen ist."

[13] Thom and Scott's edition [10] highlights the great importance of early medieval logic in general and of Kilwardby's logic in particular. In their standard historiography of logic, the Kneales had already devoted a short chapter to Kilwardby [cf. 4, pp. 274–277], but their exposition suffers from being based on only a few extracts of the *Notule libri Priorum* as edited by Ivo Thomas in [12].

AT1 and AT2, against various objections that had been put forward by other medieval logicians. These objections were mainly based on the principles *ex impossibili quodlibet* and/or *necessarium ex quodlibet*, EIQ and NEQ, but also on the rules of *disjunction introduction*, DISJ1 and DISJ2.

Doubt 1 was directed against principle AT1 and consisted of three objections. The last and strongest of them made use of DISJ1, DISJ2 by deriving $p \rightarrow (p \vee {\sim}p)$ and ${\sim}p \rightarrow (p \vee {\sim}p)$ as substitution instances. Hence, in contradiction to AT1, one and the same consequent, $(p \vee {\sim}p)$, follows both from the antecedent p and from its negation.

In a last, brave attempt to save AT1, Kilwardby introduced the *ad hoc* requirement that, given that a consequent q follows from two antecedents p_1 and p_2, the implications have to obtain "by virtue of the same thing". Johnston apparently contented himself with this dubious reply, and he spent his whole energy on developing a formal semantics that would fit Kilwardby's allegedly connexive implication. This procedure, however, is very dubious because the remaining considerations of "Lesson 55" (especially those related to Doubt 3) strongly speak in favour of the assumption that Kilwardby eventually recognized that the unrestricted ("hardcore") version of Aristotle's *Second* Thesis, AT2, is bound to fail.

7. Epilogue

Since, in the last section, I blamed Johnston for having ignored some important passages from Kilwardby's text which appear to contradict his (Johnston's) conclusions, I do not want to give anyone a chance to blame *me* for ignoring, or even wilfully suppressing, further passages which might contradict *my* own views.

It is true, though, that the decisive "[Solution] So it should be granted that from the impossible its opposite follows, and that the necessary follows from its opposite" is the *last* of a series of "Solutions" which Kilwardby had offered during the discussion of "Aristotle's Theses". It is also true that, in general, when Kilwardby unfolds a dialectical treatment of an issue, his remark 'Solution' (written on the margin of the manuscript) indicates that this passage represents his *final view* of the matter. Yet Lesson 55 does not literally *end* with the above-quoted "Solution" but rather with two additional reflections:

> To the first objection it might be said that one opposite destroys another without qualification, but posits it under a condition, and thus it follows from it under a condition. But this does not suffice, because just as it posits it under a condition it destroys it under a condition, because from one opposite's being so it always follows that the other is not so. On this account it should be said that one opposite destroys the other and does not posit it. But the reason why in such inferences one opposite follows from the other is not that one posits the other, but the consequent is posited only on account of its own necessity and not on account of its antecedent. So in natural inferences the antecedent posits its consequent, but in incidental inferences this is not necessary.
>
> From this the reply to the other point is clear. For it is only in natural inferences that the consequent has to be actually understood in the antecedent, and this does not have to be so in incidental inferences. [10, p. 1145]

The *function* of these considerations remains rather unclear, above all because the structure of Kilwardby's discussion of Doubt 3 is highly complex. Remember, the main task of Lesson 55 was to investigate whether Aristotle's Theses are valid or not. In order to defend AT1 and AT2, Kilwardby normally first presented a counterargument ("Doubt"), then replied to this doubt, and finally drew a conclusion ("Solution"). In the case of Doubt 1 and Doubt 2, these conclusions ended up in favour of the *connexive* principles AT1 and AT2.

Doubt 3, in contrast, consists of a main argument (suggesting that *ex impossibili quodlibet* and *necessarium ex quodlibet* should be restricted to EIQ* and NEQ*), supplemented by two considerations, [1] and [2], which must be meant to *support* the former argument since they are introduced by saying that "this is true for the following reason". Next follows a reply to the main argument ("To the contrary") which ends up with the *anti-connexive* conclusion that "in such inferences one opposite does follow from another". Although this conclusion is afterwards re-affirmed and explicitly emphasized as the "Solution", Kilwardby continues (and finishes) Lesson 55 with the above quoted remarks *against* [1] and [2]. These remarks, however, appear to have a slightly connexive feel, after all.

Thus the remark that "only in natural inferences [...] the consequent has to be actually understood in the antecedent" might be taken to indicate that Kilwardby considered counterexamples of the type $(p \rightarrow \sim p)$ (with an impossible antecedent p) not as natural implications. *Maybe* — in Kilwardby's opinion — the tautological consequent $\sim p$ is not "actually understood" in the contradictory antecedent p. *Maybe* Kilwardby wanted to refuse the character of "naturalness" to such implications because an impossible antecedent p does not "posit" the tautological consequent $\sim p$. Thus is seems possible that future investigations will come up with the claim that a connexive logic might be defended along Kilwardbyan lines, after all. Until then, however, historians of logic should accept the lesson that Kilwardby taught us in Lesson 55: An impossible antecedent *does* entail its own negation, although perhaps this consequent "is posited only on account of its own necessity and not on account of its antecedent".

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