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ω -Circularity of Yablo's Paradox

Abstract. In this paper, we strengthen Hardy's [1995] and Ketland's [2005] arguments on the issues surrounding the self-referential nature of Yablo's paradox [1993]. We first begin by observing that Priest's [1997] construction of the binary satisfaction relation in revealing a fixed point relies on impredicative definitions. We then show that Yablo's paradox is ' ω -circular', based on ω -inconsistent theories, by arguing that the paradox is not self-referential in the classical sense but rather admits circularity at the least transfinite countable ordinal. Hence, we both strengthen arguments for the ω -inconsistency of Yablo's paradox and present a compromise solution of the problem emerging from Yablo's and Priest's conflicting theses.

Keywords: self-reference; Yablo's paradox; ω -circularity; ω -inconsistent theories; impredicativity

Introduction

Many paradoxes in pure logic are believed to be caused by certain kind of circularity. Stephen Yablo [1993], however, proposed a very interesting paradox which is not self-referential or circular in any way. In this paper, we give arguments for strengthening Hardy's [1995] and Ketland's [2005] claims on the issues surrounding the self-referential nature of Yablo's paradox. We first begin by pointing out that Priest's [1997] construction of the binary satisfaction relation in revealing a fixed point relies on impredicative definitions. We then show that Yablo's paradox is ' ω -circular', based on ω -inconsistent theories. We argue that Yablo's paradox is not self-referential in the classical sense but rather admits circularity at the least transfinite countable ordinal. In this way, we demonstrate that Yablo's paradox is not circular in the classical sense but is circular in the *limit*. Hence, we aim to improve Hardy's and Ket-

land's theses for the ω -inconsistency of Yablo's paradox and present a compromise solution of the problem emerging from Yablo's and Priest's conflicting claims. Throughout the paper, the term 'Yablo's paradox' may occasionally be used, depending on the context, to refer to the set of sentences or the contradiction that arises from these set of sentences.

1. Self-reference and impredicative definitions

As stated in Yablo's original paper, since Russell, the reason behind all paradoxes in logic has been traditionally believed to be caused by self-reference. Clearly, in the liar paradox for instance, the sentence "This statement is false" is self-referential and hence it is circular. This first type of circularity involves only a single sentence. The barber paradox is another famous example. There is a barber in the town who shaves every people who don't shave themselves. But then whom does the barber shave? We may take "The next statement is true. The previous statement is false." as another example of a liar-like paradox. This example gives us a second type of circularity involving more than one sentence. Many of the paradoxes that we know belong to one of these two types and self-reference has usually been taken as the main reason behind them. Nevertheless, not every self-referential statement leads to a paradox. For example, there is nothing wrong with the proposition "This statement is true" for the obvious reason that the statement may just be taken as a true proposition.

It is usually *impredicativity* that allows one to construct self-referential propositions. Roughly speaking, a definition is called *impredicative* if a part or the whole of the object being defined is referred to in a totality containing the object. Paul Cohen [1966, p. 86] gives a very nice example. Let us suppose that we accept integers and the operations for arithmetic. We can consider a set of integers defined by a property $P(n)$ which has all its bound variables ranging over integers. Such set is said to be *predicatively* defined in terms of the integers. Consider now the following property. Let us say that an integer n satisfies $P(n)$ if there is a partition of the set of integers into n disjoint sets none of which contains arithmetic progressions of arbitrary length. Let S be the set of all n such that $P(n)$ holds. Now in order to determine whether any integer, say 5, belongs to S , we have to consider *all* partitions of the set of integers into 5 sets possibly including partitions in which S occurs.

Therefore, in order to define S we are forced to assume a meaningful totality of the set of all sets of integers, i.e., the power set of the set of integers. This is an example of an impredicative definition. Another example is the definition of greatest lower bound in lattice theory.

Let A be a set. If

- (i) for every $x \in A$, $y \leq x$, and
- (ii) for every $x \in A$, $z \leq y$ whenever $z \leq x$,

then y is called the *greatest lower bound* of A . Notice that in (ii), we quantify over the set of *all* lower bounds of A , one of which is actually the greatest. We refer to a totality in which the defined object exists *a priori*. The existence of this totality of course may not be a problem for a mathematical realist, but it might be seen rather as an act of metaphysical faith for a mathematical constructivist.

Impredicativity, after all, was the major problem behind Russell's paradox when Frege's abstraction rule called *Basic Law V* was used in the language of naive set theory. It was Whitehead and Russell who later proposed *type theory* on a predicative basis in *Principia Mathematica* to overcome these problems caused by impredicativity.

Frege's *Basic Laws of Arithmetic* formally allows one to construct impredicative definitions when working with naive set theory. Let us revisit Basic Law V solely by using *concepts* and *extensions*. We may think of *concepts* as unsaturated predicates, i.e. functions. The *extension* of a concept is the collection of all objects falling under that concept. For instance, if we let K be the concept "integers x satisfying $x^2 = 4$ ", then the objects which fall under K are 2 and -2 . So the extension of K is the collection $\{2, -2\}$. Let us denote an object a falling under a concept F by $F(a)$, and let $Ext(F)$ denote the extension of F . Basic Law V can be summarized as follows: Let F and G be two concepts. The extension of F is equivalent (equinumerous) to the extension of G if and only if $F(a)$ is true if and only if $G(a)$ is true. That is,

$$Ext(F) = Ext(G) \leftrightarrow \forall a(F(a) \leftrightarrow G(a)).$$

It is a known fact that Basic Law V, when used in the form of the axiom schema of comprehension in naive set theory, leads to Russell's paradox. The reason why we have a paradoxical class $R = \{x : x \notin x\}$ is due to the assumption that the extension of R *includes* the class R . Any predicative definition however bans the use of such assumptions. We leave our discussion of predicativity here and will refer to it later when

needed. The reader may refer to [Feferman, 2005] for a good introduction to predicativism.

2. Yablo's paradox

According to Stephen Yablo [1993], in defiance of the traditional view, paradoxes without self-reference do exist. If a proposition A is true let us denote this by $T(A)$. It would perhaps be fairer to refer to it as the *Visser-Yablo paradox* given that a very similar antimony occurs in Visser [1989], is an infinite conjunction of statements S_1, S_2, S_3, \dots defined as follows:

$$\begin{aligned} S_1 &: \forall k > 1 \neg T(S_k), \\ S_2 &: \forall k > 2 \neg T(S_k), \\ S_3 &: \forall k > 3 \neg T(S_k), \\ &\vdots \end{aligned}$$

Of course we may think of these statements as an infinite sequence $\{S_i\}$. We shall use this notation later on. Now let us assume that $T(S_n)$ holds for some n . Then $\forall k > n \neg T(S_k)$. Therefore, in particular, $\neg T(S_{n+1})$ holds. But if $\forall k > n \neg T(S_k)$, then

$$\forall k > n + 1 \neg T(S_k) \tag{*}$$

holds. But then $(*)$ really says the same thing as S_{n+1} . Then $T(S_{n+1})$ must hold. A contradiction. So our assumption that $T(S_n)$ holds cannot be true. That is, $\neg T(S_n)$ must be true. Moreover, since n is arbitrary, S_n must be false for all n . Then, every statement comes after S_n must be false as well. Therefore, it must be the case that $T(S_n)$ since S_n actually says that any statement comes after itself is false. Again, a contradiction. Therefore, we cannot assign any truth value to any of S_n . At the end of his paper, Yablo says:

I conclude that self-reference is neither necessary nor sufficient for Liar-like paradox. [Yablo, 1993, p. 252]

3. Priest's inclosure schema and ω -inconsistency

We shall now look at Priest's criticism of Yablo's claim that his paradox does not rely on self-reference. Priest claims that all liar-like paradoxes contain a *fixed point*. He writes:

To put the discussion into context, think, first, of the standard Liar paradox, 'This sentence is not true'. Writing T as the truth predicate, then the Liar sentence is one, t , such that $t = \neg Tt$. The fact that ' t ' occurs on both sides of the equation, makes it a fixed point of a certain kind, and, in this context, codes the self-reference.

[Priest, 1997, p. 236]

Priest objects, in Yablo's proof, to the derivation

$$T(S_n) \rightarrow \forall k > n \neg T(S_k) \quad (\dagger)$$

He writes:

What is their justification? It is natural to suppose that this is the T -schema, but it is not. The n involved in each step of the reductio argument is a free variable, since we apply universal generalization to it a little later; and the T -schema applies only to sentences, not to things, with free variables in. It is nonsense to say, for example, T ' x is white' iff x is white. What is necessary is, of course, the generalization of the T -schema formulas containing free variables. (For the purpose of this paper, I will call such things 'predicates'.) This involves the notion of satisfaction. For the line marked (\dagger) to work, it should therefore read:

$$S(n, \dot{s}) \rightarrow \forall k > n \neg T_{S_k}$$

where S is the two-place satisfaction relation between numbers and predicates, and \dot{s} is the predicate $\forall k > x, \neg T_{S_k}$.

[Priest, 1997, p. 237]

Essentially, Priest suggests using satisfiability in place of the truth schema. In this case we use the symbol ' \dot{s} ' to denote the formula ' $\forall k > x \neg S(k, \dot{s})$ '.

We observe here that, though, using \dot{s} as a predicate object defines the binary satisfaction relation S impredicatively as both objects refer to the totality of the same class of entities, namely the class of *all* n -ary relations. The usage of impredicative definitions, which is a special case of circularity, is therefore critical and required in revealing a fixed point. Even though an explicit circularity is not present, the existence of a fixed point is shown under the assumption of an existing totality of all n -ary relations possibly containing the fixed point itself as a satisfaction relation. The passage from Priest below verifies how the fixed point is being used as a predicate and its connection to the satisfaction relation:

[...] it focuses attention on the fact that the paradox concerns a predicate, \dot{s} , of the form $\forall k > x \neg S(k, \dot{s})$; and the fact that $\dot{s} = \forall k > x,$

$\neg S(k, \dot{s})$ shows that we have a fixed point, \dot{s} , here, of exactly the same self-referential kind as in the liar paradox. In a nutshell, \dot{s} is the predicate ‘no number greater than x satisfies this predicate’. The circularity is now manifest. [Priest, 1997, p. 238]

Priest also claims that all liar-like paradoxes, including Yablo’s paradox, are bound to have a structure that he calls the *inclosure schema*. An *inclosure* is a triple $\langle \delta, \Omega, \theta \rangle$, where Ω is a set of objects, θ is a property defined on subsets of Ω , such that $\theta(\Omega)$, and δ is a partial function from subsets of Ω to Ω , defined on the sets of which θ is true, and such that if $X \subseteq \Omega$:

$$\begin{array}{ll} \delta(X) \notin X & \text{(Transcendence)} \\ \delta(X) \in \Omega & \text{(Closure)} \end{array}$$

must hold [Priest, 1997, p. 240]. Priest showed that Yablo’s paradox satisfies the transcendence and closure conditions in the inclosure schema after which he concludes the following:

As we can see, then, Yablo’s paradox does involve circularity of a self-referential kind. However one formulates it, it has the characteristic fixed-point structure. [Priest, 1997, p. 242]

An interesting question to ask here is that whether or not it would be possible to assert the existence of a fixed point without defining the satisfaction relation *impredicatively*, i.e., restricting S and \dot{s} to have different types, so as to not use one in place for another. A positive solution would genuinely manifest the circularity since the construction would not rely on an impredicative definition.

We shall now turn to Hardy [1995, p. 198] who pointed out two basic differences between the liar paradox and Yablo’s paradox. The first obvious difference is that in the liar paradox the contradiction arises from a single statement, whereas in Yablo’s paradox, the contradiction is obtained from the totality of infinitely many sentences. The second difference, an apparent consequence of the first, is that there are infinitely many T -schema conditions in Yablo’s paradox. Nevertheless, Hardy argued that no contradiction arises from finitely many Yablo sentences S_1, \dots, S_n .

Ketland [2005], similarly to Hardy, also suggests that Yablo’s paradox forms an ‘ ω -inconsistent’ theory. We claim that we can strengthen Hardy’s and Ketland’s arguments by showing that Yablo’s paradox admits circularity at the least transfinite ordinal.

Let $\mathbb{N} := \{1, 2, 3, \dots\}$ be the set of positive natural numbers.¹ Let $P_1 \wedge P_2 \wedge P_3 \wedge \dots$ be an infinite conjunction of propositions.² We use the notation $\bigwedge_{i=1}^n P_i$ to denote $P_1 \wedge P_2 \wedge \dots \wedge P_n$, and use $\bigwedge_{i \in \mathbb{N}} P_i$ to denote the infinite conjunction of all P_i 's. Taking each P_i as an individual statement, we may call a set of statements a *theory*. Let $\mathcal{F} = \{P_i\}_{i \in \mathbb{N}}$ be a theory. If no contradiction is obtained from any finite subset of \mathcal{F} , then \mathcal{F} is called *consistent*. If for each $n \in \mathbb{N}$, we are able to show that $\bigwedge_{i=1}^n P_i$ is consistent and that there is a $j \in \mathbb{N}$ such that $\neg P_j$ holds, in symbols $\exists j \neg P_j$, then we say that \mathcal{F} is *ω-inconsistent*. If \mathcal{F} is not ω-inconsistent, then it is called *ω-consistent*. An *n-circular theory* is a finite set of statements $\{P_i\}_{i \in \mathbb{N}}$ such that for some $n \in \mathbb{N}$, $\bigwedge_{i=1}^n P_i$ implies some $\neg P_e$, where $e \leq n$. We remind the reader that the classical liar paradox forms an *n-circular theory*, particularly for $n = 1$, by definition. Define $P_\omega = \bigwedge_{n \in \mathbb{N}} P_n$. A theory $\{P_i\}_{i \in \mathbb{N}}$ is called *ω-circular* if P_ω implies $\neg P_e$ for some $e \in \mathbb{N}$. Now P_ω is not a finitary statement. Using transfinite induction, it is safe to assume that logical implication can be generalized to indices of transfinite ordinals since we are concerned with infinite theories and that Yablo's paradox has ingredients of infinitary logic. In particular, the paradox involves an infinite conjunction of sentences. This is a straightforward application of infinitary logic since no finite conjunction of any of these sentences causes any contradiction.

An ω-circular theory has to be necessarily ω-inconsistent, but not necessarily vice versa. We can easily define an ω-inconsistent theory that is not ω-circular in the following way. Since an ω-inconsistent theory proves $\exists j \neg P_j$, it must be the case that for some $k > j$, the theory also proves $\bigwedge_{i=1}^k P_i$ is consistent. Since for every j there exists an index $k > j$, the theory cannot be ω-circular.

We argue that Yablo's paradox not only is ω-inconsistent, but it also forms an ω-circular theory.

In fact, any set of Yabloesque sentences is ω-circular. This way we hope to establish a compromise solution to the debate between the supporters who argue for the circularity of the paradox, like [Sorensen, 1998] and of course [Yablo, 1993] himself, and those who argue for the opposite

¹ We did not include 0 in \mathbb{N} so as to be consistent with the indices used in the definition of the paradox in Yablo's original paper. We only use \mathbb{N} for the purpose of indexing without loss of generality.

² We may also denote them by an infinite sequence $\{P_i\}_{i \in \mathbb{N}}$, but for the purpose of emphasizing the infinitary logical nature of Yablo's paradox, we are explicitly writing the conjunctions.

view, like [Beall, 2001; Cook, 2006; Priest, 1997]. Consequently, we will be holding the position that Yablo's paradox is only circular in the limit sense by emphasizing the connection between ω -inconsistent theories and ω -circularity.

Now Hardy has already shown that any finite set of Yablo sentences is consistent. The paradox arises when there are infinitely many sentences $\{S_n\}_{n \in \mathbb{N}}$. Since no contradiction is entailed by $\bigwedge_{i=1}^n S_i$, this finite theory is consistent.

In fact, an ω -circular theory is sufficient to produce a Yabloesque type paradox. It follows by definition that if \mathcal{T} is an ω -circular theory, then \mathcal{T} cannot be finite, i.e., the cardinality of \mathcal{T} cannot be equal to any natural number. So an ω -circular theory cannot be circular in the classical finite sense and this means it is possible to define a set of Yabloesque sentences.

Let us now argue that Yablo's paradox forms an ω -circular theory. Now consider, for every $n \in \mathbb{N}$ a sentence:

$$S_n: \forall_{k>n} \neg S_k.$$

Assume for a contradiction that the set $\{S_n\}_{n \in \mathbb{N}}$ is not ω -circular. In this case, for every $m \in \mathbb{N}$, no truth value can be assigned to any S_i in $\bigwedge_{i=1}^m S_i$. However, we know that Yablo's paradox cannot be derived from a finite number of sentences. Moreover, since m is arbitrary, using induction on the size of the set $\{S_i\}_{i<j}$, we see that since $\bigwedge_{i=1}^{j-1} S_i$ is consistent, then so is $\bigwedge_{i=1}^j S_i$, and so $\bigwedge_{n \in \mathbb{N}} S_n$ must be consistent. However, the latter totality of infinite conjunction of sentences, i.e. $\bigwedge_{n \in \mathbb{N}} S_n$, actually gives us Yablo's paradox. Then $\bigwedge_{n \in \mathbb{N}} S_n$ must be ω -inconsistent. The fact that $\bigwedge_{n \in \mathbb{N}} S_n$ is also ω -circular follows from the fact that if $\bigwedge_{n \in \mathbb{N}} S_n$ were not ω -circular then it would not be ω -inconsistent since $\bigwedge_{n \in \mathbb{N}} S_n$ would imply $\neg P_e$ for no $e \in \mathbb{N}$ and then there would be no contradiction.

We conclude that Yablo's paradox is not circular in the classical sense but admits circularity in the limit, i.e., the contradiction arises at the first transfinite countable ordinal ω . The reason why we have used ω -inconsistent theories is because Yablo's paradox cannot be explained using proper inconsistent theories. Taking Yablo's paradox in the form of an ω -circular theory allows us to present a compromise compromised solution to the debate on the self-referential nature of the paradox. The controversy of course comes from the obvious fact that Yablo's paradox uses infinitary logic. A rather interesting question, to ask, therefore, is whether there exists a finite set of paradoxical sentences without self-

reference. The answer to this interesting question seems to be that there is not, to judge by a recent paper [Beringer and Schindler, 2017].

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