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ON A MULTILATTICE ANALOGUE OF A HYPERSEQUENT S5 CALCULUS

Abstract. In this paper, we present a logic $\mathbf{MML}_n^{\mathbf{S5}}$ which is a combination of multilattice logic and modal logic **S5**. $\mathbf{MML}_n^{\mathbf{S5}}$ is an extension of Kamide and Shramko's modal multilattice logic which is a multilattice analogue of **S4**. We present a cut-free hypersequent calculus for $\mathbf{MML}_n^{\mathbf{S5}}$ in the spirit of Restall's one for **S5** and develop a Kripke semantics for $\mathbf{MML}_n^{\mathbf{S5}}$, following Kamide and Shramko's approach. Moreover, we prove theorems for embedding $\mathbf{MML}_n^{\mathbf{S5}}$ into **S5** and vice versa. As a result, we obtain completeness, cut elimination, decidability, and interpolation theorems for $\mathbf{MML}_n^{\mathbf{S5}}$. Besides, we show the duality principle for $\mathbf{MML}_n^{\mathbf{S5}}$. Additionally, we introduce a modification of Kamide and Shramko's sequent calculus for their multilattice version of **S4** which (in contrast to Kamide and Shramko's original one) proves the interdefinability of necessity and possibility operators. Last, but not least, we present Hilbert-style calculi for all the logics in question as well as for a larger class of modal multilattice logics.

Keywords: multilattice logic; modal logic; hypersequent calculus; cut elimination; Hilbert-style calculus; embedding theorem; interpolation theorem; generalized truth values

1. Introduction

Shramko's multilattice logic $\mathbf{ML}_n$ [51] is an algebraic generalization of many-valued logics related to Belnap and Dunn's four-valued logic **FDE** (**F**irst **D**egree **E**ntailment) [5, 6, 12]. Among such logics are Arieli and Avron's four-valued bilattice logics [1], Shramko and Wansing's sixteen-valued trilattice logics [52, 53], Zaitsev's eight-valued tetralattice logics

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[59], and, as argued in [51], logics of generalized truth values [52, 54] in whole. The notion of multilattice (or n-lattice, where $n > 1$) itself generalizes the notions of bilattice [16, 17], trilattice [52], and tetralattice [59].

Modal multilattice logic is a generalization of modal many-valued logics which are based on **FDE**-related logics. Among the first papers regarding modal many-valued logics are Fitting's ones [14, 15]. Modal extensions of **FDE** and related logics were studied by Goble [18], Priest [44, 45], Odintsov and Wansing [41, 40] as well as Odintsov, Skurt, and Wansing [37], Odintsov and Latkin [36], Odintsov and Speranski [38, 39], Sedlár [55], Rivieccio, Jung, and Jansana [49].

The family of multilattice logics consists of Shramko's multilattice logic itself [51], its first-order extension $\mathbf{FML}_n$ [27], Kamide, Shramko, and Wansing's bi-intuitionistic multilattice logic $\mathbf{BML}_n$ and its connexive version $\mathbf{CML}_n$ [29], Kamide's linear multilattice logics $\mathbf{LML}_n$ and $\mathbf{EML}_n$ [26] as well as Kamide and Shramko's modal multilattice logic $\mathbf{MML}_n$ [28] which is a combination of $\mathbf{ML}_n$ and **S4**.

All these logics, except $\mathbf{MML}_n^{\mathbf{S5}}$, were formalized via cut-free standard sequent calculi. However, for the case of $\mathbf{MML}_n^{\mathbf{S5}}$ we need a *hyper*sequent calculus which is a generalization of a ordinary sequent calculus. This feature of $\mathbf{MML}_n^{\mathbf{S5}}$ is a legacy of **S5** which requires a hypersequent framework, if we want to have the cut elimination property. For standard sequent calculi for **S5** see papers by Ohnishi and Matsumoto [42, 43], Sato [50], Mints [34], Fitting [13], Takano [57], and Braüner [8]. For non-standard sequent calculi (which are not hypersequent ones, but propose some solution) for **S5** see works by Kanger [30], Belnap [7], Negri [35], Indrzejczak [20], and Stouppa [56]. The first hypersequent calculus itself and for **S5** in particular was designed by Pottinger [47]. Afterwards, the framework of hypersequent calculus was independently rediscovered by Avron [2] who also presented his own hypersequent calculus for **S5** [3]. Next, yet another hypersequent calculus for **S5** was introduced by Restall [48]. We will use this system as a basis for our hypersequent formalization of $\mathbf{MML}_n^{\mathbf{S5}}$. Finally, other hypersequent calculi for **S5** were developed by Poggiolesi [46], Lahav [33], Kurokawa [32] as well as Bednarska and Indrzejczak [4]. The paper [4] contains also a comparison of all the abovementioned hypersequent calculi for **S5** as well as the discussion of the methods of cut elimination.

The problem of the formulation of multilattice analogue of **S5** was posed by Kamide and Shramko [28]. Moreover, they suggested the way to obtain a *non*-cut-free ordinary sequent calculus for this logic. They conclude that “finding thus a natural formulation of a cut-free modal multilattice logic with S5-modalities is an interesting task deserving special investigation” [28, p. 342]. In this paper, we present such a formalization which is cut-free due to the use of a hypersequent framework (see Section 4.2). Besides, we introduce a Kripke semantics for $\mathbf{MML}_n^{\mathbf{S5}}$ (see Section 4.3). Furthermore, we prove syntactical and semantical embeddings from $\mathbf{MML}_n^{\mathbf{S5}}$ into **S5** and vice versa (Section 5.1). As a consequence, we obtain completeness, cut elimination, decidability, and interpolation theorems (see Sections 5.2 and 5.3). For the proof of Craig's interpolation property we use the technique which differs from Kamide and Shramko's one [28]. Moreover, we prove syntactical embedding from $\mathbf{MML}_n^{\mathbf{S5}}$ into itself which gives us the duality principle in the spirit of [28] (see Section 5.1). In Sections 3 and 6, we discuss Kamide and Shramko's sequent calculus for $\mathbf{MML}_n$ and argue that it does not prove the interdefinability of necessity and possibility operators. Then we present a modification of their calculus which does not have this drawback. In Section 7, we present Hilbert-style calculi for $\mathbf{ML}_n$ and $\mathbf{MML}_n^{\mathbf{S5}}$. Additionally, we formulate a basic modal multilattice logic $\mathbf{MML}_n^{\mathbf{K}}$ as well as its extension by a multilattice analogue of the scheme $G^{a,b,c,d}$ (see [10]). These logics are introduced both semantically and via Hilbert-style calculi. Section 2 is devoted to the description of multilattice logic $\mathbf{ML}_n$.

2. What is multillatice logic?

The multilattice (n-lattice ($n > 1$)) logic $\mathbf{ML}_n$ [51, 27, 28] is built over the language $\mathscr{L}_{\mathrm{N}}$ with the following alphabet: $\langle \Pi, \mathscr{C}, (,) \rangle$, where $\Pi = \{p, q, r, s, p_1, \ldots\}$ is a set of propositional variables and $\mathscr{C}$ is the following set of propositional connectives: $\{\neg_1, \ldots, \neg_n, \wedge_1, \ldots, \wedge_n, \vee_1, \ldots, \vee_n, \rightarrow_1, \ldots, \rightarrow_n, \leftarrow_1, \ldots, \leftarrow_n\}$. However, the original formulation of multilattice logic [51] does not deal with implications and co-implications. In order to distinguish the original formulation of multilattice logic [51] and its later version [27, 28], we will use the name $\mathbf{FDE}_n^n$ for the original one and the name $\mathbf{ML}_n$ for the later version. The name $\mathbf{FDE}_n^n$ was used by Shramko [51] for the proof system for multilattice logic. Thus, $\mathbf{FDE}_n^n$ is built in the $\{\neg_1, \ldots, \neg_n, \wedge_1, \ldots, \wedge_n, \vee_1, \ldots, \vee_n\}$-fragment of $\mathscr{L}_{\mathrm{N}}$.

The modal multilattice (n-lattice) logic $\mathbf{MML}_n$ (see [28]) is built over the language $\mathscr{L}_{\mathrm{M}}$ with the alphabet $\langle \Pi, \mathscr{C}_{\mathrm{M}}, (,) \rangle$, where $\mathscr{C}_{\mathrm{M}}$ is the following set of propositional connectives and modal operators:

$$\mathscr{C} \cup \{\Box_1, \ldots, \Box_n, \Diamond_1, \ldots, \Diamond_n\}.$$

Besides, let $\Pi^j := \{\pi^j \mid \pi \in \Pi\}$, for each $j \leqslant n$. Then we define the modal language $\mathscr{L}$ with the following alphabet: $\langle \Pi, \Pi^1, \ldots, \Pi^n, \neg, \wedge, \vee, \rightarrow, \leftarrow, \Box, \Diamond, (,) \rangle$. In fact, $\mathscr{L}$ is not a standard modal language due to the use of the additional sets $\Pi^1, \ldots, \Pi^n$ of propositional variables. However, Kamide and Shramko [28] use $\mathscr{L}$ in order to formulate the modal logic **S4**. We will use it to formulate the modal logic **S5**. The sets $\mathscr{F}_{\mathrm{N}}$, $\mathscr{F}_{\mathrm{M}}$, and $\mathscr{F}$, respectively, of all $\mathscr{L}_{\mathrm{N}}$'s, $\mathscr{L}_{\mathrm{M}}$'s, and $\mathscr{L}$'s formulas are defined in a standard way.

We define the notion of a *sequent* $\Gamma \Rightarrow \Delta$, where Γ and Δ are *finite sets* (possibly, empty) of formulas, as a metalanguage expression which means that if each $\gamma \in \Gamma$ is valid, then some $\delta \in \Delta$ is valid. The notion of validity in $\mathbf{ML}_n$ with respect to multilattices is given in Definitions 2.6 and 2.7. For a Kripke semantics for $\mathbf{MML}_n^{\mathbf{S5}}$ see Section 4.3. Following Restall [48], we understand a *hypersequent* $H = \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$ as a finite *multiset* of sequents. The symbol '|' is interpreted as a metalanguage disjunction.

We write $\boldsymbol{L} \vdash \Gamma \Rightarrow \Delta$ and $\boldsymbol{L} \models \Gamma \Rightarrow \Delta$, respectively, when a sequent $\Gamma \Rightarrow \Delta$ is provable in a (hyper)sequent calculus for a logic $\boldsymbol{L}$ and $\Gamma \Rightarrow \Delta$ is valid in $\boldsymbol{L}$. We write $\vdash_{\boldsymbol{L}} \Gamma \Rightarrow \Delta$ and $\models_{\boldsymbol{L}} \Gamma \Rightarrow \Delta$, respectively, in order to express that $\boldsymbol{L} \vdash \Gamma \Rightarrow \Delta$ and $\boldsymbol{L} \models \Gamma \Rightarrow \Delta$. We write $\boldsymbol{L} \models \Gamma \Leftrightarrow \Delta$, if both $\boldsymbol{L} \models \Gamma \Rightarrow \Delta$ and $\boldsymbol{L} \models \Delta \Rightarrow \Gamma$ (similarly for $\vdash$). We understand the expressions $\boldsymbol{L} \vdash H$ and $\boldsymbol{L} \models H$ in a similar way, where H is a hypersequent.

Definition 2.1. [28, p. 319, Definitions 2.1 and 2.2] An n-dimensional *multilattice* (or just multilattice or n-lattice) is a structure $\mathscr{M}_n = \langle \mathscr{S}, \leq_1, \ldots, \leq_n \rangle$, where $n > 1$, $\mathscr{S} \neq \emptyset$, $\leq_1, \ldots, \leq_n$ are partial orders such that $\langle \mathscr{S}, \leq_1 \rangle, \ldots, \langle \mathscr{S}, \leq_n \rangle$ are lattices with the corresponding pairs of meet and join operations $\langle \cap_1, \cup_1 \rangle, \ldots, \langle \cap_n, \cup_n \rangle$ as well as the corresponding j-inversion operations $-_1, \ldots, -_n$ which satisfy the following conditions, for each $j \leqslant n$, $k \leqslant n$, $j \neq k$, and $x, y \in \mathscr{S}$:

$$x \leq_j y \text{ implies } -_j\, y \leq_j -_j x; \tag{anti}$$

$$x \leq_k y \text{ implies } -_j\, x \leq_k -_j y; \tag{iso}$$

$$-_j -_j\, x = x. \tag{per2}$$

Definition 2.2. [28, p. 319, Definition 2.3] Let $\mathscr{M}_n = \langle \mathscr{S}, \leq_1, \ldots, \leq_n \rangle$ be an n-lattice. Then $\mathscr{U}_n \subset \mathscr{S}$ is an *n-ultrafilter* (ultramultifilter) on $\mathscr{M}_n$ iff it satisfies the following conditions, for each $j, k \leqslant n$, $j \neq k$, and $x, y \in \mathscr{S}$:

- $x \cap_i y \in \mathscr{U}_n$ iff $x \in \mathscr{U}_n$ and $y \in \mathscr{U}_n$ ($\mathscr{U}_n$ is an n-filter (multifilter) on $\mathscr{M}_n$);
- $x \cup_i y \in \mathscr{U}_n$ iff $x \in \mathscr{U}_n$ or $y \in \mathscr{U}_n$ ($\mathscr{U}_n$ is a prime n-filter on $\mathscr{M}_n$);
- $x \in \mathscr{U}_n$ iff $-_j -_k x \notin \mathscr{U}_n$.

Definition 2.3. [28, p. 319, Definition 2.4] A pair $\langle \mathscr{M}_n, \mathscr{U}_n \rangle$ is called an ultralogical n-lattice (ultralogical multilattice) iff $\mathscr{M}_n$ is a multilattice and $\mathscr{U}_n$ is an ultramultifilter on $\mathscr{M}_n$.

Although in [27, 28] a multilattice logic with implications and co-implications is formulated (in the form of a sequent calculus and a two-valued semantics), the appropriate definitions of multilattice operations which correspond to implications and co-implications are not presented. We introduce the following one (see Proposition 2.2 for its correctness).

Definition 2.4. Let $\mathscr{M}_n = \langle \mathscr{S}, \leq_1, \ldots, \leq_n \rangle$ be a multilattice. Then we can define for all $\langle \mathscr{S}, \leq_1 \rangle, \ldots, \langle \mathscr{S}, \leq_n \rangle$ the corresponding pseudo-complement operations $\supset_1, \ldots, \supset_n$ as well as pseudo-difference ones $\subset_1, \ldots, \subset_n$ as follows ($x, y \in \mathscr{S}$, $j \leqslant n$, $k \leqslant n$ is fixed and $j \neq k$):

$$\begin{aligned}
x \supset_j y &= -_k -_j x \cup_j y; \\
-_j(x \supset_j y) &= -_k -_j -_j x \cap_j -_j y; \\
-_k(x \supset_j y) &= -_k -_j -_k x \cup_j -_k y; \\
x \subset_j y &= x \cap_j -_k -_j y; \\
-_j(x \subset_j y) &= -_j x \cup_j -_k -_j -_j y; \\
-_k(x \subset_j y) &= -_k x \cap_j -_k -_j -_k y.
\end{aligned}$$

Definition 2.5. Let $\mathscr{M}_n = \langle \mathscr{S}, \leq_1, \ldots, \leq_n \rangle$ be a multilattice. Let v be a function from Π to $\mathscr{S}$. Then we call v a *valuation* and extend it for any $\alpha, \beta \in \mathscr{F}_{\mathrm{N}}$ as follows:

(1) $v(\neg_j \alpha) = -_j v(\alpha)$;
(2) $v(\alpha \wedge_j \beta) = v(\alpha) \cap_j v(\beta)$;
(3) $v(\alpha \vee_j \beta) = v(\alpha) \cup_j v(\beta)$;
(4) $v(\alpha \rightarrow_j \beta) = v(\alpha) \supset_j v(\beta)$;
(5) $v(\alpha \leftarrow_j \beta) = v(\alpha) \subset_j v(\beta)$.

The conditions for $\neg_k\neg_j\alpha$, $\neg_j(\alpha \wedge_j \beta)$, $\neg_k(\alpha \wedge_j \beta)$, $\neg_j(\alpha \vee_j \beta)$, and $\neg_k(\alpha \vee_j \beta)$ follow from points (1)–(3) which were presented in [51, p. 206, Definition 4.5]. Similarly, the conditions for $\neg_j(\alpha \rightarrow_j \beta)$, $\neg_k(\alpha \rightarrow_j \beta)$, $\neg_j(\alpha \leftarrow_j \beta)$, and $\neg_k(\alpha \leftarrow_j \beta)$ follow from (1), (4), and (5).

DEFINITION 2.6. [51, p. 206, Definitions 4.6 and 5.3] The entailment relation in multilattice logic $\mathbf{ML}_n$ is defined as follows, for each $\Gamma, \Delta \subseteq \mathscr{F}_{\mathrm{N}}$ and $\alpha, \beta \in \mathscr{F}_{\mathrm{N}}$:

(1) $\alpha \models_j \beta$ iff for each multilattice $\mathscr{M}_n$ and each valuation v, it holds that $v(\alpha) \leq_j v(\beta)$.
(2) $\Gamma \models_{\mathbf{ML}_n} \Delta$ iff for each ultralogical multilattice $\langle \mathscr{M}_n, \mathscr{U}_n \rangle$ and each valuation v, it holds that if $v(\gamma) \in \mathscr{U}_n$ (for each $\gamma \in \Gamma$), then $v(\delta) \in \mathscr{U}_n$ (for some $\delta \in \Delta$).

DEFINITION 2.7. Let Γ, Γ_1, …, Γ_m, Δ, Δ_1, …, Δ_m be finite sets of $\mathscr{L}_{\mathrm{N}}$-formulas. Then:

(1) $\models_{\mathbf{ML}_n} \Gamma \Rightarrow \Delta$ iff $\Gamma \models_{\mathbf{ML}_n} \Delta$;
(2) $\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_m \Rightarrow \Delta_m \models_{\mathbf{ML}_n} \Gamma \Rightarrow \Delta$ iff $\models_{\mathbf{ML}_n} \Gamma_1 \Rightarrow \Delta_1$, …, $\models_{\mathbf{ML}_n} \Gamma_m \Rightarrow \Delta_m$ implies $\models_{\mathbf{ML}_n} \Gamma \Rightarrow \Delta$.

Using Theorem 5.4 from [51, p. 208] we obtain:

FACT 2.1. *Let $\alpha_1, \ldots \alpha_l, \beta_1, \ldots, \beta_m \in \mathscr{F}_{\mathrm{N}}$. Then, for each $j \leqslant n$, it holds that*:

$$\alpha_1 \wedge_j \ldots \wedge_j \alpha_l \models_j \beta_1 \vee_j \ldots \vee_j \beta_m \;\; \textit{iff} \;\; \alpha_1, \ldots, \alpha_l \models_{\mathbf{ML}_n} \beta_1, \ldots, \beta_m.$$

Although in [27, 28] $\mathbf{ML}_n$ is presented in the form of sequent calculus, we introduce here its hypersequent formulation, since we will need it to formalize $\mathbf{MML}_n^{\mathbf{S5}}$. If $H = G = \emptyset$, then we deal with the ordinary sequent calculus for $\mathbf{ML}_n$ [27, 28]. The $\{\neg_1, ..., \neg_n, \wedge_1, ..., \wedge_n, \vee_1, ..., \vee_n\}$-fragment of sequent calculus for $\mathbf{ML}_n$ (= sequent calculus for $\mathbf{FDE}_n^n$) was introduced in [51]. Recall that $n > 1$, $j, k \leqslant n$, and $j \neq k$.

The axioms of the hypersequent calculus for $\mathbf{ML}_n$ are as follows, for each $\pi \in \Pi$:

$$(\mathrm{Ax})\ \pi \Rightarrow \pi \qquad\qquad (\mathrm{Ax}_{\neg})\ \neg_j\pi \Rightarrow \neg_j\pi$$

The internal structural rules of the hypersequent calculus for $\mathbf{ML}_n$ are as follows:

$$(\mathrm{Cut})\ \frac{\Gamma \Rightarrow \Delta, \alpha \mid H \qquad \alpha, \Theta \Rightarrow \Lambda \mid G}{\Gamma, \Theta \Rightarrow \Delta, \Lambda \mid H \mid G}$$

$$(\mathrm{IW}{\Rightarrow})\ \frac{\Gamma \Rightarrow \Delta \mid H}{\alpha, \Gamma \Rightarrow \Delta \mid H} \qquad ({\Rightarrow}\mathrm{IW})\ \frac{\Gamma \Rightarrow \Delta \mid H}{\Gamma \Rightarrow \Delta, \alpha \mid H}$$

The external structural rules are as follows:

$$(\mathrm{EW}{\Rightarrow})\ \frac{H}{\alpha \Rightarrow \mid H} \qquad ({\Rightarrow}\mathrm{EW})\ \frac{H}{\Rightarrow \alpha \mid H}$$

The non-negated logical rules are as follows:

$$(\wedge_j{\Rightarrow})\ \frac{\alpha, \beta, \Gamma \Rightarrow \Delta \mid H}{\alpha \wedge_j \beta, \Gamma \Rightarrow \Delta \mid H} \qquad ({\Rightarrow}\wedge_j)\ \frac{\Gamma \Rightarrow \Delta, \alpha \mid H \qquad \Gamma \Rightarrow \Delta, \beta \mid G}{\Gamma \Rightarrow \Delta, \alpha \wedge_j \beta \mid H \mid G}$$

$$(\vee_j{\Rightarrow})\ \frac{\alpha, \Gamma \Rightarrow \Delta \mid H \qquad \beta, \Gamma \Rightarrow \Delta \mid G}{\alpha \vee_j \beta, \Gamma \Rightarrow \Delta \mid H \mid G} \qquad ({\Rightarrow}\vee_j)\ \frac{\Gamma \Rightarrow \Delta, \alpha, \beta \mid H}{\Gamma \Rightarrow \Delta, \alpha \vee_j \beta \mid H}$$

$$(\rightarrow_j{\Rightarrow})\ \frac{\Gamma \Rightarrow \Delta, \alpha \mid H \qquad \beta, \Theta \Rightarrow \Lambda \mid G}{\alpha \rightarrow_j \beta, \Gamma, \Theta \Rightarrow \Delta, \Lambda \mid H \mid G} \qquad ({\Rightarrow}\rightarrow_j)\ \frac{\alpha, \Gamma \Rightarrow \Delta, \beta \mid H}{\Gamma \Rightarrow \Delta, \alpha \rightarrow_j \beta \mid H}$$

$$(\leftarrow_j{\Rightarrow})\ \frac{\alpha, \Gamma \Rightarrow \Delta, \beta \mid H}{\alpha \leftarrow_j \beta, \Gamma \Rightarrow \Delta \mid H} \qquad ({\Rightarrow}\leftarrow_j)\ \frac{\Gamma \Rightarrow \Delta, \alpha \mid H \qquad \beta, \Theta \Rightarrow \Lambda \mid G}{\Gamma, \Theta \Rightarrow \Delta, \Lambda, \alpha \leftarrow_j \beta \mid H \mid G}$$

The jj-negated logical rules are as follows:

$$(\neg_j\wedge_j{\Rightarrow})\ \frac{\neg_j\alpha, \Gamma \Rightarrow \Delta \mid H \qquad \neg_j\beta, \Gamma \Rightarrow \Delta \mid G}{\neg_j(\alpha \wedge_j \beta), \Gamma \Rightarrow \Delta \mid H \mid G} \qquad ({\Rightarrow}\neg_j\wedge_j)\ \frac{\Gamma \Rightarrow \Delta, \neg_j\alpha, \neg_j\beta \mid H}{\Gamma \Rightarrow \Delta, \neg_j(\alpha \wedge_j \beta) \mid H}$$

$$(\neg_j\vee_j{\Rightarrow})\ \frac{\neg_j\alpha, \neg_j\beta, \Gamma \Rightarrow \Delta \mid H}{\neg_j(\alpha \vee_j \beta), \Gamma \Rightarrow \Delta \mid H} \qquad ({\Rightarrow}\neg_j\vee_j)\ \frac{\Gamma \Rightarrow \Delta, \neg_j\alpha \mid H \qquad \Gamma \Rightarrow \Delta, \neg_j\beta \mid G}{\Gamma \Rightarrow \Delta, \neg_j(\alpha \vee_j \beta) \mid H \mid G}$$

$$(\neg_j\rightarrow_j{\Rightarrow})\ \frac{\neg_j\beta, \Gamma \Rightarrow \Delta, \neg_j\alpha \mid H}{\neg_j(\alpha \rightarrow_j \beta), \Gamma \Rightarrow \Delta \mid H} \qquad ({\Rightarrow}\neg_j\rightarrow_j)\ \frac{\Gamma \Rightarrow \Delta, \neg_j\beta \mid H \qquad \neg_j\alpha, \Theta \Rightarrow \Lambda \mid G}{\Gamma, \Theta \Rightarrow \Delta, \Lambda, \neg_j(\alpha \rightarrow_j \beta) \mid H \mid G}$$

$$(\neg_j\leftarrow_j{\Rightarrow})\ \frac{\Gamma \Rightarrow \Delta, \neg_j\beta \mid H \qquad \neg_j\alpha, \Theta \Rightarrow \Lambda \mid G}{\neg_j(\alpha \leftarrow_j \beta), \Gamma, \Theta \Rightarrow \Delta, \Lambda \mid H \mid G} \qquad ({\Rightarrow}\neg_j\leftarrow_j)\ \frac{\neg_j\beta, \Gamma \Rightarrow \Delta, \neg_j\alpha \mid H}{\Gamma \Rightarrow \Delta, \neg_j(\alpha \leftarrow_j \beta) \mid H}$$

$$(\neg_j\neg_j{\Rightarrow})\ \frac{\alpha, \Gamma \Rightarrow \Delta \mid H}{\neg_j\neg_j\alpha, \Gamma \Rightarrow \Delta \mid H} \qquad ({\Rightarrow}\neg_j\neg_j)\ \frac{\Gamma \Rightarrow \Delta, \alpha \mid H}{\Gamma \Rightarrow \Delta, \neg_j\neg_j\alpha \mid H}$$

The kj-negated logical rules are as follows:

$$(\neg_k\wedge_j{\Rightarrow})\ \frac{\neg_k\alpha, \neg_k\beta, \Gamma \Rightarrow \Delta \mid H}{\neg_k(\alpha \wedge_j \beta), \Gamma \Rightarrow \Delta \mid H} \qquad ({\Rightarrow}\neg_k\wedge_j)\ \frac{\Gamma \Rightarrow \Delta, \neg_k\alpha \mid H \qquad \Gamma \Rightarrow \Delta, \neg_k\beta \mid G}{\Gamma \Rightarrow \Delta, \neg_k(\alpha \wedge_j \beta) \mid H \mid G}$$

$$(\neg_k\vee_j{\Rightarrow})\ \frac{\neg_k\alpha, \Gamma \Rightarrow \Delta \mid H \qquad \neg_k\beta, \Gamma \Rightarrow \Delta \mid G}{\neg_k(\alpha \vee_j \beta), \Gamma \Rightarrow \Delta \mid H \mid G} \qquad ({\Rightarrow}\neg_k\vee_j)\ \frac{\Gamma \Rightarrow \Delta, \neg_k\alpha, \neg_k\beta \mid H}{\Gamma \Rightarrow \Delta, \neg_k(\alpha \vee_j \beta) \mid H}$$

$$(\neg_k\rightarrow_j{\Rightarrow})\ \frac{\Gamma \Rightarrow \Delta, \neg_k\alpha \mid H \qquad \neg_k\beta, \Theta \Rightarrow \Lambda \mid G}{\neg_k(\alpha \rightarrow_j \beta), \Gamma, \Theta \Rightarrow \Delta, \Lambda \mid H \mid G} \qquad ({\Rightarrow}\neg_k\rightarrow_j)\ \frac{\neg_k\alpha, \Gamma \Rightarrow \Delta, \neg_k\beta \mid H}{\Gamma \Rightarrow \Delta, \neg_k(\alpha \rightarrow_j \beta) \mid H}$$

$$(\neg_k\leftarrow_j{\Rightarrow})\ \frac{\neg_k\alpha, \Gamma \Rightarrow \Delta, \neg_k\beta \mid H}{\neg_k(\alpha \leftarrow_j \beta), \Gamma \Rightarrow \Delta \mid H} \qquad (\neg_k\leftarrow_j{\Rightarrow})\ \frac{\Gamma \Rightarrow \Delta, \neg_k\alpha \mid H \qquad \neg_k\beta, \Theta \Rightarrow \Lambda \mid G}{\Gamma, \Theta \Rightarrow \Delta, \Lambda, \neg_k(\alpha \leftarrow_j \beta) \mid H \mid G}$$

$$(\neg_k\neg_j\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \alpha \mid H}{\neg_k\neg_j\alpha, \Gamma \Rightarrow \Delta \mid H} \qquad (\Rightarrow\neg_k\neg_j)\ \frac{\alpha, \Gamma \Rightarrow \Delta \mid H}{\Gamma \Rightarrow \Delta, \neg_k\neg_j\alpha \mid H}$$

The notion of a proof in the hypersequent calculus for $\mathbf{ML}_n$ is defined in a standard way. As follows from [27], this calculus enjoys cut elimination. The rules of *internal weakening* are redundant, if we formulate axioms in the following way:

$$(\mathrm{Ax}')\ \Gamma, \pi \Rightarrow \Delta, \pi \qquad (\mathrm{Ax}'_{\neg})\ \Gamma, \neg_j\pi \Rightarrow \Delta, \neg_j\pi$$

By a routine check, the formulation with these generalized axioms is equivalent to the original one. In order to shorten the proofs we will use the one with (Ax') and $(\mathrm{Ax}'_{\neg})$.

Proposition 2.2. *The following sequents are provable in* $\mathbf{ML}_n$:

(1) $\alpha \rightarrow_j \beta \Leftrightarrow \neg_k\neg_j\alpha \vee_j \beta$;
(2) $\neg_j(\alpha \rightarrow_j \beta) \Leftrightarrow \neg_k\neg_j\neg_j\alpha \wedge_j \neg_j\beta$;
(3) $\neg_k(\alpha \rightarrow_j \beta) \Leftrightarrow \neg_k\neg_j\neg_k\alpha \vee_j \neg_k\beta$;
(4) $\alpha \leftarrow_j \beta \Leftrightarrow \alpha \wedge_j \neg_k\neg_j\beta$;
(5) $\neg_j(\alpha \leftarrow_j \beta) \Leftrightarrow \neg_j\alpha \vee_j \neg_k\neg_j\neg_j\beta$;
(6) $\neg_k(\alpha \leftarrow_j \beta) \Leftrightarrow \neg_k\alpha \wedge_j \neg_k\neg_j\neg_k\beta$.

Proof. As an example, we prove the cases (1)–(3).

$$\dfrac{\dfrac{\dfrac{\alpha \Rightarrow \alpha \qquad \beta \Rightarrow \beta}{\alpha, \alpha \rightarrow_j \beta \Rightarrow \beta}\,(\rightarrow_j\Rightarrow)}{\alpha \rightarrow_j \beta \Rightarrow \neg_k\neg_j\alpha, \beta}\,(\Rightarrow\neg_k\neg_j)}{\alpha \rightarrow_j \beta \Rightarrow \neg_k\neg_j\alpha \vee_j \beta}\,(\Rightarrow\vee_j)$$

$$\dfrac{\dfrac{\dfrac{\alpha \Rightarrow \beta, \alpha}{\alpha, \neg_k\neg_j\alpha \Rightarrow \beta}\,(\neg_k\neg_j\Rightarrow) \qquad \alpha, \beta \Rightarrow \beta}{\alpha, \neg_k\neg_j\alpha \vee_j \beta \Rightarrow \beta}\,(\Rightarrow\vee_j)}{\neg_k\neg_j\alpha \vee_j \beta \Rightarrow \alpha \rightarrow_j \beta}\,(\Rightarrow\rightarrow_j)$$

$$\dfrac{\dfrac{\dfrac{\neg_j\alpha, \neg_j\beta \Rightarrow \neg_j\alpha}{\neg_j\beta \Rightarrow \neg_k\neg_j\neg_j\alpha, \neg_j\alpha}\,(\Rightarrow\neg_k\neg_j) \qquad \neg_j\beta \Rightarrow \neg_j\beta, \neg_j\alpha}{\neg_j\beta \Rightarrow \neg_k\neg_j\neg_j\alpha \wedge_j \neg_j\beta, \neg_j\alpha}\,(\Rightarrow\wedge_j)}{\neg_j(\alpha \rightarrow_j \beta) \Rightarrow \neg_k\neg_j\neg_j\alpha \wedge_j \neg_j\beta}\,(\neg_j\rightarrow_j\Rightarrow)$$

$$\dfrac{\dfrac{\dfrac{\neg_j\beta \Rightarrow \neg_j\beta \qquad \neg_j\alpha \Rightarrow \neg_j\alpha}{\neg_j\beta \Rightarrow \neg_j(\alpha \rightarrow_j \beta), \neg_j\alpha}\,(\Rightarrow\neg_j\rightarrow_j)}{\neg_k\neg_j\neg_j\alpha, \neg_j\beta \Rightarrow \neg_j(\alpha \rightarrow_j \beta)}\,(\neg_k\neg_j\Rightarrow)}{\neg_k\neg_j\neg_j\alpha \wedge_j \neg_j\beta \Rightarrow \neg_j(\alpha \rightarrow_j \beta)}\,(\wedge_j\Rightarrow)$$

$$\dfrac{\dfrac{\dfrac{\neg_k\alpha\Rightarrow\neg_k\alpha \qquad \neg_k\beta\Rightarrow\neg_k\beta}{\neg_k\alpha, \neg_k(\alpha\rightarrow_j\beta)\Rightarrow\neg_k\beta}\,(\neg_k\!\rightarrow_j\Rightarrow)}{\neg_k(\alpha\rightarrow_j\beta)\Rightarrow\neg_k\neg_j\neg_k\alpha, \neg_k\beta}\,(\Rightarrow\neg_k\neg_j)}{\neg_k(\alpha\rightarrow_j\beta)\Rightarrow\neg_k\neg_j\neg_k\alpha\vee_j\neg_k\beta}\,(\Rightarrow\vee_j)$$

$$\dfrac{\dfrac{\dfrac{\neg_k\alpha\Rightarrow\neg_k\beta, \neg_k\alpha}{\neg_k\neg_j\neg_k\alpha, \neg_k\alpha\Rightarrow\neg_k\beta}\,(\neg_k\neg_j\Rightarrow) \qquad \neg_k\beta\Rightarrow\neg_k\beta}{\neg_k\alpha, \neg_k\neg_j\neg_k\alpha\vee_j\neg_k\beta\Rightarrow\neg_k\beta}\,(\vee_j\Rightarrow)}{\neg_k\neg_j\neg_k\alpha\vee_j\neg_k\beta\Rightarrow\neg_k(\alpha\rightarrow_j\beta)}\,(\Rightarrow\neg_k\!\rightarrow_j)$$

⊣

3. What is modal multillatice logic?

Definition 3.1. [28, p. 320, Definition 2.5] A multilattice $\mathscr{M}_n = \langle \mathscr{S}, \leq_1, \dots, \leq_n\rangle$ is called *S4-modal* iff for each $j \leqslant n$ the unary operations of *interior* I_j and *closure* C_j can be defined on $\mathscr{S}$, satisfying the following conditions ($x, y \in \mathscr{S}$):

$$I_j(x) \leq_j x; \qquad \text{(decreasing)}$$
$$I_j(x) = I_jI_j(x); \qquad \text{(idempotent)}$$
$$I_j(x \cap_j y) \leq_j I_j(x) \cap_j I_j(y); \qquad \text{(sub-multiplicative)}$$
$$x \leq_j C_j(x); \qquad \text{(increasing)}$$
$$C_j(x) = C_jC_j(x); \qquad \text{(idempotent)}$$
$$C_j(x) \cup_j C_j(y) \leq_j C_j(x \cup_j y). \qquad \text{(sub-additive)}$$

Definition 3.2. Let $\mathscr{M}_n = \langle \mathscr{S}, \leq_1, \dots, \leq_n\rangle$ be an S4-modal multilattice. Let v be a valuation as in Definition 2.5. Then we extend it for modal formulas as follows:

(1) $v(\Box_j\alpha) = I_jv(\alpha)$,
(2) $v(\Diamond_j\alpha) = C_jv(\alpha)$.

An algebraic completeness of $\mathbf{MML}_n$ with respect to an S4-modal multilattices is left as an open problem in [28]. However, it was solved in [19]. But the formulation of the notion of S4-modal multilattice was changed in [19, Definition 41]. The conditions (sub-multiplicative) and (sub-additive) were strengthened as follows:

$$I_j(x \cap_j y) = I_j(x) \cap_j I_j(y); \qquad \text{(multiplicative)}$$
$$C_j(x) \cup_j C_j(y) = C_j(x \cup_j y). \qquad \text{(additive)}$$

Moreover, the following six conditions were added:

$$x \leq_j y \text{ implies } I_j(x) \leq_j I_j(y); \qquad (I\text{-monotonicity})$$
$$x \leq_j y \text{ implies } C_j(x) \leq_j C_j(y); \qquad (C\text{-monotonicity})$$
$$-_j I_j(x) = C_j(-_j x); \qquad (-_j I_j\text{-definition})$$
$$-_j C_j(x) = I_j(-_j x); \qquad (-_j C_j\text{-definition})$$
$$-_k I_j(x) = I_j(-_k x); \qquad (-_k I_j\text{-definition})$$
$$-_k C_j(x) = C_j(-_k x). \qquad (-_k C_j\text{-definition})$$

We present below the notion of S5-modal multilattice.

Definition 3.3. An S4-modal multilattice $\mathscr{M}_n = \langle \mathscr{S}, \leq_1, \ldots, \leq_n \rangle$ (in a sense of [19, Definition 41]) is called *S5-modal* iff for each $j \leqslant n$ and $x \in \mathscr{S}$ it satisfies the following condition:

$$C_j(x) \leq_j I_j(C_j(x)). \tag{5}$$

Let us clarify some important issues regarding $\mathbf{MML}_n$. Kamide and Shramko formulate a sequent calculus for $\mathbf{MML}_n$ in the spirit of the one for **S4** which is based on Ohnishi and Matsumoto's works [42, 43]. The standard **S4**-style rules for $\Box$ are as follows:

$$(\Box\Rightarrow)\ \frac{\alpha, \Gamma \Rightarrow \Delta}{\Box\alpha, \Gamma \Rightarrow \Delta} \qquad\qquad (\Rightarrow\Box)\ \frac{\Box\Gamma \Rightarrow \alpha}{\Box\Gamma \Rightarrow \Box\alpha}$$

If one considers a formulation of **S4** having $\Box$ as the only modal operator, then, as follows from [42, 43], a sequent calculus for classical logic supplied with these rules gives a sound and complete sequent calculus for **S4**.

If one considers a formulation of **S4** having $\Diamond$ as the only modal operator, then, one should consider the following rules instead of $(\Rightarrow\Box)$ and $(\Box\Rightarrow)$:

$$(\Diamond\Rightarrow)\ \frac{\alpha \Rightarrow \Diamond\Gamma}{\Diamond\alpha \Rightarrow \Diamond\Gamma} \qquad\qquad (\Rightarrow\Diamond)\ \frac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma \Rightarrow \Delta, \Diamond\alpha}$$

However, as follows from Kripke's paper [31], if one wants to deal with **S4** having both $\Box$ and $\Diamond$, then its not enough to just add the rules $(\Box\Rightarrow)$, $(\Rightarrow\Box)$, $(\Diamond\Rightarrow)$, and $(\Rightarrow\Diamond)$ to a sequent formulation of classical logic, since in this system the sequents $\Box\alpha \Leftrightarrow \neg\Diamond\neg\alpha$ and $\Diamond\alpha \Leftrightarrow \neg\Box\neg\alpha$ are unprovable. In order to avoid such situation, one should consider the following rules instead of $(\Rightarrow\Box)$ and $(\Diamond\Rightarrow)$:

$$(\Rightarrow\Box^4)\ \frac{\Box\Gamma\Rightarrow\Diamond\Delta,\alpha}{\Box\Gamma\Rightarrow\Diamond\Delta,\Box\alpha} \qquad\qquad (\Diamond^4\Rightarrow)\ \frac{\alpha,\Box\Gamma\Rightarrow\Diamond\Delta}{\Diamond\alpha,\Box\Gamma\Rightarrow\Diamond\Delta}$$

A similar situation holds for the other modal logics. For more details one may consult Fitting's monograph [13] or Indrzejczak's one [21].

What is the most important for us in this story is that Kamide and Shramko [28] use a *non*-complete version of a sequent calculus of **S4**. They just add the rules $(\Box\Rightarrow)$, $(\Rightarrow\Box)$, $(\Diamond\Rightarrow)$, and $(\Rightarrow\Diamond)$ to a classical sequent calculus. Then they present a Kripke semantics for $\mathbf{MML}_n$ and provide semantical and syntactical embeddings of $\mathbf{MML}_n$ into **S4** and conclude that $\mathbf{MML}_n$ is sound a complete with respect to its Kripke semantics due to embeddings and the completeness theorem for **S4**. However, as we have already noted, they use a non-complete version of a sequent calculus for **S4**. Hence, Kripke completness does not hold for their version of $\mathbf{MML}_n$ as well. In particular, the sequents $\Box_j\alpha \Leftrightarrow \neg_j\neg_k\Diamond_j\neg_j\neg_k\alpha$ and $\Diamond_j\alpha \Leftrightarrow \neg_j\neg_k\Box_j\neg_j\neg_k\alpha$ (k is fixed and $k \leqslant n$) are valid, but are not provable in their calculus.

It is interesting that, as follows from [19], the algebraic completeness theorem holds for Kamide and Shramko's original formulation of $\mathbf{MML}_n$, since it is generally not the case that the equalities $I_j(x) = -_j -_k C_j(-_j -_k x)$ and $C_j(x) = -_j -_k I_j(-_j -_k x)$ hold for S4-modal multilattices.

In the next section, we present an alternative formulation of a sequent calculus for $\mathbf{MML}_n$ such that the sequents $\Box_j\alpha \Leftrightarrow \neg_j\neg_k\Diamond_j\neg_j\neg_k\alpha$ and $\Diamond_j\alpha \Leftrightarrow \neg_j\neg_k\Box_j\neg_j\neg_k\alpha$ are provable in it. We establish Kripke completeness, cut elimination, and decidability for it.

4. The formulation of $\mathbf{MML}_n^{\mathbf{S5}}$

4.1. Hypersequent calculus for S5

First of all, let us present Restall's hypersequent calculus for **S5** [48]. The only axiom of Restall's calculus is as follows, for each propositional variable π:[1]

$$(\mathrm{Ax})\ \pi \Rightarrow \pi$$

[1] In Restall's original formulation this axiom is as follows: $\alpha \Rightarrow \alpha$, for each $\alpha \in \mathscr{F}$. By induction, it is possible to show that axiom $\alpha \Rightarrow \alpha$ is provable in our formulation. However, we use axiom $\pi \Rightarrow \pi$ in order to prove embedding theorems by Kamide and Shramko's method [28].

The internal structural rules of the hypersequent calculus for **S5** are (Cut), (IW⇒), and (⇒IW). The external structural rules of hypersequent calculus for **S5** are (EW⇒) and (⇒EW) as well as the following one:

$$(\text{Merge})\ \frac{\Gamma \Rightarrow \Delta \mid \Theta \Rightarrow \Lambda \mid H}{\Gamma, \Theta \Rightarrow \Delta, \Lambda \mid H}$$

The non-modal logical rules of the hypersequent calculus for **S5** are as follows[2]:

$$(\wedge\Rightarrow)\ \frac{\alpha, \beta, \Gamma \Rightarrow \Delta \mid H}{\alpha \wedge \beta, \Gamma \Rightarrow \Delta \mid H} \qquad (\Rightarrow\wedge)\ \frac{\Gamma \Rightarrow \Delta, \alpha \mid H \qquad \Gamma \Rightarrow \Delta, \beta \mid G}{\Gamma \Rightarrow \Delta, \alpha \wedge \beta \mid H \mid G}$$

$$(\vee\Rightarrow)\ \frac{\alpha, \Gamma \Rightarrow \Delta \mid H \qquad \beta, \Gamma \Rightarrow \Delta \mid G}{\alpha \vee \beta, \Gamma \Rightarrow \Delta \mid H \mid G} \qquad (\Rightarrow\vee)\ \frac{\Gamma \Rightarrow \Delta, \alpha, \beta \mid H}{\Gamma \Rightarrow \Delta, \alpha \vee \beta \mid H}$$

$$(\rightarrow\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \alpha \mid H \qquad \beta, \Theta \Rightarrow \Lambda \mid G}{\alpha \rightarrow \beta, \Gamma, \Theta \Rightarrow \Delta, \Lambda \mid H \mid G} \qquad (\Rightarrow\rightarrow)\ \frac{\alpha, \Gamma \Rightarrow \Delta, \beta \mid H}{\Gamma \Rightarrow \Delta, \alpha \rightarrow \beta \mid H}$$

$$(\leftarrow\Rightarrow)\ \frac{\alpha, \Gamma \Rightarrow \Delta, \beta \mid H}{\alpha \leftarrow \beta, \Gamma \Rightarrow \Delta \mid H} \qquad (\Rightarrow\leftarrow)\ \frac{\Gamma \Rightarrow \Delta, \alpha \mid H \qquad \beta, \Theta \Rightarrow \Lambda \mid G}{\Gamma, \Theta \Rightarrow \Delta, \Lambda, \alpha \leftarrow \beta \mid H \mid G}$$

$$(\neg\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \alpha \mid H}{\neg\alpha, \Gamma \Rightarrow \Delta \mid H} \qquad (\Rightarrow\neg)\ \frac{\alpha, \Gamma \Rightarrow \Delta \mid H}{\Gamma \Rightarrow \Delta, \neg\alpha \mid H}$$

The modal logical rules of the hypersequent calculus for **S5** are as follows[3]:

$$(\Box\Rightarrow)\ \frac{\alpha, \Gamma \Rightarrow \Delta \mid H}{\Box\alpha \Rightarrow \mid \Gamma \Rightarrow \Delta \mid H} \qquad (\Rightarrow\Box)\ \frac{\Rightarrow \alpha \mid H}{\Rightarrow \Box\alpha \mid H}$$

$$(\Diamond\Rightarrow)\ \frac{\alpha \Rightarrow \mid H}{\Diamond\alpha \Rightarrow \mid H} \qquad (\Rightarrow\Diamond)\ \frac{\Gamma \Rightarrow \Delta, \alpha \mid H}{\Gamma \Rightarrow \Delta \mid \ \Rightarrow \Diamond\alpha \mid H}$$

Using the well-known definition (Df$_\Diamond$) $\Diamond\alpha := \neg\Box\neg\alpha$, we show the derivability of the rules for $\Diamond$ via all the other ones:

$$\dfrac{\dfrac{\dfrac{\dfrac{\alpha \Rightarrow \mid H}{\Rightarrow \neg\alpha \mid H}\,(\Rightarrow\neg)}{\Rightarrow \Box\neg\alpha \mid H}\,(\Rightarrow\Box)}{\neg\Box\neg\alpha \Rightarrow \mid H}\,(\neg\Rightarrow)}{\Diamond\alpha \Rightarrow \mid H}\,(\mathrm{Df}_\Diamond) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow \Delta, \alpha \mid H}{\neg\alpha, \Gamma \Rightarrow \Delta \mid H}\,(\neg\Rightarrow)}{\Box\neg\alpha \Rightarrow \mid \Gamma \Rightarrow \Delta \mid H}\,(\Box\Rightarrow)}{\Rightarrow \neg\Box\neg\alpha \mid \Gamma \Rightarrow \Delta \mid H}\,(\Rightarrow\neg)}{\Rightarrow \Diamond\alpha \mid \Gamma \Rightarrow \Delta \mid H}\,(\mathrm{Df}_\Diamond)}{\Gamma \Rightarrow \Delta \mid \ \Rightarrow \Diamond\alpha \mid H}\,(\mathrm{Df}_H)$$

[2] Following Kamide and Shramko's paper [28], we extend Restall's original formulation by the rules for ∨, →, and ←.

[3] We extend Restall's original formulation by the rules for $\Diamond$.

where (Df_H) stands for the definition of a hypersequent. Since a hypersequent is a multiset of sequents, we can permute sequents without using the special rule for doing it, called external permutation.

On the other hand, using the well-known definition $(\mathrm{Df}_\Box)$ $\Box\alpha := \neg\Diamond\neg\alpha$, we show the derivability of the rules for $\Box$ via all the other ones:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow \alpha \mid H}{\neg\alpha \Rightarrow \mid H}\,(\neg\Rightarrow)}{\Diamond\neg\alpha \Rightarrow \mid H}\,(\Rightarrow\Diamond)}{\Rightarrow\neg\Diamond\neg\alpha \mid H}\,(\Rightarrow\neg)}{\Rightarrow\Box\alpha \mid H}\,(\mathrm{Df}_\Box)
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha, \Gamma \Rightarrow \Delta \mid H}{\Gamma \Rightarrow \Delta, \neg\alpha \mid H}\,(\Rightarrow\neg)}{\Gamma \Rightarrow \Delta \mid \ \Rightarrow\Diamond\neg\alpha \mid H}\,(\Rightarrow\Diamond)}{\Gamma \Rightarrow \Delta \mid \neg\Diamond\neg\alpha \Rightarrow \mid H}\,(\neg\Rightarrow)}{\Gamma \Rightarrow \Delta \mid \Box\alpha \Rightarrow \mid H}\,(\mathrm{Df}_\Box)}{\Box\alpha \Rightarrow \mid \Gamma \Rightarrow \Delta \mid H}\,(\mathrm{Df}_H)$$

What is important for the further exposition is that $(\mathrm{Df}_\Diamond)$ and $(\mathrm{Df}_\Box)$ are provable in the above mentioned hypersequent calculus for **S5**:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha \Rightarrow \alpha}{\neg\alpha, \alpha \Rightarrow}\,(\neg\Rightarrow)}{\Box\neg\alpha \Rightarrow \mid \alpha \Rightarrow}\,(\Box\Rightarrow)}{\Box\neg\alpha \Rightarrow \mid \Diamond\alpha \Rightarrow}\,(\Diamond\Rightarrow)}{\Box\neg\alpha, \Diamond\alpha \Rightarrow}\,(\mathrm{Merge})}{\Diamond\alpha\Rightarrow\neg\Box\neg\alpha}\,(\Rightarrow\neg)
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha \Rightarrow \alpha}{\Rightarrow \alpha, \neg\alpha}\,(\neg\Rightarrow)}{\Rightarrow\neg\alpha \mid \ \Rightarrow\Diamond\alpha}\,(\Diamond\Rightarrow)}{\Rightarrow\Box\neg\alpha \mid \ \Rightarrow\Diamond\alpha}\,(\Box\Rightarrow)}{\Rightarrow\Box\neg\alpha, \Diamond\alpha}\,(\mathrm{Merge})}{\neg\Box\neg\alpha\Rightarrow\Diamond\alpha}\,(\neg\Rightarrow)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha \Rightarrow \alpha}{\neg\alpha, \alpha \Rightarrow}\,(\neg\Rightarrow)}{\Box\alpha \Rightarrow \mid \neg\alpha \Rightarrow}\,(\Box\Rightarrow)}{\Box\alpha \Rightarrow \mid \Diamond\neg\alpha \Rightarrow}\,(\Diamond\Rightarrow)}{\Box\alpha, \Diamond\neg\alpha \Rightarrow}\,(\mathrm{Merge})}{\Box\alpha\Rightarrow\neg\Diamond\neg\alpha}\,(\neg\Rightarrow)
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha \Rightarrow \alpha}{\Rightarrow \alpha, \neg\alpha}\,(\neg\Rightarrow)}{\Rightarrow \alpha \mid \ \Rightarrow\Diamond\neg\alpha}\,(\Rightarrow\Diamond)}{\Rightarrow\Box\alpha \mid \ \Rightarrow\Diamond\neg\alpha}\,(\Rightarrow\Box)}{\Rightarrow\Box\alpha, \Diamond\neg\alpha}\,(\mathrm{Merge})}{\neg\Diamond\neg\alpha\Rightarrow\Box\alpha}\,(\Rightarrow\neg)$$

4.2. Hypersequent calculus for $\mathbf{MML}_n^{\mathbf{S5}}$

The hypersequent calculus for $\mathbf{MML}_n^{\mathbf{S5}}$ is an extension of the hypersequent calculus for $\mathbf{ML}_n$ by (Merge) and the following modal rules. The non-negated ones are as follows:

$$(\Box_j\Rightarrow)\ \dfrac{\alpha, \Gamma \Rightarrow \Delta \mid H}{\Box_j\alpha \Rightarrow \mid \Gamma \Rightarrow \Delta \mid H}
\qquad
(\Rightarrow\Box_j)\ \dfrac{\Rightarrow \alpha \mid H}{\Rightarrow\Box_j\alpha \mid H}$$

$$(\Diamond_j\Rightarrow)\ \dfrac{\alpha \Rightarrow \mid H}{\Diamond_j\alpha \Rightarrow \mid H}
\qquad
(\Rightarrow\Diamond_j)\ \dfrac{\Gamma \Rightarrow \Delta, \alpha \mid H}{\Gamma \Rightarrow \Delta \mid \ \Rightarrow\Diamond_j\alpha \mid H}$$

The jj-negated modal logical rules are as follows:

$$(\neg_j\Box_j\Rightarrow)\ \frac{\neg_j\alpha\Rightarrow\ |\ H}{\neg_j\Box_j\alpha\Rightarrow\ |\ H}\qquad(\Rightarrow\neg_j\Box_j)\ \frac{\Gamma\Rightarrow\Delta,\neg_j\alpha\ |\ H}{\Gamma\Rightarrow\Delta\ |\ \Rightarrow\neg_j\Box_j\alpha\ |\ H}$$

$$(\neg_j\Diamond_j\Rightarrow)\ \frac{\neg_j\alpha,\Gamma\Rightarrow\Delta\ |\ H}{\neg_j\Diamond_j\alpha\Rightarrow\ |\ \Gamma\Rightarrow\Delta\ |\ H}\qquad(\Rightarrow\neg_j\Diamond_j)\ \frac{\Rightarrow\neg_j\alpha\ |\ H}{\Rightarrow\neg_j\Diamond_j\alpha\ |\ H}$$

The kj-negated modal logical rules are as follows:

$$(\neg_k\Box_j\Rightarrow)\ \frac{\neg_k\alpha,\Gamma\Rightarrow\Delta\ |\ H}{\neg_k\Box_j\alpha\Rightarrow\ |\ \Gamma\Rightarrow\Delta\ |\ H}\qquad(\Rightarrow\neg_k\Box_j)\ \frac{\Rightarrow\neg_k\alpha\ |\ H}{\Rightarrow\neg_k\Box_j\alpha\ |\ H}$$

$$(\neg_k\Diamond_j\Rightarrow)\ \frac{\neg_k\alpha\Rightarrow\ |\ H}{\neg_k\Diamond_j\alpha\Rightarrow\ |\ H}\qquad(\Rightarrow\neg_k\Diamond_j)\ \frac{\Gamma\Rightarrow\Delta,\neg_k\alpha\ |\ H}{\Gamma\Rightarrow\Delta\ |\ \Rightarrow\neg_k\Diamond_j\alpha\ |\ H}$$

Notice that the sequents $\Box_j\alpha\Leftrightarrow\neg_j\neg_k\Diamond_j\neg_j\neg_k\alpha$, $\Diamond_j\alpha\Leftrightarrow\neg_j\neg_k\Box_j\neg_j\neg_k\alpha$ are provable in $\mathbf{MML}_n^{\mathbf{S5}}$ (the proofs are analogous to the proofs of $(Df_\Box)$ and $(Df_\Diamond)$ in **S5**). Besides, the sequents $\neg_j\Diamond_j\alpha\Leftrightarrow\Box_j\neg_j\alpha$, $\neg_j\Box_j\alpha\Leftrightarrow\Diamond_j\neg_j\alpha$, $\neg_k\Box_j\alpha\Leftrightarrow\Box_j\neg_k\alpha$, and $\neg_k\Diamond_j\alpha\Leftrightarrow\Diamond_j\neg_k\alpha$ are provable in $\mathbf{MML}_n^{\mathbf{S5}}$ as well.

4.3. Kripke semantics for $\mathbf{MML}_n^{\mathbf{S5}}$

We start with some standard definitions following the lines of [28].

DEFINITION 4.1. A Kripke frame $\mathcal{F}$ is a structure $(M, R_1, \ldots, R_n)$ where M is a non-empty set, each R_i is a binary relation on M, $1\leqslant i\leqslant n$.

An **S5**-Kripke frame is a Kripke frame where each R_j, $1\leqslant j\leqslant n$, is an equivalence relation.

DEFINITION 4.2. A valuation on a Kripke frame is a mapping $\models\colon\Pi\longrightarrow 2^M$ from the set of propositional variables to the power-set of M. A Kripre model is a pair $(\mathcal{F},\models)$, where $\mathcal{F}$ is a Kripke frame, $\models$ is a valuation on it.

For a model $\mathcal{M}$ we say that it is *based* on the frame $\mathcal{F}$ if $\mathcal{M}=(\mathcal{F},\models)$ for some valuation $\models$. We say that a model $\mathcal{M}$ is an **S5**-model if it is based on some **S5**-frame.

In the next definition $\Pi\cup\neg\Pi$ denotes the set of propositional variables joined with the set of negated propositional variables, that is for each $\pi\in\Pi$ and each j, $1\leqslant j\leqslant n$, $\neg_j\pi\in\Pi$.

DEFINITION 4.3. A paraconsistent valuation $\models^p$ on a Kripke frame is a mapping $\models^p : \Pi \cup \neg\Pi \longrightarrow 2^M$ from the set of propositional variables and negated propositional variables to the power-set of M. A paraconsistent Kripke model is a pair $(\mathcal{F}, \models^p)$, where $\mathcal{F}$ is a paraconsistent Kripke frame, $\models^p$ is a paraconsistent valuation. An **S5**-paraconsistent Kripke model is a paraconsistent Kripke model based on an **S5**-Kripke frame.

For an $x \in M$ and an $\pi \in \Pi$ such that $x \in \models^p (\pi)$ or $x \in \models^p (\neg_j \pi)$ we will adopt the more convenient notations $x \models^p \pi$ and $x \models^p \neg_j \pi$, respectively, in the sequel. Likewise we write $x \models \pi$ instead of $x \in \models (\pi)$. The extension of a valuation $\models$ to the set of all formulas of the language $\mathscr{L}$ is almost standard and supposed to be known to the reader.

Let us now extend the paraconsistent valuation to the set of all formulas of the language $\mathscr{L}_{\mathrm{M}}$, assuming that $j, k \leqslant n$ and $j \neq k$ in the following definition.

DEFINITION 4.4. For each paraconsistent frame $\mathcal{F}$, each $x \in \mathcal{F}$, all formulas α and β, the extended paraconsistent valuation is defined by the following expressions:

(1) $x \models^p \alpha \wedge_j \beta$ iff $x \models^p \alpha$ and $x \models^p \beta$,
(2) $x \models^p \alpha \vee_j \beta$ iff $x \models^p \alpha$ or $x \models^p \beta$,
(3) $x \models^p \alpha \rightarrow_j \beta$ iff $x \not\models^p \alpha$ or $x \models^p \beta$,
(4) $x \models^p \alpha \leftarrow_j \beta$ iff $x \models^p \alpha$ or $x \not\models^p \beta$,
(5) $x \models^p \Box_j \alpha$ iff $\forall y(R_j(x, y) \Rightarrow y \models^p \alpha)$,
(6) $x \models^p \Diamond_j \alpha$ iff $\exists y(R_j(x, y)$ and $y \models^p \alpha)$,
(7) $x \models^p \neg_j(\alpha \wedge_j \beta)$ iff $x \models^p \neg_j \alpha$ or $x \models^p \neg_j \beta$,
(8) $x \models^p \neg_j(\alpha \vee_j \beta)$ iff $x \models^p \neg_j \alpha$ and $x \models^p \neg_j \beta$,
(9) $x \models^p \neg_j(\alpha \rightarrow_j \beta)$ iff $x \models^p \neg_j \beta$ and $x \not\models^p \neg_j \alpha$,
(10) $x \models^p \neg_j(\alpha \leftarrow_j \beta)$ iff $x \models^p \neg_j \alpha$ or $x \not\models^p \neg_j \beta$,
(11) $x \models^p \neg_j \Box_j \alpha$ iff $\exists y(R_j(x, y)$ and $y \models^p \neg_j \alpha)$,
(12) $x \models^p \neg_j \Diamond_j \alpha$ iff $\forall y(R_j(x, y) \Rightarrow y \models^p \neg_j \alpha)$,
(13) $x \models^p \neg_j \neg_j \alpha$ iff $x \models^p \alpha$,
(14) $x \models^p \neg_k(\alpha \wedge_j \beta)$ iff $x \models^p \neg_k \alpha$ and $x \models^p \neg_k \beta$,
(15) $x \models^p \neg_k(\alpha \vee_j \beta)$ iff $x \models^p \neg_k \alpha$ or $x \models^p \neg_k \beta$,
(16) $x \models^p \neg_k(\alpha \rightarrow_j \beta)$ iff $x \models^p \neg_k \beta$ or $x \not\models^p \neg_k \alpha$,
(17) $x \models^p \neg_k(\alpha \leftarrow_j \beta)$ iff $x \models^p \neg_k \alpha$ and $x \not\models^p \neg_k \beta$,
(18) $x \models^p \neg_k \Box_j \alpha$ iff $\forall y(R_j(x, y) \Rightarrow y \models^p \neg_k \alpha)$,
(19) $x \models^p \neg_k \Diamond_j \alpha$ iff $\exists y(R_j(x, y)$ and $y \models^p \neg_k \alpha)$,
(20) $x \models^p \neg_k \neg_j \alpha$ iff $x \not\models^p \alpha$.

Definition 4.5. A formula α is true in a paraconsistent model $\mathcal{M}$ iff for each $x \in \mathcal{M}$, $x \models^p \alpha$. A formula α is $\mathbf{MML}_n^{\mathbf{S5}}$-valid in an $\mathbf{S5}$-frame $\mathcal{F}$ iff it is true in every paraconsistent model $\mathcal{M}$ based on $\mathcal{F}$. A formula α is $\mathbf{MML}_n^{\mathbf{S5}}$-valid iff it is $\mathbf{MML}_n^{\mathbf{S5}}$-valid in every $\mathbf{S5}$-frame.

5. Embedding theorems for $\mathbf{MML}^{\mathbf{S5}}$

5.1. Syntactical embeddings

Definition 5.1. [28, p. 324–325, Definition 3.3] Let $n > 1$, $j, k \leqslant n$, and $j \neq k$. Then a mapping f from $\mathscr{L}_{\mathrm{M}}$ to $\mathscr{L}$ is defined inductively as follows:

(1) $f(\pi) := \pi$ and $f(\neg_j \pi) := \pi^j$ (where $\pi^j \in \Pi^j$), for each $\pi \in \Pi$,
(2) $f(\alpha \wedge_j \beta) := f(\alpha) \wedge f(\beta)$,
(3) $f(\alpha \vee_j \beta) := f(\alpha) \vee f(\beta)$,
(4) $f(\alpha \rightarrow_j \beta) := f(\alpha) \rightarrow f(\beta)$,
(5) $f(\alpha \leftarrow_j \beta) := f(\alpha) \leftarrow f(\beta)$,
(6) $f(\neg_j(\alpha \wedge_j \beta)) := f(\neg_j \alpha) \vee f(\neg_j \beta)$,
(7) $f(\neg_j(\alpha \vee_j \beta)) := f(\neg_j \alpha) \wedge f(\neg_j \beta)$,
(8) $f(\neg_j(\alpha \rightarrow_j \beta)) := f(\neg_j \beta) \leftarrow f(\neg_j \alpha)$,
(9) $f(\neg_j(\alpha \leftarrow_j \beta)) := f(\neg_j \beta) \rightarrow f(\neg_j \alpha)$,
(10) $f(\neg_j \neg_j \alpha) := f(\alpha)$,
(11) $f(\neg_k(\alpha \wedge_j \beta)) := f(\neg_k \alpha) \wedge f(\neg_k \beta)$,
(12) $f(\neg_k(\alpha \vee_j \beta)) := f(\neg_k \alpha) \vee f(\neg_k \beta)$,
(13) $f(\neg_k(\alpha \rightarrow_j \beta)) := f(\neg_k \alpha) \rightarrow f(\neg_k \beta)$,
(14) $f(\neg_k(\alpha \leftarrow_j \beta)) := f(\neg_k \alpha) \leftarrow f(\neg_k \beta)$,
(15) $f(\neg_k \neg_j \alpha) := \neg f(\alpha)$,
(16) $f(\Box_j \alpha) := \Box f(\alpha)$,
(17) $f(\Diamond_j \alpha) := \Diamond f(\alpha)$,
(18) $f(\neg_j \Box_j \alpha) := \Diamond f(\neg_j \alpha)$,
(19) $f(\neg_j \Diamond_j \alpha) := \Box f(\neg_j \alpha)$,
(20) $f(\neg_k \Box_j \alpha) := \Box f(\neg_k \alpha)$,
(21) $f(\neg_k \Diamond_j \alpha) := \Diamond f(\neg_k \alpha)$.

Theorem 5.1 (Weak syntactical embedding from $\mathbf{MML}_n^{\mathbf{S5}}$ into $\mathbf{S5}$). *Let f be the mapping introduced in Definition 5.1. Then, for each pair of finite sets Ξ and Σ of $\mathscr{L}_{\mathrm{M}}$-formulas and each hypersequent I, it holds that:*

(1) $\mathbf{MML}_n^{\mathbf{S5}} \vdash \Xi \Rightarrow \Sigma \mid I$ *implies* $\mathbf{S5} \vdash f(\Xi) \Rightarrow f(\Sigma) \mid f(I)$,
(2) $\mathbf{S5} \backslash (\mathrm{Cut}) \vdash f(\Xi) \Rightarrow f(\Sigma) \mid f(I)$ *implies* $\mathbf{MML}_n^{\mathbf{S5}} \backslash (\mathrm{Cut}) \vdash \Xi \Rightarrow \Sigma \mid I$.

Proof. (1) By induction on the proof $\mathscr{P}$ of $\Xi \Rightarrow \Sigma \mid I$ in the hypersequent calculus for $\mathbf{MML}_n^{\mathbf{S5}}$. We distinguish the cases according to $\mathscr{P}$'s last inference. The propositional cases are proved in [28]. We do the modal ones only. As an example, we consider the cases regarding the left rules.

1. The case ($\Box_j\Rightarrow$). The last inference of $\mathscr{P}$ has the following form:

$$\frac{\alpha, \Gamma \Rightarrow \Delta \mid H}{\Box_j\alpha \Rightarrow \mid \Gamma \Rightarrow \Delta \mid H}\ (\Box_j\Rightarrow)$$

Using the induction hypothesis, we obtain the following proof in hypersequent calculus for **S5**:

$$\frac{\begin{array}{c}\vdots\\ f(\alpha), f(\Gamma) \Rightarrow f(\Delta) \mid f(H)\end{array}}{\Box f(\alpha) \Rightarrow \mid f(\Gamma) \Rightarrow f(\Delta) \mid f(H)}\ (\Box\Rightarrow)$$

where $\Box f(\alpha) = f(\Box_j\alpha)$, by the definition of f.

2. The case ($\Diamond_j\Rightarrow$). The last inference of $\mathscr{P}$ has the following form:

$$\frac{\alpha \Rightarrow \mid H}{\Diamond_j\alpha \Rightarrow \mid H}\ (\Diamond_j\Rightarrow)$$

Using the induction hypothesis, we have

$$\frac{\begin{array}{c}\vdots\\ f(\alpha) \Rightarrow \mid f(H)\end{array}}{\Diamond f(\alpha) \Rightarrow \mid f(H)}\ (\Diamond\Rightarrow)$$

where $\Diamond f(\alpha) = f(\Diamond_j\alpha)$, by the definition of f.

3. The case ($\neg_j\Box_j\Rightarrow$). The last inference of $\mathscr{P}$ has the following form:

$$\frac{\neg_j\alpha \Rightarrow \mid H}{\neg_j\Box_j\alpha \Rightarrow \mid H}\ (\neg_j\Box_j\Rightarrow)$$

Using the induction hypothesis, we have

$$\frac{\begin{array}{c}\vdots\\ f(\neg_j\alpha) \Rightarrow \mid f(H)\end{array}}{\Diamond f(\neg_j\alpha) \Rightarrow \mid f(H)}\ (\Diamond\Rightarrow)$$

where $\Diamond f(\neg_j\alpha) = f(\neg_j\Box_j\alpha)$, by the definition of f.

4. The case ($\neg_j\Diamond_j\Rightarrow$). The last inference of $\mathscr{P}$ has the following form:

$$\frac{\neg_j\alpha, \Gamma \Rightarrow \Delta \mid H}{\neg_j\Diamond_j\alpha \Rightarrow \mid \Gamma \Rightarrow \Delta \mid H}\ (\neg_j\Diamond_j\Rightarrow)$$

Using the induction hypothesis, we have

$$\dfrac{\begin{array}{c}\vdots\\ f(\neg_j\alpha), f(\Gamma) \Rightarrow f(\Delta) \mid f(H)\end{array}}{\Box f(\neg_j\alpha) \Rightarrow \mid f(\Gamma) \Rightarrow f(\Delta) \mid f(H)}\,(\Box\Rightarrow)$$

where $\Box f(\neg_j\alpha) = f(\neg_j\Diamond_j\alpha)$, by the definition of f.

5. The case $(\neg_k\Box_j\Rightarrow)$. The last inference of $\mathscr{P}$ has the following form:

$$\dfrac{\neg_k\alpha, \Gamma \Rightarrow \Delta \mid H}{\neg_k\Box_j\alpha \Rightarrow \mid \Gamma \Rightarrow \Delta \mid H}\,(\neg_k\Box_j\Rightarrow)$$

Using the induction hypothesis, we have

$$\dfrac{\begin{array}{c}\vdots\\ f(\neg_k\alpha), f(\Gamma) \Rightarrow f(\Delta) \mid f(H)\end{array}}{\Box f(\neg_k\alpha) \Rightarrow \mid f(\Gamma) \Rightarrow f(\Delta) \mid f(H)}\,(\Box\Rightarrow)$$

where $\Box f(\neg_k\alpha) = f(\neg_k\Box_j\alpha)$, by the definition of f.

6. The case $(\neg_k\Diamond_j\Rightarrow)$. The last inference of $\mathscr{P}$ has the following form:

$$\dfrac{\neg_k\alpha \Rightarrow \mid H}{\neg_k\Diamond_j\alpha \Rightarrow \mid H}\,(\neg_k\Diamond_j\Rightarrow)$$

Using the induction hypothesis, we have

$$\dfrac{\begin{array}{c}\vdots\\ f(\neg_k\alpha) \Rightarrow \mid f(H)\end{array}}{\Diamond f(\neg_k\alpha) \Rightarrow \mid f(H)}\,(\Diamond\Rightarrow)$$

where $\Diamond f(\neg_k\alpha) = f(\neg_k\Diamond_j\alpha)$, by the definition of f.

(2) By induction on the proof $\mathscr{Q}$ of $f(\Xi) \Rightarrow f(\Sigma) \mid f(I)$ in **S5** $\setminus$ (Cut). We distinguish the cases according to $\mathscr{Q}$'s last inference. Since the propositional cases are proved in [28], so we do the modal ones only. As an example, we consider the cases regarding the left rules.

1. The case $(\Box\Rightarrow)$. The last inference of $\mathscr{Q}$ is an application of $(\Box\Rightarrow)$.

Subcase (1.1): The last inference of $\mathscr{Q}$ has the following form:

$$\dfrac{f(\alpha), f(\Gamma) \Rightarrow f(\Delta) \mid f(H)}{\Box f(\alpha) \Rightarrow \mid f(\Gamma) \Rightarrow f(\Delta) \mid f(H)}\,(\Box\Rightarrow)$$

where $\Box f(\alpha) = f(\Box_j\alpha)$, by the definition of f.

Using the induction hypothesis, we have

$$\frac{\begin{array}{c}\vdots\\ \alpha, \Gamma \Rightarrow \Delta \mid H\end{array}}{\Box_j\alpha \Rightarrow \mid \Gamma \Rightarrow \Delta \mid H}\,(\Box_j{\Rightarrow})$$

Subcase (1.2): The last inference of $\mathscr{Q}$ has the following form:

$$\frac{f(\neg_j\alpha), f(\Gamma) \Rightarrow f(\Delta) \mid f(H)}{\Box f(\neg_j\alpha) \Rightarrow \mid f(\Gamma) \Rightarrow f(\Delta) \mid f(H)}\,(\Box{\Rightarrow})$$

where $\Box f(\neg_j\alpha) = f(\Box_j\neg_j\alpha) = f(\neg_j\Diamond_j\alpha)$, by the definition of f. Using the induction hypothesis, we can obtain the required fact in at least two ways (recall that $\mathbf{MML}_n^{\mathbf{S5}} \vdash \Box_j\neg_j\alpha \Leftrightarrow \neg_j\Diamond_j\alpha$):

$$\frac{\begin{array}{c}\vdots\\ \neg_j\alpha, \Gamma \Rightarrow \Delta \mid H\end{array}}{\Box_j\neg_j\alpha \Rightarrow \mid \Gamma \Rightarrow \Delta \mid H}\,(\Box_j{\Rightarrow}) \qquad \frac{\begin{array}{c}\vdots\\ \neg_j\alpha, \Gamma \Rightarrow \Delta \mid H\end{array}}{\neg_j\Diamond_j\alpha \Rightarrow \mid \Gamma \Rightarrow \Delta \mid H}\,(\neg_j\Diamond_j{\Rightarrow})$$

Subcase (1.3): The last inference of $\mathscr{Q}$ has the following form:

$$\frac{f(\neg_k\alpha), f(\Gamma) \Rightarrow f(\Delta) \mid f(H)}{\Box f(\neg_k\alpha) \Rightarrow \mid f(\Gamma) \Rightarrow f(\Delta) \mid f(H)}\,(\Box{\Rightarrow})$$

where $\Box f(\neg_k\alpha) = f(\Box_j\neg_k\alpha) = f(\neg_k\Box_j\alpha)$, by the definition of f. Using the induction hypothesis, we can obtain the required fact in at least two ways (recall that $\mathbf{MML}_n^{\mathbf{S5}} \vdash \Box_j\neg_k\alpha \Leftrightarrow \neg_k\Box_j\alpha$):

$$\frac{\begin{array}{c}\vdots\\ \neg_k\alpha, \Gamma \Rightarrow \Delta \mid H\end{array}}{\Box_j\neg_k\alpha \Rightarrow \mid \Gamma \Rightarrow \Delta \mid H}\,(\Box_j{\Rightarrow}) \qquad \frac{\begin{array}{c}\vdots\\ \neg_k\alpha, \Gamma \Rightarrow \Delta \mid H\end{array}}{\neg_k\Box_j\alpha \Rightarrow \mid \Gamma \Rightarrow \Delta \mid H}\,(\neg_k\Box_j{\Rightarrow})$$

2. The case $(\Diamond{\Rightarrow})$. The last inference of $\mathscr{Q}$ is an application of $(\Diamond{\Rightarrow})$.

Subcase (2.1): The last inference of $\mathscr{Q}$ has the following form:

$$\frac{f(\alpha) \Rightarrow \mid f(H)}{\Diamond f(\alpha) \Rightarrow \mid f(H)}\,(\Diamond{\Rightarrow})$$

where $\Diamond f(\alpha) = f(\Diamond_j\alpha)$, by the definition of f.

Using the induction hypothesis, we have

$$\frac{\begin{array}{c}\vdots\\ \alpha \Rightarrow \mid H\end{array}}{\Diamond_j\alpha \Rightarrow \mid H}\,(\Diamond_j{\Rightarrow})$$

Subcase (2.2): The last inference of $\mathscr{Q}$ has the following form:

$$\frac{f(\neg_j\alpha) \Rightarrow \mid f(H)}{\Diamond f(\neg_j\alpha) \Rightarrow \mid f(H)}\ (\Diamond\Rightarrow)$$

where $\Diamond f(\neg_j\alpha) = f(\Diamond_j\neg_j\alpha) = f(\neg_j\Box_j\alpha)$, by the definition of f.

Using the induction hypothesis, we can obtain the required fact in at least two ways (recall that $\mathbf{MML}_n^{\mathbf{S5}} \vdash \Diamond_j\neg_j\alpha \Leftrightarrow \neg_j\Box_j\alpha$):

$$\frac{\begin{matrix}\vdots\\ \neg_j\alpha \Rightarrow \mid H\end{matrix}}{\Diamond_j\neg_j\alpha \Rightarrow \mid H}\ (\Diamond_j\Rightarrow) \qquad \frac{\begin{matrix}\vdots\\ \neg_j\alpha \Rightarrow \mid H\end{matrix}}{\neg_j\Box_j\alpha \Rightarrow \mid H}\ (\neg_j\Box_j\Rightarrow)$$

Subcase (2.3): The last inference of $\mathscr{Q}$ has the following form:

$$\frac{f(\neg_k\alpha) \Rightarrow \mid f(H)}{\Diamond f(\neg_k\alpha) \Rightarrow \mid f(H)}\ (\Diamond\Rightarrow)$$

where $\Diamond f(\neg_k\alpha) = f(\Diamond_j\neg_k\alpha) = f(\neg_k\Diamond_j\alpha)$, by the definition of f.

Using the induction hypothesis, we can obtain the required fact in at least two ways (recall that $\mathbf{MML}_n^{\mathbf{S5}} \vdash \Diamond_j\neg_k\alpha \Leftrightarrow \neg_k\Diamond_j\alpha$):

$$\frac{\begin{matrix}\vdots\\ \neg_k\alpha \Rightarrow \mid H\end{matrix}}{\Diamond_j\neg_k\alpha \Rightarrow \mid H}\ (\Diamond_j\Rightarrow) \qquad \frac{\begin{matrix}\vdots\\ \neg_k\alpha \Rightarrow \mid H\end{matrix}}{\neg_k\Diamond_j\alpha \Rightarrow \mid H}\ (\neg_k\Diamond_j\Rightarrow)$$

$\dashv$

THEOREM 5.2 (Cut elimination for $\mathbf{MML}_n^{\mathbf{S5}}$). *The rule* (Cut) *is admissible in the cut-free hypersequent calculus for* $\mathbf{MML}_n^{\mathbf{S5}}$.

PROOF. Suppose $\mathbf{MML}_n^{\mathbf{S5}} \vdash \Gamma \Rightarrow \Delta$. By Theorem 5.1, $\mathbf{S5} \vdash f(\Gamma) \Rightarrow f(\Delta)$. By the cut elimination theorem for $\mathbf{S5}$, it holds that $\mathbf{S5} \setminus (\mathrm{Cut}) \vdash f(\Gamma) \Rightarrow f(\Delta)$. By Theorem 5.1, $\mathbf{MML}_n^{\mathbf{S5}} \setminus (\mathrm{Cut}) \vdash \Gamma \Rightarrow \Delta$. $\dashv$

THEOREM 5.3 (Syntactical embedding from $\mathbf{MML}_n^{\mathbf{S5}}$ into $\mathbf{S5}$). *Let f be the mapping introduced in Definition 5.1. Then, for each pair of finite sets Ξ and Σ of $\mathscr{L}_{\mathrm{M}}$-formulas and each hypersequent I, it holds that*:

(1) $\mathbf{MML}_n^{\mathbf{S5}} \vdash \Xi \Rightarrow \Sigma \mid I$ *iff* $\mathbf{S5} \vdash f(\Xi) \Rightarrow f(\Sigma) \mid f(I)$;
(2) $\mathbf{MML}_n^{\mathbf{S5}} \setminus (\mathrm{Cut}) \vdash \Xi \Rightarrow \Sigma \mid I$ *iff* $\mathbf{S5} \setminus (\mathrm{Cut}) \vdash f(\Xi) \Rightarrow f(\Sigma) \mid f(I)$.

PROOF. (1) By Theorem 5.1, $\mathbf{MML}_n^{\mathbf{S5}} \vdash \Xi \Rightarrow \Sigma \mid I$ implies $\mathbf{S5} \vdash f(\Xi) \Rightarrow f(\Sigma) \mid f(I)$. Suppose $\mathbf{S5} \vdash f(\Xi) \Rightarrow f(\Sigma) \mid f(I)$. By the cut elimination theorem for $\mathbf{S5}$, it holds that $\mathbf{S5} \setminus (\mathrm{Cut}) \vdash f(\Xi) \Rightarrow f(\Sigma) \mid f(I)$. By Theorem 5.1, $\mathbf{MML}_n^{\mathbf{S5}} \setminus (\mathrm{Cut}) \vdash \Xi \Rightarrow \Sigma \mid I$. Hence, $\mathbf{MML}_n^{\mathbf{S5}} \vdash \Xi \Rightarrow \Sigma \mid I$.

(2) Suppose $\mathbf{MML}_n^{\mathbf{S5}} \setminus (\mathrm{Cut}) \vdash \Xi \Rightarrow \Sigma \mid I$. Then $\mathbf{MML}_n^{\mathbf{S5}} =\vdash \Xi \Rightarrow \Sigma \mid I$. By Theorem 5.1, $\mathbf{S5} \vdash f(\Xi) \Rightarrow f(\Sigma) \mid f(I)$. By the cut elimination theorem for **S5**, it holds that $\mathbf{S5} \setminus (\mathrm{Cut}) \vdash f(\Xi) \Rightarrow f(\Sigma) \mid f(I)$. By Theorem 5.1, $\mathbf{S5} \setminus (\mathrm{Cut}) \vdash f(\Xi) \Rightarrow f(\Sigma) \mid f(I)$ implies $\mathbf{MML}_n^{\mathbf{S5}} \setminus (\mathrm{Cut}) \vdash \Xi \Rightarrow \Sigma \mid I$. ⊣

Theorem 5.4 (Decidability for $\mathbf{MML}_n^{\mathbf{S5}}$). *$\mathbf{MML}_n^{\mathbf{S5}}$ is decidable.*

Proof. By the decidability of **S5**, for each α, it is possible to decide whether $\Rightarrow f(\alpha)$ is provable in **S5**. Thus, by Theorem 5.3, $\mathbf{MML}_n^{\mathbf{S5}}$ is decidable. ⊣

Definition 5.2. [28, p. 325–326, Definition 3.4] Let $n > 1$. Then a mapping g from $\mathscr{L}$ to $\mathscr{L}_{\mathrm{M}}$ is defined inductively as follows:

(1) $g(\pi) := \pi$ and $g(\pi^j) := \neg_j \pi$, for each $\pi \in \Pi$, $\pi^j \in \Pi^j$, and $j \leqslant n$,
(2) $g(\alpha \wedge \beta) := g(\alpha) \wedge_j g(\beta)$, where j is fixed such that $0 < j \leqslant n$,
(3) $g(\alpha \vee \beta) := g(\alpha) \vee_j g(\beta)$, where j is fixed such that $0 < j \leqslant n$,
(4) $g(\alpha \rightarrow \beta) := g(\alpha) \rightarrow_j g(\beta)$, where j is fixed such that $0 < j \leqslant n$,
(5) $g(\alpha \leftarrow \beta) := g(\alpha) \leftarrow_j g(\beta)$, where j is fixed such that $0 < j \leqslant n$,
(6) $g(\neg\alpha) := \neg_k \neg_j g(\alpha)$, where j and k are fixed such that $0 < j, k \leqslant n$ and $j \neq k$,
(7) $g(\Box\alpha) := \Box_j g(\alpha)$, where j is fixed such that $0 < j \leqslant n$;
(8) $g(\Diamond\alpha) := \Diamond_j g(\alpha)$, where j is fixed such that $0 < j \leqslant n$.

Theorem 5.5 (Weak syntactical embedding from **S5** into $\mathbf{MML}_n^{\mathbf{S5}}$). *Let g be the mapping introduced in Definition 5.2. Then, for each pair of finite sets Ξ and Σ of $\mathscr{L}$-formulas and each hypersequent I, it holds that*:

(1) $\mathbf{S5} \vdash \Xi \Rightarrow \Sigma \mid I$ *implies* $\mathbf{MML}_n^{\mathbf{S5}} \vdash g(\Xi) \Rightarrow g(\Sigma) \mid g(I)$;
(2) $\mathbf{MML}_n^{\mathbf{S5}} \setminus (\mathrm{Cut}) \vdash g(\Xi) \Rightarrow g(\Sigma) \mid g(I)$ *implies* $\mathbf{S5} \setminus (\mathrm{Cut}) \vdash \Xi \Rightarrow \Sigma \mid I$.

Proof. (1) By induction on the proof $\mathscr{P}$ of $\Xi \Rightarrow \Sigma \mid I$ in the hypersequent calculus for **S5**. We distinguish the cases according to $\mathscr{P}$'s last inference. The propositional cases are proved in [28]. We do the modal ones only. As an example, we consider the cases regarding the left rules.
1. The case ($\Box\Rightarrow$). The last inference of $\mathscr{P}$ has the following form:

$$\dfrac{\alpha, \Gamma \Rightarrow \Delta \mid H}{\Box\alpha \Rightarrow \mid \Gamma \Rightarrow \Delta \mid H}\ (\Box\Rightarrow)$$

Using the induction hypothesis, we obtain the following proof in hypersequent calculus for $\mathbf{MML}_n^{\mathbf{S5}}$:

$$\dfrac{\begin{array}{c}\vdots\\ g(\alpha), g(\Gamma) \Rightarrow g(\Delta) \mid g(H)\end{array}}{\Box_j g(\alpha) \Rightarrow \mid g(\Gamma) \Rightarrow g(\Delta) \mid g(H)}\ (\Box_j{\Rightarrow})$$

where $\Box_j g(\alpha) = g(\Box\alpha)$, by the definition of g.

2. The case $(\Diamond{\Rightarrow})$. The last inference of $\mathscr{P}$ has the following form:

$$\dfrac{\alpha \Rightarrow \mid H}{\Diamond\alpha \Rightarrow \mid H}\ (\Diamond{\Rightarrow})$$

Using the induction hypothesis, we have

$$\dfrac{\begin{array}{c}\vdots\\ g(\alpha) \Rightarrow \mid g(H)\end{array}}{\Diamond_j g(\alpha) \Rightarrow \mid g(H)}\ (\Diamond_j{\Rightarrow})$$

where $\Diamond_j g(\alpha) = g(\Diamond\alpha)$, by the definition of g.

(2) By induction on the proof $\mathscr{Q}$ of $f(\Xi) \Rightarrow f(\Sigma) \mid f(I)$ in $\mathbf{MML}_n^{\mathbf{S5}} \setminus$ (Cut). We distinguish the cases according to $\mathscr{Q}$'s last inference. Since the propositional cases are proved in [28], we do the modal ones only. As an example, we consider the rules $(\Box_j{\Rightarrow})$ and $(\Diamond_j \Rightarrow)$.

1. The case $(\Box_j{\Rightarrow})$. The last inference of $\mathscr{Q}$ has the following form:

$$\dfrac{g(\alpha), g(\Gamma) \Rightarrow g(\Delta) \mid g(H)}{\Box_j g(\alpha) \Rightarrow \mid g(\Gamma) \Rightarrow g(\Delta) \mid g(H)}\ (\Box_j{\Rightarrow})$$

where $\Box_j g(\alpha) = g(\Box\alpha)$, by the definition of g.

Using the induction hypothesis, we have

$$\dfrac{\begin{array}{c}\vdots\\ \alpha, \Gamma \Rightarrow \Delta \mid H\end{array}}{\Box\alpha \Rightarrow \mid \Gamma \Rightarrow \Delta \mid H}\ (\Box{\Rightarrow})$$

2. The case $(\Diamond_j{\Rightarrow})$. The last inference of $\mathscr{Q}$ has the following form:

$$\dfrac{g(\alpha) \Rightarrow \mid g(H)}{\Diamond_j g(\alpha) \Rightarrow \mid g(H)}\ (\Diamond_j{\Rightarrow})$$

where $\Diamond_j g(\alpha) = g(\Diamond\alpha)$, by the definition of g.

Using the induction hypothesis, we have

$$\dfrac{\begin{array}{c}\vdots\\ \alpha \Rightarrow \mid H\end{array}}{\Diamond\alpha \Rightarrow \mid H}\ (\Diamond{\Rightarrow})$$

$\dashv$

Theorem 5.6 (Syntactical embedding from **S5** into $\mathbf{MML}_n^{\mathbf{S5}}$). *Let g be the mapping introduced in Definition 5.2. Then, for each pair of finite sets Ξ and Σ of $\mathscr{L}$-formulas and each hypersequent I, it holds that*:

(1) $\mathbf{S5} \vdash \Xi \Rightarrow \Sigma \mid I$ *iff* $\mathbf{MML}_n^{\mathbf{S5}} \vdash g(\Xi) \Rightarrow g(\Sigma) \mid g(I)$;
(2) $\mathbf{S5} \setminus (\mathrm{Cut}) \vdash \Xi \Rightarrow \Sigma \mid I$ *iff* $\mathbf{MML}_n^{\mathbf{S5}} \setminus (\mathrm{Cut}) \vdash g(\Xi) \Rightarrow g(\Sigma) \mid g(I)$.

Proof. Similarly to Theorem 5.3. Use Theorems 5.5 and 5.2. ⊣

Definition 5.3. [28, p. 336, Definition 6.1] Let $n > 1$ and $j \leqslant n$. Then a mapping h from $\mathscr{L}_{\mathrm{M}}$ to $\mathscr{L}_{\mathrm{M}}$ is defined inductively as follows:[4]

(1) $h(\pi) := \pi$, for each $\pi \in \Pi$,
(2) $h(\alpha \wedge_j \beta) := h(\alpha) \vee_j h(\beta)$,
(3) $h(\alpha \vee_j \beta) := h(\alpha) \wedge_j h(\beta)$,
(4) $h(\alpha \rightarrow_j \beta) := h(\alpha) \leftarrow_j h(\beta)$,
(5) $h(\alpha \leftarrow_j \beta) := h(\alpha) \rightarrow_j h(\beta)$,
(6) $h(\neg_j \alpha) := \neg_j h(\alpha)$,
(7) $h(\Box_j \alpha) := \Diamond_j h(\alpha)$,
(8) $h(\Diamond_j \alpha) := \Box_j h(\alpha)$.

The following theorem represents the duality principle in the spirit of [28] for $\mathbf{MML}_n^{\mathbf{S5}}$.

Theorem 5.7 (Syntactical embedding from $\mathbf{MML}_n^{\mathbf{S5}}$ into $\mathbf{MML}_n^{\mathbf{S5}}$). *Let h be the mapping introduced in Definition 5.3. Then, for each pair of finite sets Ξ and Σ of $\mathscr{L}$-formulas and each hypersequent I, it holds that*:

(1) $\mathbf{MML}_n^{\mathbf{S5}} \vdash \Xi \Rightarrow \Sigma \mid I$ *iff* $\mathbf{MML}_n^{\mathbf{S5}} \vdash h(\Sigma) \Rightarrow h(\Xi) \mid h(I)$;
(2) $\mathbf{MML}_n^{\mathbf{S5}} \setminus (\mathrm{Cut}) \vdash \Xi \Rightarrow \Sigma \mid I$ *iff* $\mathbf{MML}_n^{\mathbf{S5}} \setminus (\mathrm{Cut}) \vdash h(\Sigma) \Rightarrow h(\Xi) \mid h(I)$.

Proof. Since (1) follows from (2), so we prove (2) only.

Suppose $\mathbf{MML}_n^{\mathbf{S5}} \setminus (\mathrm{Cut}) \vdash \Xi \Rightarrow \Sigma \mid I$. We use induction on the proof $\mathscr{P}$ of $\Xi \Rightarrow \Sigma \mid I$ in $\mathbf{MML}_n^{\mathbf{S5}} \setminus (\mathrm{Cut})$. We distinguish the cases according to $\mathscr{P}$'s last inference. The propositional cases are proved in [28]. We do the cases regarding the left modal rules only.

1. The case ($\Box_j\Rightarrow$). The last inference of $\mathscr{P}$ has the following form:

$$\frac{\alpha, \Gamma \Rightarrow \Delta \mid H}{\Box_j \alpha \Rightarrow \mid \Gamma \Rightarrow \Delta \mid H}\,(\Box_j\Rightarrow)$$

[4] In [28], h is called a *dualization function*.

Using the induction hypothesis, we obtain the following proof in $\mathbf{MML}_n^{\mathbf{S5}}\setminus$ (Cut):

$$\dfrac{\vdots \atop h(\Delta) \Rightarrow h(\Gamma), h(\alpha) \mid h(H)}{h(\Delta) \Rightarrow h(\Gamma) \mid\ \Rightarrow \Diamond_j h(\alpha) \mid h(H)}\,(\Rightarrow\Diamond_j)$$

where $\Diamond_j h(\alpha) = h(\Box_j \alpha)$, by the definition of h.

2. The case $(\Diamond_j\Rightarrow)$. The last inference of $\mathscr{P}$ has the following form:

$$\dfrac{\alpha \Rightarrow \mid H}{\Diamond_j \alpha \Rightarrow \mid H}\,(\Diamond_j\Rightarrow)$$

Using the induction hypothesis, we have

$$\dfrac{\vdots \atop \Rightarrow h(\alpha) \mid h(H)}{\Rightarrow \Box_j h(\alpha) \mid h(H)}\,(\Rightarrow\Box_j)$$

where $\Box_j h(\alpha) = h(\Diamond_j \alpha)$, by the definition of h.

3. The case $(\neg_j\Box_j\Rightarrow)$. The last inference of $\mathscr{P}$ has the following form:

$$\dfrac{\neg_j \alpha \Rightarrow \mid H}{\neg_j \Box_j \alpha \Rightarrow \mid H}\,(\neg_j\Box_j\Rightarrow)$$

Using the induction hypothesis, we have

$$\dfrac{\vdots \atop \Rightarrow \neg_j h(\alpha) \mid h(H)}{\Rightarrow \neg_j \Diamond_j h(\alpha) \mid h(H)}\,(\Rightarrow\neg_j\Diamond_j)$$

where $\neg_j h(\alpha) = h(\neg_j \alpha)$ and $\neg_j \Diamond_j h(\alpha) = h(\neg_j \Box_j \alpha)$, by the definition of h.

4. The case $(\neg_j\Diamond_j\Rightarrow)$. The last inference of $\mathscr{P}$ has the following form:

$$\dfrac{\neg_j \alpha, \Gamma \Rightarrow \Delta \mid H}{\neg_j \Diamond_j \alpha \Rightarrow \mid \Gamma \Rightarrow \Delta \mid H}\,(\neg_j\Diamond_j\Rightarrow)$$

Using the induction hypothesis, we have

$$\dfrac{\vdots \atop h(\Delta) \Rightarrow h(\Gamma), \neg_j h(\alpha) \mid h(H)}{h(\Delta) \Rightarrow h(\Gamma) \mid \Rightarrow \neg_j \Box_j h(\alpha) \mid h(H)}\,(\Rightarrow\neg_j\Box_j)$$

where $\neg_j h(\alpha) = h(\neg_j \alpha)$ and $\neg_j \Box_j h(\alpha) = h(\neg_j \Diamond_j \alpha)$, by the definition of h.

5. The case $(\neg_k\Box_j{\Rightarrow})$. The last inference of $\mathscr{P}$ has the following form:

$$\dfrac{\neg_k\alpha, \varGamma \Rightarrow \varDelta \mid H}{\neg_k\Box_j\alpha \Rightarrow \mid \varGamma \Rightarrow \varDelta \mid H}\ (\neg_k\Box_j{\Rightarrow})$$

Using the induction hypothesis, we have

$$\dfrac{\begin{array}{c}\vdots\\ h(\varDelta) \Rightarrow h(\varGamma), \neg_k h(\alpha) \mid h(H)\end{array}}{h(\varDelta) \Rightarrow h(\varGamma) \mid \Rightarrow \neg_k\Diamond_j h(\alpha) \mid h(H)}\ ({\Rightarrow}\neg_k\Diamond_j)$$

where $\neg_k h(\alpha) = h(\neg_k\alpha)$ and $\neg_k\Diamond_j h(\alpha) = h(\neg_k\Box_j\alpha)$, by the definition of h.

6. The case $(\neg_k\Diamond_j{\Rightarrow})$. The last inference of $\mathscr{P}$ has the following form:

$$\dfrac{\neg_k\alpha \Rightarrow \mid H}{\neg_k\Diamond_j\alpha \Rightarrow \mid H}\ (\neg_k\Diamond_j{\Rightarrow})$$

Using the induction hypothesis, we have

$$\dfrac{\begin{array}{c}\vdots\\ \Rightarrow \neg_k h(\alpha) \mid h(H)\end{array}}{\Rightarrow \neg_k\Box_j h(\alpha) \mid h(H)}\ ({\Rightarrow}\neg_k\Box_j)$$

where $\neg_k h(\alpha) = h(\neg_k\alpha)$ and $\neg_k\Box_j h(\alpha) = h(\neg_k\Diamond_j\alpha)$, by the definition of h.

Suppose $\mathbf{MML}_n^{\mathbf{S5}} \setminus (\mathrm{Cut}) \vdash h(\varSigma) \Rightarrow h(\varXi) \mid h(I)$. We use induction on the proof $\mathscr{Q}$ of $h(\varSigma) \Rightarrow h(\varXi) \mid h(I)$ in $\mathbf{MML}_n^{\mathbf{S5}} \setminus (\mathrm{Cut})$. We distinguish the cases according to $\mathscr{Q}$'s last inference. The propositional cases are proved in [28]. We do the cases regarding the left modal rules only.

1. The case $(\Box_j{\Rightarrow})$. The last inference of $\mathscr{Q}$ has the following form:

$$\dfrac{h(\alpha), h(\varGamma) \Rightarrow h(\varDelta) \mid h(H)}{h(\Diamond_j\alpha) \Rightarrow \mid h(\varGamma) \Rightarrow h(\varDelta) \mid h(H)}\ (\Box_j{\Rightarrow})$$

where $h(\Diamond_j\alpha) = \Box_j h(\alpha)$, by the definition of h.

Using the induction hypothesis, we obtain the following proof in $\mathbf{MML}_n^{\mathbf{S5}} \setminus (\mathrm{Cut})$:

$$\dfrac{\begin{array}{c}\vdots\\ \varDelta \Rightarrow \varGamma, \alpha \mid H\end{array}}{\varDelta \Rightarrow \varGamma \mid\ \Rightarrow \Diamond_j\alpha \mid H}\ ({\Rightarrow}\Diamond_j)$$

2. The case $(\Diamond_j{\Rightarrow})$. The last inference of $\mathscr{Q}$ has the following form:

$$\frac{h(\alpha) \Rightarrow \mid h(H)}{h(\Box_j \alpha) \Rightarrow \mid h(H)}\ (\Diamond_j{\Rightarrow})$$

where $h(\Box_j\alpha) = \Diamond_j h(\alpha)$, by the definition of h.

Using the induction hypothesis, we have

$$\frac{\begin{array}{c}\vdots\\ \Rightarrow \alpha \mid H\end{array}}{\Rightarrow \Box_j \alpha \mid H}\ ({\Rightarrow}\Box_j)$$

3. The case $(\neg_j\Box_j{\Rightarrow})$. The last inference of $\mathscr{Q}$ has the following form:

$$\frac{h(\neg_j \alpha) \Rightarrow \mid h(H)}{h(\neg_j \Diamond_j \alpha) \Rightarrow \mid h(H)}\ (\neg_j\Box_j{\Rightarrow})$$

where $h(\neg_j\Diamond_j\alpha) = \neg_j\Box_j h(\alpha)$, by the definition of h.

Using the induction hypothesis, we have

$$\frac{\begin{array}{c}\vdots\\ \Rightarrow \neg_j \alpha \mid H\end{array}}{\Rightarrow \neg_j \Diamond_j \alpha \mid H}\ ({\Rightarrow}\neg_j\Diamond_j)$$

4. The case $(\neg_j\Diamond_j{\Rightarrow})$. The last inference of $\mathscr{Q}$ has the following form:

$$\frac{h(\neg_j \alpha), h(\Gamma) \Rightarrow h(\Delta) \mid h(H)}{h(\neg_j \Box_j \alpha) \Rightarrow \mid h(\Gamma) \Rightarrow h(\Delta) \mid h(H)}\ (\neg_j\Diamond_j{\Rightarrow})$$

where $h(\neg_j\Box_j\alpha) = \neg_j\Diamond_j h(\alpha)$, by the definition of h.

Using the induction hypothesis, we have

$$\frac{\begin{array}{c}\vdots\\ \Delta \Rightarrow \Gamma, \neg_j \alpha \mid H\end{array}}{\Delta \Rightarrow \Gamma \mid \Rightarrow \neg_j \Box_j \alpha \mid H}\ ({\Rightarrow}\neg_j\Box_j)$$

5. The case $(\neg_k\Box_j{\Rightarrow})$. The last inference of $\mathscr{Q}$ has the following form:

$$\frac{h(\neg_k \alpha), h(\Gamma) \Rightarrow h(\Delta) \mid h(H)}{h(\neg_k \Diamond_j \alpha) \Rightarrow \mid h(\Gamma) \Rightarrow h(\Delta) \mid h(H)}\ (\neg_k\Box_j{\Rightarrow})$$

where $h(\neg_k\Diamond_j\alpha) = \neg_k\Box_j h(\alpha)$, by the definition of h.

Using the induction hypothesis, we have

$$\frac{\begin{array}{c}\vdots\\ \Delta \Rightarrow \Gamma, \neg_k \alpha \mid H\end{array}}{\Delta \Rightarrow \Gamma \mid \Rightarrow \neg_k \Diamond_j \alpha \mid H}\ ({\Rightarrow}\neg_k\Diamond_j)$$

6. The case $(\neg_k\Diamond_j\Rightarrow)$. The last inference of $\mathscr{Q}$ has the following form:

$$\frac{h(\neg_k\alpha) \Rightarrow \mid h(H)}{h(\neg_k\Box_j\alpha) \Rightarrow \mid h(H)}\,(\neg_k\Diamond_j\Rightarrow)$$

where $h(\neg_k\Box_j\alpha) = \neg_k\Diamond_j h(\alpha)$, by the definition of h.

Using the induction hypothesis, we have

$$\begin{array}{c}\vdots\\ \dfrac{\Rightarrow \neg_k\alpha \mid H}{\Rightarrow \neg_k\Box_j\alpha \mid H}\,(\Rightarrow\neg_k\Box_j)\end{array}$$

$\dashv$

5.2. Craig interpolation

Now we prove the theorem which stipulates the presence of the Craig interpolation property for the logic $\mathbf{MML}_n^{\mathbf{S5}}$, an analogue of Theorem 4.5 from [28] (but here we prove it quite differently). To show this fact we use the well-known result that the modal system **S5** shares the interpolation property, see, e.g., [9, Theorem 14.23]. We need some technical lemmas preceding the main theorem. Let us denote as $V(\alpha)$ the set of all propositional variables of a formula α.

LEMMA 5.8. *Let $n > 1$, $j \leqslant n$, $I_\pi = \{\pi\} \cup \{\pi^j \mid 1 \leqslant j \leqslant n\}$ for $\pi \in \Pi$ and let f be the mapping defined in Definition 5.1. Then for any propositional variable π and any formulas $\gamma \in \mathscr{F}$, $\alpha \in \mathscr{F}_{\mathrm{M}}$:*

(1) *$\pi \in V(g(\gamma))$ iff $\pi' \in V(\gamma)$ for some $\pi' \in I_\pi$,*
(2) *$\pi \in V(g(f(\alpha)))$ iff $\pi' \in V(f(\alpha))$ for some $\pi' \in I_\pi$.*

PROOF. By induction on the complexity of a formula γ.

Ad (1) 1. The case when $\gamma \in \Pi$ is evident, since $\gamma = g(\gamma)$. When $\gamma = \pi^l$ (for some l, $1 \leqslant l \leqslant n$, $\pi \in \Pi$), $g(\gamma) = \neg_j\pi$ for some fixed j. Thus we have $\pi \in V(g(\gamma))$ iff $\pi' \in V(\gamma)$, where $\pi' = \pi^l$.

2. Suppose γ is of the form $\alpha \wedge \beta$, so $g(\gamma) = g(\alpha) \wedge_j g(\beta)$. Thus $\pi \in V(g(\gamma))$ iff $\pi \in V(g(\alpha))$ or $\pi \in V(g(\beta))$ iff $\pi' \in V(\alpha)$ or $\pi' \in V(\beta)$ (by the induction hypotheses) for some $\pi' \in I_\pi$ iff $\pi' \in V(\alpha \wedge \beta)$.

3. Let $\gamma = \Box\alpha$. Then $g(\gamma) = \Box_j(g(\alpha))$. We have $\pi \in V(g(\Box\alpha))$ iff $\pi \in V(g(\alpha))$ iff $\pi' \in V(\alpha)$ for some $\pi' \in I_\pi$ (by the induction hypothesis) iff $\pi' \in V(\Box\alpha)$.

Ad (2) 1. Again, the case $\alpha \in \Pi$ is straightforward. Assume $\alpha = \neg_j\pi$ for some $\pi \in \Pi$ and $1 \leqslant j \leqslant n$. Then $f(\alpha) = \pi^j$, $g(f(\alpha)) = \alpha$, $V(f(\alpha)) = \{\pi^j\}$. Clearly (2) holds.

2. Suppose $\gamma = \neg_k \neg_j \alpha$, where $k \neq j$ and $k, j \leqslant n$. Then $f(\alpha) = \neg f(\alpha)$, $g(f(\alpha)) = \neg_{k'} \neg_{j'} g(f(\alpha))$ for some fixed $k', j' \leqslant n$. Then $\pi \in V(g(f(\neg_k \neg_j \alpha)))$ iff $\pi \in V(g(\neg f(\alpha)))$ (by the definition of f) iff $\pi \in V(\neg_{k'} \neg_{j'} g(f(\alpha)))$ (for some fixed $k', j' \leqslant n$, by the definition of g) iff $\pi \in V(g(f(\alpha)))$ iff $\pi' \in V(f(\alpha))$ for some $\pi' \in I_\pi$ (by the induction hypothesis) iff $\pi' \in V(\neg f(\alpha))$ iff $\pi' \in V(f(\neg_k \neg_j \alpha))$ (by the definition of f).

3. Suppose $\gamma = \neg_k(\alpha \to_j \beta)$, where k and j are distinct integers $\leqslant n$. Then $\pi \in V(g(f(\neg_k(\alpha \to_j \beta))))$ iff $\pi \in V(g(f(\neg_k \alpha) \to f(\neg_k(\beta)))$ (by the definition of f) iff $\pi \in V(g(f(\neg_k \alpha)) \to_{j'} g(f(\neg_k \beta)))$ for some fixed j' (by the definition of g) iff $\pi \in V(g(f(\neg_k \alpha))$ or $\pi \in Vg(f(\neg_k \beta))$ iff $\pi' \in V(f(\neg_k \alpha))$ or $\pi' \in V(f(\neg_k \beta))$ for some $\pi' \in I_\pi$ (by the induction hypothesis) iff $\pi \in V(f(\neg_k \alpha) \to f(\neg_k \beta))$ iff $\pi \in V(f(\neg_k(\alpha \to_j \beta)))$ (by the definition of f).

4. Assume that $\gamma = \Box_j \alpha$, where j is a positive number $\leqslant n$. Then $\pi \in V(g(f(\Box_j \alpha)))$ iff $\pi \in V(g(\Box f(\alpha)))$ (by the definition of f) iff $\pi \in V(\Box_{j'} g(f(\alpha)))$ for some fixed j' (by the definition of g) iff $\pi \in V(g(f(\alpha)))$ iff $\pi' \in V(f(\alpha))$ for some $\pi' \in I_\pi$ (by the induction hypothesis) iff $\pi' \in V(\Box f(\alpha))$ iff $\pi' \in V(f(\Box_j \alpha))$ (by the definition of f).

5. Assume that $\gamma = \neg_j \Box_j \alpha$. We have $\pi \in V(g(f(\neg_j \Box_j \alpha)))$ iff $\pi \in V(g(\Diamond f(\neg_j \alpha)))$ (by the definition of f) iff $\pi \in V(\Diamond_{j'} g(f(\neg_j \alpha)))$ (by the definition of g) iff $\pi \in V(g(f(\neg_j \alpha)))$ iff $\pi' \in V(f(\neg_j \alpha))$ for some $\pi' \in I_\pi$ (by the induction hypothesis) iff $\pi' \in V(\Diamond f(\neg_j \alpha))$ iff $\pi' \in V(f(\neg_j \Box_j \alpha))$ (by the definition of f).

6. Suppose $\gamma = \neg_k \Box_j \alpha$. Applying the definitions of the mappings f, g and induction hypothesis we get the following sequence of equivalences: $\pi \in V(g(f(\neg_k \Box_j \alpha)))$ iff $\pi \in V(g(\Box f(\neg_k \alpha)))$ iff $\pi \in V(\Box_{j'} g(f(\neg_k \alpha)))$ for some fixed j' iff $\pi \in V(g(f(\neg_k \alpha)))$ iff $\pi \in V(f(\neg_k \alpha))$ iff $\pi \in V(\Box f(\neg_k \alpha))$ iff $\pi \in V(f(\neg_k \Box_j \alpha))$. ⊣

By an easy induction on the length of formulas we obtain:

LEMMA 5.9. *Let f and g be the mappings defined in definitions 5.1 and 5.2, respectively. For any formula $\gamma \in \mathscr{F}_M$, $V(g(f(\gamma)) \subseteq V(\gamma)$.*

LEMMA 5.10. *Let $\gamma \in \mathscr{F}$, $\alpha \in \mathscr{F}_{\mathrm{M}}$ and let f and g be the mappings defined in definitions 5.1 and 5.2, respectively. Then:*

(1) *$V(\gamma) \subseteq V(f(\alpha))$ implies $V(g(\gamma)) \subseteq V(g(f(\alpha)))$,*
(2) *$V(g(\gamma)) \subseteq V(g(f(\alpha)))$ implies $V(g(\gamma)) \subseteq V(\alpha)$.*

PROOF. Let us prove (1). Suppose that its antecedent holds and assume that for some propositional variable π and some formula $\gamma \in \mathscr{F}$,

$\pi \in V(g(\gamma))$. Applying part (1) from Lemma 5.8 we infer that $\pi' \in V(\gamma)$ for some $\pi' \in I_\pi$. Then $\pi' \in V(f(\alpha))$ and, using part (2) from Lemma 5.8, we conclude that $\pi \in V(g(f(\alpha)))$.

To prove (2) suppose $\pi \in V(g(\gamma))$ for some $\pi \in \Pi$ and $\gamma \in \mathscr{F}$. Then $\pi \in V(g(f(\alpha)))$ by assumption and $\pi \in V(\alpha)$ by Lemma 5.9. $\dashv$

LEMMA 5.11. *Let f and g be the mappings defined in definitions 5.1 and 5.2, respectively. Then for any finite sets Ξ and Σ:*

(1) $\mathbf{MML}_n^{\mathbf{S5}} \vdash g(f(\Xi)) \Rightarrow \Sigma \mid I$ *implies* $\mathbf{MML}_n^{\mathbf{S5}} \vdash \Xi \Rightarrow \Sigma \mid I$,

(2) $\mathbf{MML}_n^{\mathbf{S5}} \vdash \Xi \Rightarrow g(f(\Sigma)) \mid I$ *implies* $\mathbf{MML}_n^{\mathbf{S5}} \vdash \Xi \Rightarrow \Sigma \mid I$.

PROOF. Let us show (1) by the induction on the proof $\mathscr{P}$ of $g(f(\Xi)) \Rightarrow \Sigma \mid I$ in $\mathbf{MML}_n^{\mathbf{S5}}$. Only left side introduction rules really matter in this case. We explore some of the cases.

1. The case $(\Box_j\Rightarrow)$. The last inference of $\mathscr{P}$ has the following form:

$$\dfrac{\begin{matrix}\vdots\\ g(f(\alpha)), g(f(\Gamma)) \Rightarrow \Delta \mid H\end{matrix}}{\Box_j g(f(\alpha)) \Rightarrow \mid g(f(\Gamma)) \Rightarrow \Delta \mid H}\ (\Box_j\Rightarrow)$$

where $\Box_j g(f(\alpha)) = g(\Box f(\alpha)) = g(f(\Box_j\alpha))$, by the definitions of g and f.

Using the induction hypothesis, we have

$$\dfrac{\begin{matrix}\vdots\\ \alpha, \Gamma \Rightarrow \Delta \mid H\end{matrix}}{\Box_j\alpha \Rightarrow \mid \Gamma \Rightarrow \Delta \mid H}\ (\Box_j\Rightarrow)$$

2. The case $(\Diamond_j\Rightarrow)$. As in the previous case, just replacing $\Box$ with $\Diamond$ and using the equations $g(f(\alpha)) = g(\Diamond f(\alpha)) = g(f(\Diamond_j\alpha))$.

Proof for (2) uses similar techniques and is left to the reader. $\dashv$

THEOREM 5.12 (Craig Interpolation). *Let $n > 1$, $j, n \leqslant n$ and $j \neq k$. For any formulas α and β, if $\mathbf{MML}_n^{\mathbf{S5}} \vdash \alpha \Rightarrow \beta$ and $V(\alpha) \cap V(\beta) \neq \emptyset$, there exists a formula γ such that*

(1) $\mathbf{MML}_n^{\mathbf{S5}} \vdash \alpha \Rightarrow \gamma$ *and* $\mathbf{MML}_n^{\mathbf{S5}} \vdash \gamma \Rightarrow \beta$,

(2) $V(\gamma) \subseteq V(\alpha) \cap V(\beta)$.

PROOF. Suppose that for some formulas α and β, $\mathbf{MML}_n^{\mathbf{S5}} \vdash \alpha \Rightarrow \beta$ and $V(\alpha) \cap V(\beta) \neq \emptyset$. Then, according to Theorem 5.1, $\mathbf{S5} \vdash f(\alpha) \Rightarrow f(\beta)$. Since $\mathbf{S5}$ is known to have the interpolation property, we stipulate that if $V(f(\alpha)) \cap V(f(\beta)) \neq \emptyset$, then there exists a formula δ such that $\mathbf{S5} \vdash f(\alpha) \Rightarrow \delta$ and $\mathbf{S5} \vdash \delta \Rightarrow f(\beta)$, $V(\delta) \subseteq V(f(\alpha)) \cap V(f(\beta))$. Suppose that

$V(f(\alpha)) \cap V(f(\beta)) \neq \emptyset$ and δ is an interpolant. Then, by Lemma 5.6, we have $\mathbf{MML_n^{S5}} \vdash g(f(\alpha)) \Rightarrow g(\delta)$ and $\mathbf{MML_n^{S5}} \vdash g(\delta) \Rightarrow g(f(\beta))$, which in turn implies $\mathbf{MML_n^{S5}} \vdash \alpha \Rightarrow g(\delta)$ and $\mathbf{MML_n^{S5}} \vdash g(\delta) \Rightarrow \beta$ according to Lemma 5.11. To see that $V(g(\delta)) \subseteq V(\alpha) \cap V(\beta)$ we need to apply Lemma 5.9 twice. Indeed, by (1) $V(\delta) \subseteq V(f(\alpha)) \cap V(f(\beta))$ implies $V(g(\delta)) \subseteq V(g(f(\alpha))) \cap V(g(f(\beta)))$ which, by (2), entails $V(g(\delta)) \subseteq V(\alpha) \cap V(\beta)$. Thus $g(\delta)$ is a required interpolant. $\dashv$

5.3. Semantical embeddings

The next Lemma is an analogue of Lemma 5.6 from [28], but unlike the approach adopted in the cited paper we use Kripke frames with multiple relations rather then with a single one.

Lemma 5.13. *Let f be the mapping introduced in Definition 5.1. For any* **S5***-paraconsistent model $\mathcal{M} = (M, R_1, \dots, R_n, \models^p)$ we can construct an* **S5***-Kripke model $\mathcal{M}' = (M', R', \models)$ such that for any formula α and any $x \in M$ for all m ($m \leqslant n$, $1 < n$) such that $(x, m) \in \mathcal{M}'$,*

$$x \models^p \alpha \textit{ iff } (x, m) \models f(\alpha).$$

Proof. We construct a model $\mathcal{M}'$ in two stages. First we decompose the initial frame into n new frames and then join them again in one frame using disjoint union construction. So for any m ($1 \leqslant m \leqslant n$) let $M_m = M \times \{m\}$. For convenience we will write x_m instead of (x, m) for an element (x, m) of M_m. Let us define for any $x_m, y_m \in M_m$, $\hat{R}_m(x_m, y_m)$ iff $R_m(x, y)$ (where x and y are projections of x_m and y_m on their first coordinates). So, $(M_m, \hat{R}_m)$ is a Kripke frame for any m ($1 \leqslant m \leqslant n$). Also note that for any l, m ($l, m \leqslant n$, $1 < n$) such that $l \neq m$, M_l and M_m are disjoint. Let us denote by $\models^p_m$ a paraconsistent valuation on the frame $(M_m, \hat{R}_m)$. We define for each m ($1 \leqslant m \leqslant n$): $x_m \models^p_m \pi$ iff $x \models^p \pi$.

Now we construct a frame (M', R') such that M' is a union of all M_m, R' is a union of all R'_m, $1 \leqslant m \leqslant n$. It remains to define a valuation $\models$ on a new frame. Let $\models$ be a mapping from the set $\Pi \cup \bigcup_{1 \leqslant j \leqslant n} \Pi^j$ to 2^M such that for any j, m ($j, m \leqslant n$, $1 < n$), $x \in M$ and any $\pi \in \Pi$:

1. $x \models^p \pi$ iff $x_m \models^p_m \pi$ iff $x_m \models \pi$,
2. $x \models^p \neg_j \pi$ iff $x_m \models^p_m \neg_j \pi$ iff $x_m \models \pi^j$.

Now we prove the assertion of lemma by the induction on the construction of a formula α. Let us consider some typical modal cases. We do not consider propositional ones, since they have been already done in [28]. Throughout the following cases we suppose for indices j, k, m that $j, k, m \leqslant n$, $1 < n$, $j \neq k$.

1. The case $\alpha := \Box_j\beta$. Let $x \models^p \Box_j\beta$. Then by Definition 4.4 we have $\forall y(R_j(x, y) \Rightarrow y \models^p \beta)$ iff $\forall y_j(\hat{R}_j(x_j, y_j) \Rightarrow y_j \models^p_j \beta)$ (by the definitions of $\hat{R}_j$ and $\models^p_j$) iff $\forall y_j(R'(x_j, y_j) \Rightarrow y_j \models \beta)$ (by the definitions of R') iff $\forall y_j(R'(x_j, y_j) \Rightarrow y_j \models f(\beta))$ (by IH) iff $x_j \models \Box f(\beta)$ iff $x_j \models f(\Box_j\beta)$ (by Definition 5.1).

2. The case $\alpha := \neg_j\Box_j\beta$. Assume $x \models^p \neg_j\Box_j\beta$. We have $\exists y(R_j(x, y)$ and $y \models^p \neg_j\beta)$ (by Definition 4.4) iff $\exists y_j(\hat{R}_j(x_j, y_j)$ and $y_j \models^p_j \neg_j\beta)$ (by the definitions of $\hat{R}_j$ and $\models^p_j$) iff $\exists y_j(R'(x_j, y_j)$ and $y_j \models \neg_j\beta)$ (by the definitions of R') iff $\exists y_j(R'(x_j, y_j)$ and $y_j \models f(\neg_j\beta))$ (by IH) iff $x_j \models \Diamond f(\neg_j\beta)$ iff $x_j \models f(\neg_j\Box_j\beta)$ (by Definition 5.1).

3. The case $\alpha := \neg_k\Box_j\beta$. $x \models^p \neg_k\Box_j\beta$ iff $\forall y(R_j(x, y) \Rightarrow y \models^p \neg_k\beta)$ (by Definition 4.4) iff $\forall y_j(\hat{R}_j(x_j, y_j) \Rightarrow y_j \models^p_j \neg_k\beta)$ (by the definitions of $\hat{R}_j$ and $\models^p_j$) iff $\forall y_j(R'(x_j, y_j) \Rightarrow y_j \models \neg_k\beta)$ (by the definitions of R') iff $\forall y_j(R'(x_j, y_j) \Rightarrow y_j \models f(\neg_k\beta))$ (by IH) iff $x_j \models \Box f(\neg_k\beta)$ iff $x_j \models f(\neg_k\Box_j\beta)$ (by Definition 5.1). ⊣

Lemma 5.14. *Let f be the mapping introduced in Definition 5.1. For any* **S5***-Kripke model $\mathcal{M} = (M, R, \models)$ we can construct an* **S5***-paraconsistent Kripke model $\mathcal{M}' = (M, R_1, \dots, R_n, \models^p)$ such that for any $\alpha \in \mathscr{F}$ and any $x \in M$,*

$$x \models^p \alpha \textit{ iff } x \models f(\alpha).$$

Proof. Given an **S5**-Kripke model $(M, R, \models)$ we convert it into the corresponding **S5**-paraconsistent Kripke model by taking a tuple $(R_1, ..., R_n)$ consisting of n copies of R and then defining a paraconsistent valuation on the resulting frame $(M, R_1, \dots, R_n)$ such that for each $\pi \in \Pi$ the following holds:

1. $x \models \pi$ iff $x \models^p \pi$,
2. $x \models \pi^j$ iff $x \models^p \neg_j\pi$.

Next we proceed by the induction on the construction of an $\mathscr{L}$-formula. The proof is essentially the same as in the previous lemma. ⊣

From Lemmas 5.14 and 5.13 we obtain:

Theorem 5.15. *Let f be the mapping defined in Definition 5.1. For any $\alpha \in \mathscr{F}_{\mathrm{M}}$, α is $\mathbf{MML}^{\mathbf{S5}}_n$-valid iff $f(\alpha)$ is* **S5***-valid.*

THEOREM 5.16. *For any* $\alpha \in \mathscr{F}_{\mathrm{M}}$, $\mathbf{MML}_n^{\mathbf{S5}} \vdash \Rightarrow \alpha$ *iff* α *is* $\mathbf{MML}_n^{\mathbf{S5}}$*-valid.*

PROOF. The assertion of the theorem is justified by the following sequence of equivalences: $\mathbf{MML}_n^{\mathbf{S5}} \vdash \alpha$ iff $\mathbf{S5} \vdash \Rightarrow f(\alpha)$ (by Theorem 5.3) iff $f(\alpha)$ is **S5**-valid (by the completeness theorem for **S5**) iff α is $\mathbf{MML}_n^{\mathbf{S5}}$-valid (by Theorem 5.15). ⊣

Let us turn to the semantic embedding of **S5** to $\mathbf{MML}_n^{\mathbf{S5}}$.

LEMMA 5.17. *Let* g *be the mapping specified in Definition 5.2. For any* **S5***-Kripke model* $\mathcal{M} = (M, R, \models)$ *we can construct an* **S5***-paraconsistent Kripke model* $\mathcal{M}' = (M, R_1, \ldots, R_n, \models^p)$ *such that for any* $\alpha \in \mathscr{F}$ *and any* $x \in M$,

$$x \models \alpha \textit{ iff } x \models^p g(\alpha).$$

PROOF. Let $\Pi \cup {}^{\neg}\Pi$ be the set of propositional variables joined with the set of negated propositional variables, $\Pi^j = \{p^j : p \in \Pi, 1 \leqslant j \leqslant n\}$. Suppose that $\mathcal{M} = (M, R, \models)$ is a Kripke model, $\models$is a mapping from $\Pi \cup \bigcup_{1 \leqslant j \leqslant n} \Pi^j$ to 2^M. Suppose $\mathcal{M}' = (M, R_1, \ldots, R_n, \models^p)$ is an **S5**-paraconsistent Kripke model where $R_i = R$ for all i $(1 \leqslant i \leqslant n)$, $\models^p$ is a paraconsistent valuation such that for all $x \in M$, any $\pi \in \Pi$,

1. $x \models \pi$ iff $x \models^p \pi$,
2. $x \models \pi^j$ iff $x \models^p \neg_j \pi$.

We proceed by the induction on the construction of a formula α. Let us explore some of the cases.

1. $\alpha := \pi$, where $\pi \in \Pi$. We have $x \models \pi$ iff $x \models^p \pi$ (by the assumption) iff $x \models g(\pi)$ (by Definition 5.2).

2. $\alpha := \pi^j$, where $\pi \in \Pi$. We have $x \models \pi^j$ iff $x \models^p \neg_j \pi$ (by the assumption) iff $x \models^p g(\pi^j)$ (by Definition 5.2).

3. The case $\alpha := \Box\beta$. Assume that $x \models \Box\beta$. Then $\forall y(R(x,y) \Rightarrow y \models \beta)$ (by the definition of **S5**-Kripke model) iff $\forall y(R_j(x,y) \Rightarrow y \models^p g(\beta))$ (by IH and the definition of R_j) iff $x \models^p \Box_j(g(\beta))$ (by the definition of **S5**-paraconsistent Kripke model) iff $x \models^p g(\Box\beta)$ (by Definition 5.2). ⊣

LEMMA 5.18. *Let* g *be the mapping specified in Definition 5.2. For any* **S5***-paraconsistent Kripke model* $\mathcal{M} = (M, R_1, \ldots, R_n, \models^p)$ *we can construct an* **S5***-Kripke model* $\mathcal{M}' = (M', R', \models)$ *such that for any* $\alpha \in \mathscr{F}$*, any* $x \in M$ *and for any* m $(m \leqslant n,\ 1 < n)$,

$$x \models^p g(\alpha) \textit{ iff } (x, m) \models \alpha.$$

PROOF. To construct $\mathcal{M}'$ from $\mathcal{M}$ we exploit the same technique as in Lemma 5.14. Next we proceed by the induction to the construction of α. Further details of the proof are left to the reader. $\dashv$

By Lemmas 5.17 and 5.18 we obtain:

THEOREM 5.19. *Let g be the mapping specified in Definition 5.2. For any formula α, α is* **S5***-valid iff $g(\alpha)$ is* $\mathbf{MML}_n^{\mathbf{S5}}$*-valid.*

6. An alternative formulation of $\mathbf{MML}_n$

First of all, in order to avoid confusion let us write $\mathbf{MML}_n$ for Kamide and Shramko's original formulation [28] of a multilattice version of **S4** and $\mathbf{MML}_n^{\mathbf{S4}}$ for its modified version which has $\Box_j\alpha \Leftrightarrow \neg_j\neg_k\Diamond_j\neg_j\neg_k\alpha$ and $\Diamond_j\alpha \Leftrightarrow \neg_j\neg_k\Box_j\neg_j\neg_k\alpha$ as provable sequents. Let us present a sequent calculus for $\mathbf{MML}_n^{\mathbf{S4}}$. It is an extension of a sequent calculus for $\mathbf{ML}_n$ (i.e. a hypersequent calculus for $\mathbf{ML}_n$ with $H = G = \emptyset$) by the following sequent rules. The non-negated modal logical rules are as follows (we add the sign '$*$' to indicate those which do not coincide with Kamide and Shramko's original ones):

$$(\Box_j\Rightarrow)\ \frac{\alpha, \Gamma \Rightarrow \Delta}{\Box_j\alpha, \Gamma \Rightarrow \Delta} \qquad\qquad (\Rightarrow\Diamond_j)\ \frac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma \Rightarrow \Delta, \Diamond_j\alpha}$$

$$(\Rightarrow\Box_j)^*\ \frac{\Box_j\Gamma, \neg_j\Diamond_j\Delta, \neg_k\Box_j\Theta \Rightarrow \Diamond_j\Lambda, \neg_j\Box_j\Xi, \neg_k\Diamond_j\Sigma, \alpha}{\Box_j\Gamma, \neg_j\Diamond_j\Delta, \neg_k\Box_j\Theta \Rightarrow \Diamond_j\Lambda, \neg_j\Box_j\Xi, \neg_k\Diamond_j\Sigma, \Box_j\alpha}$$

$$(\Diamond_j\Rightarrow)^*\ \frac{\alpha, \Box_j\Gamma, \neg_j\Diamond_j\Delta, \neg_k\Box_j\Theta \Rightarrow \Diamond_j\Lambda, \neg_j\Box_j\Xi, \neg_k\Diamond_j\Sigma}{\Diamond_j\alpha, \Box_j\Gamma, \neg_j\Diamond_j\Delta, \neg_k\Box_j\Theta \Rightarrow \Diamond_j\Lambda, \neg_j\Box_j\Xi, \neg_k\Diamond_j\Sigma}$$

The jj-negated modal inference rules are as follows:

$$(\Rightarrow\neg_j\Box_j)\ \frac{\Gamma \Rightarrow \Delta, \neg_j\alpha}{\Gamma \Rightarrow \Delta, \neg_j\Box_j\alpha} \qquad\qquad (\neg_j\Diamond_j\Rightarrow)\ \frac{\neg_j\alpha, \Gamma \Rightarrow \Delta}{\neg_j\Diamond_j\alpha, \Gamma \Rightarrow \Delta}$$

$$(\neg_j\Box_j\Rightarrow)^*\ \frac{\neg_j\alpha, \Box_j\Gamma, \neg_j\Diamond_j\Delta, \neg_k\Box_j\Theta \Rightarrow \Diamond_j\Lambda, \neg_j\Box_j\Xi, \neg_k\Diamond_j\Sigma}{\neg_j\Box_j\alpha, \Box_j\Gamma, \neg_j\Diamond_j\Delta, \neg_k\Box_j\Theta \Rightarrow \Diamond_j\Lambda, \neg_j\Box_j\Xi, \neg_k\Diamond_j\Sigma}$$

$$(\Rightarrow\neg_j\Diamond_j)^*\ \frac{\Box_j\Gamma, \neg_j\Diamond_j\Delta, \neg_k\Box_j\Theta \Rightarrow \Diamond_j\Lambda, \neg_j\Box_j\Xi, \neg_k\Diamond_j\Sigma, \neg_j\alpha}{\Box_j\Gamma, \neg_j\Diamond_j\Delta, \neg_k\Box_j\Theta \Rightarrow \Diamond_j\Lambda, \neg_j\Box_j\Xi, \neg_k\Diamond_j\Sigma, \neg_j\Diamond_j\alpha}$$

The kj-negated modal inference rules are as follows:

$$(\neg_k\Box_j\Rightarrow)\ \frac{\neg_k\alpha, \Gamma \Rightarrow \Delta}{\neg_k\Box_j\alpha, \Gamma \Rightarrow \Delta} \qquad\qquad (\Rightarrow\neg_k\Diamond_j)\ \frac{\Gamma \Rightarrow \Delta, \neg_k\alpha}{\Gamma \Rightarrow \Delta, \neg_k\Diamond_j\alpha}$$

$$(\Rightarrow\neg_k\Box_j)^* \ \dfrac{\Box_j\mathit{\Gamma}, \neg_j\Diamond_j\mathit{\Delta}, \neg_k\Box_j\mathit{\Theta}\Rightarrow\Diamond_j\mathit{\Lambda}, \neg_j\Box_j\mathit{\Xi}, \neg_k\Diamond_j\mathit{\Sigma}, \neg_k\alpha}{\Box_j\mathit{\Gamma}, \neg_j\Diamond_j\mathit{\Delta}, \neg_k\Box_j\mathit{\Theta}\Rightarrow\Diamond_j\mathit{\Lambda}, \neg_j\Box_j\mathit{\Xi}, \neg_k\Diamond_j\mathit{\Sigma}, \Box_j\alpha}$$

$$(\neg_k\Diamond_j\Rightarrow)^* \ \dfrac{\neg_k\alpha, \Box_j\mathit{\Gamma}, \neg_j\Diamond_j\mathit{\Delta}, \neg_k\Box_j\mathit{\Theta}\Rightarrow\Diamond_j\mathit{\Lambda}, \neg_j\Box_j\mathit{\Xi}, \neg_k\Diamond_j\mathit{\Sigma}}{\neg_k\Diamond_j\alpha, \Box_j\mathit{\Gamma}, \neg_j\Diamond_j\mathit{\Delta}, \neg_k\Box_j\mathit{\Theta}\Rightarrow\Diamond_j\mathit{\Lambda}, \neg_j\Box_j\mathit{\Xi}, \neg_k\Diamond_j\mathit{\Sigma}}$$

PROPOSITION 6.1. *The following sequents are provable in* $\mathbf{MML}_n^{\mathbf{S4}}$:

(1) $\Box_j\alpha \Leftrightarrow \neg_j\neg_k\Diamond_j\neg_j\neg_k\alpha$ *and* $\Diamond_j\alpha \Leftrightarrow \neg_j\neg_k\Box_j\neg_j\neg_k\alpha$;
(2) $\neg_j\Box_j\alpha \Leftrightarrow \Diamond_j\neg_j\alpha$ *and* $\neg_j\Diamond_j\alpha \Leftrightarrow \Box_j\neg_j\alpha$;
(3) $\neg_k\Box_j\alpha \Leftrightarrow \Box_j\neg_k\alpha$ *and* $\neg_k\Diamond_j\alpha \Leftrightarrow \Diamond_j\neg_k\alpha$.

PROOF. We prove $\Box_j\alpha \Leftrightarrow \neg_j\neg_k\Diamond_j\neg_j\neg_k\alpha$ and $\neg_j\Diamond_j\alpha \Leftrightarrow \Box_j\neg_j\alpha$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\alpha\Rightarrow\alpha}{\neg_j\neg_k\alpha, \alpha\Rightarrow}\,(\neg_j\neg_k\Rightarrow)}{\neg_j\neg_k\alpha, \Box_j\alpha\Rightarrow}\,(\Box_j\Rightarrow)}{\Diamond_j\neg_j\neg_k\alpha, \Box_j\alpha\Rightarrow}\,(\Diamond_j\Rightarrow)}{\Box_j\alpha\Rightarrow\neg_j\neg_k\Diamond_j\neg_j\neg_k\alpha}\,(\Rightarrow\neg_j\neg_k)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\alpha\Rightarrow\alpha}{\Rightarrow\alpha, \neg_j\neg_k\alpha}\,(\Rightarrow\neg_j\neg_k)}{\Rightarrow\alpha, \Diamond_j\neg_j\neg_k\alpha}\,(\Diamond_j\Rightarrow)}{\Rightarrow\Box_j\alpha, \Diamond_j\neg_j\neg_k\alpha}\,(\Box_j\Rightarrow)}{\neg_j\neg_k\Diamond_j\neg_j\neg_k\alpha\Rightarrow\Box_j\alpha}\,(\neg_j\neg_k\Rightarrow)$$

$$\dfrac{\dfrac{\neg_j\alpha\Rightarrow\neg_j\alpha}{\neg_j\Diamond_j\alpha\Rightarrow\neg_j\alpha}\,(\neg_j\Diamond_j\Rightarrow)}{\neg_j\Diamond_j\alpha\Rightarrow\Box_j\neg_j\alpha}\,(\Rightarrow\Box_j) \qquad \dfrac{\dfrac{\neg_j\alpha\Rightarrow\neg_j\alpha}{\Box_j\neg_j\alpha\Rightarrow\neg_j\alpha}\,(\Box_j\Rightarrow)}{\Box_j\neg_j\alpha\Rightarrow\neg_j\Diamond_j\alpha}\,(\Rightarrow\neg_j\Diamond_j) \qquad \dashv$$

Let us present now syntactical and semantical embeddings for $\mathbf{MML}_n^{\mathbf{S4}}$ and **S4**.

THEOREM 6.2 (Weak syntactical embedding from $\mathbf{MML}_n^{\mathbf{S4}}$ into **S4**). *Let f be the mapping introduced in Definition 5.1. Then, for each pair of finite sets $\mathit{\Phi}$ and $\mathit{\Psi}$ of $\mathscr{L}_{\mathrm{M}}$-formulas, it holds that*:

(1) $\mathbf{MML}_n^{\mathbf{S4}} \vdash \mathit{\Phi} \Rightarrow \mathit{\Psi}$ *implies* $\mathbf{S4} \vdash f(\mathit{\Phi}) \Rightarrow f(\mathit{\Psi})$;
(2) $\mathbf{S4} \setminus (\mathrm{Cut}) \vdash f(\mathit{\Phi}) \Rightarrow f(\mathit{\Psi})$ *implies* $\mathbf{MML}_n^{\mathbf{S4}} \setminus (\mathrm{Cut}) \vdash \mathit{\Phi} \Rightarrow \mathit{\Psi}$.

PROOF. (1) By induction on the proof $\mathscr{P}$ of $\mathit{\Phi} \Rightarrow \mathit{\Psi}$ in the sequent calculus for $\mathbf{MML}_n^{\mathbf{S4}}$. We distinguish the cases according to $\mathscr{P}$'s last inference. The propositional cases are proved in [28]. The cases regarding the rules $(\Box_j\Rightarrow)$, $(\Rightarrow\Diamond_j)$, $(\Rightarrow\neg_j\Box_j)$, $(\neg_j\Diamond_j\Rightarrow)$, $(\neg_k\Box_j\Rightarrow)$, and $(\Rightarrow\neg_k\Diamond_j)$ are considered in [28] as well. We examine the ones only which are based on the rules which do not coincide with Kamide and Shramko's original ones. As an example, we consider the rules $(\Diamond_j\Rightarrow)^*$ and $(\Rightarrow\neg_k\Box_j)^*$.

1. The case $(\Diamond_j\Rightarrow)^*$. The last inference of $\mathscr{P}$ has the following form:

$$\dfrac{\alpha, \Box_j\Gamma, \neg_j\Diamond_j\Delta, \neg_k\Box_j\Theta \Rightarrow \Diamond_j\Lambda, \neg_j\Box_j\Xi, \neg_k\Diamond_j\Sigma}{\Diamond_j\alpha, \Box_j\Gamma, \neg_j\Diamond_j\Delta, \neg_k\Box_j\Theta \Rightarrow \Diamond_j\Lambda, \neg_j\Box_j\Xi, \neg_k\Diamond_j\Sigma}\ (\Diamond_j\Rightarrow)^*$$

Using the induction hypothesis, we have (where (DF_f) stands for the definition of f and $f(\Diamond_j\alpha) = \Diamond_j f(\alpha)$):

$$\dfrac{\dfrac{\vdots}{f(\alpha), f(\Box_j\Gamma), f(\neg_j\Diamond_j\Delta), f(\neg_k\Box_j\Theta) \Rightarrow f(\Diamond_j\Lambda), f(\neg_j\Box_j\Xi), f(\neg_k\Diamond_j\Sigma)}}{\dfrac{f(\alpha), \Box_j f(\Gamma), \Box_j\neg_j f(\Delta), \Box_j\neg_k f(\Theta) \Rightarrow \Diamond_j f(\Lambda), \Diamond_j\neg_j f(\Xi), \Diamond_j\neg_k f(\Sigma)}{\Diamond_j f(\alpha), \Box_j f(\Gamma), \Box_j\neg_j f(\Delta), \Box_j\neg_k f(\Theta) \Rightarrow \Diamond_j f(\Lambda), \Diamond_j\neg_j f(\Xi), \Diamond_j\neg_k f(\Sigma)}\ (\Diamond^4\Rightarrow)}\ (DF_f)$$

2. The case $(\Rightarrow\neg_k\Box_j)^*$. The last inference of $\mathscr{P}$ has the following form:

$$\dfrac{\Box_j\Gamma, \neg_j\Diamond_j\Delta, \neg_k\Box_j\Theta \Rightarrow \Diamond_j\Lambda, \neg_j\Box_j\Xi, \neg_k\Diamond_j\Sigma, \neg_k\alpha}{\Box_j\Gamma, \neg_j\Diamond_j\Delta, \neg_k\Box_j\Theta \Rightarrow \Diamond_j\Lambda, \neg_j\Box_j\Xi, \neg_k\Diamond_j\Sigma, \neg_k\Box_j\alpha}\ (\Rightarrow\neg_k\Box_j)^*$$

Using the induction hypothesis and the definition of f, we have:

$$\dfrac{\dfrac{\vdots}{f(\Box_j\Gamma), f(\neg_j\Diamond_j\Delta), f(\neg_k\Box_j\Theta) \Rightarrow f(\Diamond_j\Lambda), f(\neg_j\Box_j\Xi), f(\neg_k\Diamond_j\Sigma), f(\neg_k\alpha)}}{f(\Box_j\Gamma), f(\neg_j\Diamond_j\Delta), f(\neg_k\Box_j\Theta) \Rightarrow f(\Diamond_j\Lambda), f(\neg_j\Box_j\Xi), f(\neg_k\Diamond_j\Sigma), \Box f(\neg_k\alpha)}\ (\Rightarrow\Box^4)$$

(2) By induction on the proof $\mathscr{Q}$ of $f(\Phi) \Rightarrow f(\Psi)$ in **S4** $\setminus$ (Cut). We distinguish the cases according to $\mathscr{Q}$'s last inference. The propositional cases as well as the ones which concern $(\Box\Rightarrow)$ and $(\Rightarrow\Diamond)$ were proved in [28]. However, in contrast to [28], instead of the rules $(\Rightarrow\Box)$ and $(\Diamond\Rightarrow)$ we have $(\Rightarrow\Box^4)$ and $(\Diamond^4\Rightarrow)$, respectively. As an example, we consider the case $(\Diamond^4\Rightarrow)$. The last inference of $\mathscr{Q}$ is an application of the rule $(\Diamond^4\Rightarrow)$.

Subcase (2.1): The last inference of $\mathscr{Q}$ has the following form:

$$\dfrac{\dfrac{f(\alpha), \Box f(\Gamma), \Box f(\neg_j\Delta), \Box f(\neg_k\Theta) \Rightarrow \Diamond f(\Lambda), \Diamond f(\neg_j\Xi), \Diamond f(\neg_k\Sigma)}{\Diamond f(\alpha), \Box f(\Gamma), \Box f(\neg_j\Delta), \Box f(\neg_k\Theta) \Rightarrow \Diamond f(\Lambda), \Diamond f(\neg_j\Xi), \Diamond f(\neg_k\Sigma)}\ (\Diamond^4\Rightarrow)}{f(\Diamond_j\alpha), f(\Box_j\Gamma), f(\neg_j\Diamond_j\Delta), f(\neg_k\Box_j\Theta) \Rightarrow f(\Diamond_j\Lambda), f(\neg_j\Box_j\Xi), f(\neg_k\Diamond_j\Sigma)}\ (DF_f)$$

Using the induction hypothesis, we have

$$\dfrac{\dfrac{\vdots}{\alpha, \Box_j\Gamma, \neg_j\Diamond_j\Delta, \neg_k\Box_j\Theta \Rightarrow \Diamond_j\Lambda, \neg_j\Box_j\Xi, \neg_k\Diamond_j\Sigma}}{\Diamond_j\alpha, \Box_j\Gamma, \neg_j\Diamond_j\Delta, \neg_k\Box_j\Theta \Rightarrow \Diamond_j\Lambda, \neg_j\Box_j\Xi, \neg_k\Diamond_j\Sigma}\ (\Diamond_j\Rightarrow)^*$$

Subcase (2.2): The last inference of $\mathscr{Q}$ has the following form:

$$\cfrac{\cfrac{f(\neg_j\alpha), \Box f(\Gamma), \Box f(\neg_j\Delta), \Box f(\neg_k\Theta) \Rightarrow \Diamond f(\Lambda), \Diamond f(\neg_j\Xi), \Diamond f(\neg_k\Sigma)}{\Diamond f(\neg_j\alpha), \Box f(\Gamma), \Box f(\neg_j\Delta), \Box f(\neg_k\Theta) \Rightarrow \Diamond f(\Lambda), \Diamond f(\neg_j\Xi), \Diamond f(\neg_k\Sigma)}\,(\Diamond^4\Rightarrow)}{f(\neg_j\Box_j\alpha), f(\Box_j\Gamma), f(\neg_j\Diamond_j\Delta), f(\neg_k\Box_j\Theta) \Rightarrow f(\Diamond_j\Lambda), f(\neg_j\Box_j\Xi), f(\neg_k\Diamond_j\Sigma)}\,(DF_f)$$

Using the induction hypothesis, we can obtain the required fact as follows (one can obtain the same result by the rule $(\Diamond_j \Rightarrow)^*$ as well, since $\mathbf{MML}_n^{\mathbf{S4}} \vdash \Diamond_j\neg_j\alpha \Leftrightarrow \neg_j\Box_j\alpha$):

$$\cfrac{\begin{matrix}\vdots\\ \neg_j\alpha, \Box_j\Gamma, \neg_j\Diamond_j\Delta, \neg_k\Box_j\Theta \Rightarrow \Diamond_j\Lambda, \neg_j\Box_j\Xi, \neg_k\Diamond_j\Sigma\end{matrix}}{\neg_j\Box_j\alpha, \Box_j\Gamma, \neg_j\Diamond_j\Delta, \neg_k\Box_j\Theta \Rightarrow \Diamond_j\Lambda, \neg_j\Box_j\Xi, \neg_k\Diamond_j\Sigma}\,(\neg_j\Box_j\Rightarrow)^*$$

Subcase (2.3): The last inference of $\mathscr{Q}$ has the following form:

$$\cfrac{\cfrac{f(\neg_k\alpha), \Box f(\Gamma), \Box f(\neg_j\Delta), \Box f(\neg_k\Theta) \Rightarrow \Diamond f(\Lambda), \Diamond f(\neg_j\Xi), \Diamond f(\neg_k\Sigma)}{\Diamond f(\neg_k\alpha), \Box f(\Gamma), \Box f(\neg_j\Delta), \Box f(\neg_k\Theta) \Rightarrow \Diamond f(\Lambda), \Diamond f(\neg_j\Xi), \Diamond f(\neg_k\Sigma)}\,(\Diamond^4\Rightarrow)}{f(\neg_k\Diamond_j\alpha), f(\Box_j\Gamma), f(\neg_j\Diamond_j\Delta), f(\neg_k\Box_j\Theta) \Rightarrow f(\Diamond_j\Lambda), f(\neg_j\Box_j\Xi), f(\neg_k\Diamond_j\Sigma)}\,(DF_f)$$

Using the induction hypothesis, we can obtain the required fact as follows (one can obtain the same result by the rule $(\Diamond_j\Rightarrow)^*$ as well, since $\mathbf{MML}_n^{\mathbf{S4}} \vdash \Diamond_j\neg_k\alpha \Leftrightarrow \neg_k\Diamond_j\alpha$):

$$\cfrac{\begin{matrix}\vdots\\ \neg_k\alpha, \Box_j\Gamma, \neg_j\Diamond_j\Delta, \neg_k\Box_j\Theta \Rightarrow \Diamond_j\Lambda, \neg_j\Box_j\Xi, \neg_k\Diamond_j\Sigma\end{matrix}}{\neg_k\Diamond_j\alpha, \Box_j\Gamma, \neg_j\Diamond_j\Delta, \neg_k\Box_j\Theta \Rightarrow \Diamond_j\Lambda, \neg_j\Box_j\Xi, \neg_k\Diamond_j\Sigma}\,(\neg_k\Diamond_j\Rightarrow)^* \qquad \dashv$$

From Theorem 6.2 and cut elimination for **S4** we obtain the following two theorems.

THEOREM 6.3 (Cut elimination for $\mathbf{MML}_n^{\mathbf{S4}}$). *The rule* (Cut) *is admissible in the cut-free sequent calculus for* $\mathbf{MML}_n^{\mathbf{S4}}$.

THEOREM 6.4 (Syntactical embedding from $\mathbf{MML}_n^{\mathbf{S4}}$ into **S4**). *Let f be the mapping introduced in Definition 5.1. Then, for each pair of finite sets Γ and Δ of $\mathscr{L}_{\mathrm{M}}$-formulas, it holds that*:

(1) $\mathbf{MML}_n^{\mathbf{S4}} \vdash \Gamma \Rightarrow \Delta$ *iff* $\mathbf{S4} \vdash f(\Gamma) \Rightarrow f(\Delta)$;
(2) $\mathbf{MML}_n^{\mathbf{S4}} \setminus (\mathrm{Cut}) \vdash \Gamma \Rightarrow \Delta$ *iff* $\mathbf{S4} \setminus (\mathrm{Cut}) \vdash f(\Gamma) \Rightarrow f(\Delta)$.

From Theorem 6.4 and decidability of **S4** we obtain:

THEOREM 6.5 (Decidability for $\mathbf{MML}_n^{\mathbf{S4}}$). $\mathbf{MML}_n^{\mathbf{S4}}$ *is decidable.*

Theorem 6.6 (Weak syntactical embedding from **S4** into $\mathbf{MML}_n^{\mathbf{S4}}$). *Let g be the mapping introduced in Definition 5.2. Then, for each pair of finite sets Φ and Ψ of $\mathscr{L}$-formulas, it holds that*:

(1) $\mathbf{S4} \vdash \Phi \Rightarrow \Psi$ *implies* $\mathbf{MML}_n^{\mathbf{S4}} \vdash g(\Phi) \Rightarrow g(\Psi)$;

(2) $\mathbf{MML}_n^{\mathbf{S4}} \setminus (\mathrm{Cut}) \vdash g(\Phi) \Rightarrow g(\Psi)$ *implies* $\mathbf{S4} \setminus (\mathrm{Cut}) \vdash \Phi \Rightarrow \Psi$.

Proof. (1) By induction on the proof $\mathscr{P}$ of $\Phi \Rightarrow \Psi$ in the sequent calculus for **S4**. We distinguish the cases according to $\mathscr{P}$'s last inference. The propositional cases as well as the ones regarding the rules ($\Box\Rightarrow$) and ($\Rightarrow\Diamond$) are proved in [28]. As an example, we consider the case regarding the rule ($\Rightarrow\Box^4$). The last inference of $\mathscr{P}$ has the following form:

$$\frac{\Box\Gamma\Rightarrow\Diamond\Delta, \alpha}{\Box\Gamma\Rightarrow\Diamond\Delta, \Box\alpha}\ (\Rightarrow\Box^4)$$

Using the induction hypothesis, we obtain the following proof in the sequent calculus for $\mathbf{MML}_n^{\mathbf{S4}}$ (where (DF_g) stands for the definition of g):

$$\cfrac{\cfrac{\vdots}{\cfrac{g(\Box\Gamma) \Rightarrow g(\Diamond\Delta), g(\alpha)}{\Box_j g(\Gamma)\Rightarrow\Diamond_j g(\Delta), g(\alpha)}\ (DF_g)}}{\Box_j g(\Gamma)\Rightarrow\Diamond_j g(\Delta), \Box_j g(\alpha)}\ (\Rightarrow\Box_j)^*$$

where $\Box_j g(\alpha) = g(\Box\alpha)$, by (DF_g).

(2) By induction on the proof $\mathscr{Q}$ of $g(\Phi) \Rightarrow g(\Psi)$ in $\mathbf{MML}_n^{\mathbf{S4}} \setminus (\mathrm{Cut})$. We distinguish the cases according to $\mathscr{Q}$'s last inference. As an example, we consider the case $(\Rightarrow\Box_j)^*$. The last inference of $\mathscr{Q}$ has the following form:

$$\frac{g(\Box\Gamma), \Rightarrow g(\Diamond\Delta), g(\alpha)}{g(\Box\Gamma), \Rightarrow g(\Diamond\Delta), \Box_j g(\alpha)}\ (\Rightarrow\Box_j)^*$$

where $g(\Box\Gamma) = \Box_j g(\Gamma)$, $g(\Diamond\Delta) = \Diamond_j g(\Delta)$, $\Box_j g(\alpha) = g(\Box\alpha)$, by (DF_g).

Using the induction hypothesis, we have

$$\cfrac{\vdots}{\cfrac{\Box\Gamma \Rightarrow \Diamond\Delta, \alpha}{\Box\Gamma \Rightarrow \Diamond\Delta, \Box\alpha}\ (\Rightarrow\Box^4)}$$

$\dashv$

Similarly to Theorem 5.3, using Theorems 5.5 and 5.2, we obtain:

THEOREM 6.7 (Syntactical embedding from **S4** into $\mathbf{MML}_n^{\mathbf{S4}}$). *Let g be the mapping introduced in Definition 5.2. Then, for each finite sets Φ and Ψ of $\mathscr{L}$-formulas, it holds that*:

(1) $\mathbf{S4} \vdash \Phi \Rightarrow \Psi$ *iff* $\mathbf{MML}_n^{\mathbf{S4}} \vdash g(\Phi) \Rightarrow g(\Psi)$;
(2) $\mathbf{S4} \setminus (\mathrm{Cut}) \vdash \Phi \Rightarrow \Psi$ *iff* $\mathbf{MML}_n^{\mathbf{S4}} \setminus (\mathrm{Cut}) \vdash g(\Phi) \Rightarrow g(\Psi)$.

It seems that the duality principle based on the function h which is introduced in Definition 5.3 does not hold for $\mathbf{MML}_n^{\mathbf{S4}}$. Consider, for example, the the case of the rule $(\Rightarrow\Box_j)^*$:

$$\frac{\Box_j\Gamma, \neg_j\Diamond_j\Delta, \neg_k\Box_j\Theta \Rightarrow \Diamond_j\Lambda, \neg_j\Box_j\Xi, \neg_k\Diamond_j\Sigma, \alpha}{\Box_j\Gamma, \neg_j\Diamond_j\Delta, \neg_k\Box_j\Theta \Rightarrow \Diamond_j\Lambda, \neg_j\Box_j\Xi, \neg_k\Diamond_j\Sigma, \Box_j\alpha}\ (\Rightarrow\Box_j)^*$$

Let $\Phi \Rightarrow \Psi$ be an abbreviation for the conclusion of this application of $(\Rightarrow\Box_j)^*$. We need to obtain $h(\Psi) \Rightarrow h(\Phi)$, i.e. $\Diamond_j\alpha, \Diamond_j\Gamma, \neg_j\Box_j\Delta, \neg_k\Diamond_j\Theta \Rightarrow \Box_j\Lambda, \neg_j\Diamond_j\Xi, \neg_k\Box_j\Sigma$. However, it is not clear which modal rule may help in this situation, since there is no modal rule in the sequent calculus for $\mathbf{MML}_n^{\mathbf{S4}}$ which deals with such context of the rule application.

A Kripke semantics for $\mathbf{MML}_n^{\mathbf{S4}}$ is similar to the one for $\mathbf{MML}_n^{\mathbf{S5}}$. The only difference is that we postulate that the relations $R_1, \ldots, R_n$ are reflexive and transitive.

Similarly to Theorem 5.15 we obtain:

THEOREM 6.8. *Let f be the mapping introduced in Definition 5.1. For any $\alpha \in \mathscr{F}_{\mathrm{M}}$, α is $\mathbf{MML}_n^{\mathbf{S4}}$-valid iff $f(\alpha)$ is* **S4***-valid.*

Finally, from Theorems 6.4 and 6.8 as well as the completeness theorem for **S4** we obtain:

THEOREM 6.9. *For any $\alpha \in \mathscr{F}_{\mathrm{M}}$, $\mathbf{MML}_n^{\mathbf{S4}} \vdash\ \Rightarrow \alpha$ iff α is $\mathbf{MML}_n^{\mathbf{S4}}$-valid.*

7. Hilbert-style calculi for modal multilattice logics

7.1. Hilbert-style calculus for $\mathbf{ML}_n$

Let $\alpha \leftrightarrow_j \beta := (\alpha \to_j \beta) \wedge_j (\beta \to_j \alpha)$. Let us present a Hilbert-style calculus for $\mathbf{ML}_n$. It has the following schemes of axioms (in what follows we will write just axioms) and rules:

(A1) $\alpha \to_j (\beta \to_j \alpha)$
(A2) $(\alpha \to_j (\beta \to_j \gamma)) \to_j ((\alpha \to_j \beta) \to_j (\alpha \to_j \gamma))$

(A3) $((\alpha \rightarrow_j \beta) \rightarrow_j \alpha) \rightarrow_j \alpha$
(A4) $\alpha \rightarrow_j (\alpha \vee_j \beta)$
(A5) $\beta \rightarrow_j (\alpha \vee_j \beta)$
(A6) $(\alpha \rightarrow_j \gamma) \rightarrow_j ((\beta \rightarrow_j \gamma) \rightarrow_j ((\alpha \vee_j \beta) \rightarrow_j \gamma))$
(A7) $(\alpha \wedge_j \beta) \rightarrow_j \alpha$
(A8) $(\alpha \wedge_j \beta) \rightarrow_j \beta$
(A9) $\alpha \rightarrow_j (\beta \rightarrow_j (\alpha \wedge_j \beta))$
(A10) $\neg_j(\alpha \wedge_j \beta) \leftrightarrow_j (\neg_j\alpha \vee_j \neg_j\beta)$
(A11) $\neg_j(\alpha \vee_j \beta) \leftrightarrow_j (\neg_j\alpha \wedge_j \neg_j\beta)$
(A12) $\neg_k(\alpha \wedge_j \beta) \leftrightarrow_j (\neg_k\alpha \wedge_j \neg_k\beta)$
(A13) $\neg_k(\alpha \vee_j \beta) \leftrightarrow_j (\neg_k\alpha \vee_j \neg_k\beta)$
(A14) $\alpha \leftrightarrow_j \neg_j\neg_j\alpha$
(A15) $(\neg_k\neg_j\beta \rightarrow_j \neg_k\neg_j\alpha) \rightarrow_j ((\neg_k\neg_j\beta \rightarrow_j \alpha) \rightarrow \beta)$
(A16) $\neg_j(\alpha \rightarrow_j \beta) \leftrightarrow_j (\neg_k\neg_j\neg_j\alpha \wedge_j \neg_j\beta)$
(A17) $\neg_k(\alpha \rightarrow_j \beta) \leftrightarrow_j (\neg_k\neg_j\neg_k\alpha \vee_j \neg_k\beta)$
(A18) $(\alpha \leftarrow_j \beta) \leftrightarrow_j (\alpha \wedge_j \neg_k\neg_j\beta)$
(A19) $\neg_j(\alpha \leftarrow_j \beta) \leftrightarrow_j (\neg_j\alpha \vee_j \neg_k\neg_j\neg_j\beta)$
(A20) $\neg_k(\alpha \leftarrow_j \beta) \leftrightarrow_j (\neg_k\alpha \wedge_j \neg_k\neg_j\neg_k\beta)$

(MP) $\dfrac{\alpha \quad \alpha \rightarrow_j \beta}{\beta}$

Lemma 7.1. *For each $\alpha \in \mathscr{F}_{\mathrm{N}}$, it holds that if α is provable in the Hilbert-style calculus for $\mathbf{ML}_n$, then α is provable in the hypersequent calculus for $\mathbf{ML}_n$.*

Proof. By induction of the proof of α in the Hilbert-style calculus for $\mathbf{ML}_n$. We need to show that each axiom/rule of the Hilbert-style calculus for $\mathbf{ML}_n$ is provable/derivable in the hypersequent calculus for this logic. In Proposition 2.2, we proved axioms (A16)–(A20). The other cases are considered similarly. ⊣

Lemma 7.2. *For each $\alpha \in \mathscr{F}_{\mathrm{N}}$, it holds that if α is provable in the hypersequent calculus for $\mathbf{ML}_n$, then α is provable in the Hilbert-style calculus for $\mathbf{ML}_n$.*

Proof. By induction of the proof of α in the hypersequent calculus for $\mathbf{ML}_n$. We need to show that all the axioms/rules of the hypersequent calculus for $\mathbf{ML}_n$ are provable/derivable in the Hilbert-style calculus for this logic. First of all, note that the following rule is derivable the hypersequent calculus for $\mathbf{ML}_n$ by (A1), (A2), and (MP):

$$(\mathrm{Tr})\ \frac{\alpha \rightarrow_j \beta \qquad \beta \rightarrow_j \gamma}{\alpha \rightarrow_j \gamma}$$

(Ax) and (Ax$_\neg$) are obviously provable in the Hilbert-style calculus for $\mathbf{ML}_n$. The rule (Cut) (which is a generalization of the rule (Tr)) is provable in a similar way. Using (Tr) and (A8) as well as (Tr) and (A4), respectively, we obtain that (IW$\Rightarrow$) as well as ($\Rightarrow$IW) are derivable in the Hilbert-style calculus for $\mathbf{ML}_n$. Using (Tr) and (A4)–(A5), we obtain that (EW$\Rightarrow$) as well as ($\Rightarrow$EW) are derivable in the Hilbert-style calculus for $\mathbf{ML}_n$. Besides, (MP), (A1)–(A9), and (A15) formalize classical logic. Moreover, (Ax), internal structural rules as well as ($\nabla\Rightarrow$) and ($\Rightarrow\nabla$), where $\nabla \in \{\wedge, \vee, \rightarrow, \neg_k\neg_j\}$, also formalize classical logic. Thus, the rules ($\nabla\Rightarrow$) and ($\Rightarrow\nabla$) are derivable in the Hilbert-style calculus for $\mathbf{ML}_n$.

Furthermore, using (A10), (A11) and (A14), one can easily show the derivability of the rules ($\neg_j\nabla\Rightarrow$) and ($\Rightarrow\neg_j\nabla$), where $\nabla \in \{\wedge_j, \vee_j, \neg_j\}$. Similarly, the derivability of the rules ($\neg_k\nabla\Rightarrow$) and ($\Rightarrow\neg_k\nabla$), where $\nabla \in \{\wedge_j, \vee_j\}$, is shown due to (A12) and (A13). (A16)–(A20) introduce the following connectives and combinations of connectives via their definitions: $\neg_j(\alpha \rightarrow_j \beta)$, $\neg_k(\alpha \rightarrow_j \beta)$, $\alpha \leftarrow_j \beta$, $\neg_j(\alpha \leftarrow_j \beta)$, and $\neg_k(\alpha \leftarrow_j \beta)$. Since these types of formulas are expressed via all the other ones, so all the tautologies of such forms are provable in both Hilbert-style and hypersequent calculi for $\mathbf{ML}_n$. Moreover, hypersequent rules for these types of formulas are also derivible in the Hilbert-style calculus for $\mathbf{ML}_n$. $\dashv$

Immediately from Lemmas 7.1 and 7.2 we obtain:.

Theorem 7.3. *For each $\alpha \in \mathscr{F}_{\mathrm{N}}$, it holds that α is provable in the Hilbert-style calculus for $\mathbf{ML}_n$ iff α is provable in the hypersequent calculus for $\mathbf{ML}_n$.*

Corollary 7.4. *For each $\alpha \in \mathscr{F}_{\mathrm{N}}$, it holds that α is provable in the Hilbert-style calculus for $\mathbf{ML}_n$ iff α is $\mathbf{ML}_n$-valid.*

7.2. Hilbert-style calculi for $\mathbf{MML}_n^{\mathbf{S4}}$ and $\mathbf{MML}_n^{\mathbf{S5}}$. Generalized approach to modal multilattice logic

The Hilbert-style calculus for $\mathbf{MML}_n^{\mathbf{S4}}$ is an extension of the Hilbert-style calculus for $\mathbf{ML}_n$ by the following axioms and inference rules:

(Ax1) $\Box_j(\alpha \rightarrow_j \beta) \rightarrow_j (\Box_j\alpha \rightarrow_j \Box_j\beta)$
(Ax2) $\Box_j\alpha \rightarrow_j \alpha$
(Ax3) $\Box_j\alpha \rightarrow_j \Box_j\Box_j\alpha$

(Ax4) $\Diamond_j\alpha \leftrightarrow_j \neg_k\neg_j\Box_j\neg_k\neg_j\alpha$
(Ax5) $\neg_j\Diamond_j\alpha \leftrightarrow_j \Box_j\neg_j\alpha$
(Ax6) $\neg_j\Box_j\alpha \leftrightarrow_j \Diamond_j\neg_j\alpha$
(Ax7) $\neg_k\Box_j\alpha \leftrightarrow_j \Box_j\neg_k\alpha$
(Ax8) $\neg_k\Diamond_j\alpha \leftrightarrow_j \Diamond_j\neg_k\alpha$

(G) $\dfrac{\vdash \alpha}{\vdash \Box_j\alpha}$

The Hilbert-style calculus for $\mathbf{MML}_n^{\mathbf{S5}}$ is an extension of the Hilbert-style calculus for $\mathbf{MML}_n$ by the following axiom:

$$\Diamond_j\alpha \to_j \Box_j\Diamond_j\alpha.$$

Let us introduce one more logic, $\mathbf{MML}_n^{\mathbf{K}}$, which is a multilattice analogue of basic modal logic **K**. The Hilbert-style calculus for $\mathbf{MML}_n^{\mathbf{K}}$ is an extension of the Hilbert-style calculus for $\mathbf{ML}_n$ by (Ax1), (Ax4)–(Ax8), and (G). The semantics for $\mathbf{MML}_n^{\mathbf{K}}$ is similar to the one for $\mathbf{MML}_n^{\mathbf{S5}}$, but each R_i is an arbitrary binary relation.

It is easy to prove the following lemma.

Lemma 7.5. *Let $\boldsymbol{L} \in \{\mathbf{MML}_n^{\mathbf{K}}, \mathbf{MML}_n^{\mathbf{S4}}, \mathbf{MML}_n^{\mathbf{S5}}\}$. For each $\alpha \in \mathscr{F}_{\mathrm{M}}$, it holds that if α is provable in the Hilbert-style calculus for $\boldsymbol{L}$, then α is $\boldsymbol{L}$-valid.*

Now let us present a more general approach to modal multilattice logic. We adopt Definition 3.6 from [10] for our case.

Definition 7.1. Let $\mathcal{F} = \langle M, R_1, \ldots, R_n\rangle$ be a paraconsistent Kripke frame. Then we define, for each $1 \leqslant j \leqslant n$ and $x, y \in M$:

(1) $R_j^0(x, y)$ iff $x = y$,
(2) for $a > 0$, $R_j^a(x, y)$ iff for some $z \in M$, $R_j(x, z)$ and $R_j^{a-1}(z, y)$.

One can easily check that $x \models^p \Box_j^a\alpha$ iff $\forall y(R_j^a(x, y) \Rightarrow y \models^p \alpha)$, for each paraconsistent frame $\mathcal{F}$, $x \in \mathcal{F}$, $\alpha \in \mathscr{F}_{\mathrm{M}}$, and $a \geqslant 0$ (see [10, Theorem 3.7]). Similarly for other modal operators. Now let us introduce the scheme

$$\Diamond_j^a\Box_j^b\alpha \to_j \Box_j^c\Diamond_j^d, \qquad G_j^{a,b,c,d}$$

where $a, b, c, d \geqslant 0$. This scheme is a multilattice analogue of the scheme $G^{a,b,c,d}$ described in [10]. Clearly, $\Box_j\alpha \to_j \alpha$ is $G_j^{0,1,0,0}$, $\Box_j\alpha \to_j \Box_j\Box_j\alpha$ is $G_j^{0,1,2,0}$, and $\Diamond_j\alpha \to_j \Box_j\Diamond_j\alpha$ is $G_j^{1,0,1,1}$. Moreover, we can consider multilattice analogues of some other well-known modal formulas. For example, $\alpha \to_j \Box_j \Diamond_j \alpha$ is $G_j^{0,0,1,1}$, $\Box_j \alpha \to_j \Diamond_j \alpha$ is $G_j^{0,1,0,1}$, and $\Diamond_j \Box_j \alpha \to_j \Box_j \Diamond_j \alpha$ is $G_j^{1,1,1,1}$.

Let $\mathcal{F} = \langle M, R_1, \ldots, R_n \rangle$ be a paraconsistent Kripke frame. We say that R_j ($1 \leqslant j \leqslant n$) satisfies (a,b,c,d)-condition ($a,b,c,d \geqslant 0$) iff for each $x, y, z \in M$, it holds that

- if $R_j^a(x,y)$ and $R_j^b(x,z)$, then there is $t \in M$ such that $R_j^c(y,t)$ and $R_j^d(z,t)$.

Clearly, $(0,1,0,0)$-condition is reflexivity, $(0,1,2,0)$-condition is transitivity, $(1,0,1,1)$-condition is Euclideanness, $(0,0,1,1)$-condition is symmetry, $(0,1,0,1)$-condition is seriality, and $(1,1,1,1)$-condition is Church-Rosser property.

An extension of $\mathbf{MML}_n^{\mathbf{K}}$ by $G_j^{a,b,c,d}$ we will call $\mathbf{MML}_n^{G^{a,b,c,d}}$ or just $\mathbf{MML}_n^{G}$. In contrast to the semantics for $\mathbf{MML}_n^{\mathbf{K}}$, the semantics for $\mathbf{MML}_n^{G}$ requires that each R_j satisfies (a,b,c,d)-condition. $\mathbf{MML}_n^{G}$ is a multilattice analogue of $G^{a,b,c,d}$ (see, for example [10], about this logic).

Similarly to Theorem 3.8 from [10] we obtain:

Proposition 7.6. *$G_j^{a,b,c,d}$ is valid in the class of paraconsistent $G^{a,b,c,d}$-Kripke models.*

Using Proposition 7.6 we obtain:

Lemma 7.7. *For each $\alpha \in \mathscr{F}_{\mathrm{M}}$, it holds that if α is provable in the Hilbert-style calculus for $\mathbf{MML}_n^{G}$, then α is $\mathbf{MML}_n^{G}$-valid.*

Moreover, we prove that:

Lemma 7.8. *Let $\boldsymbol{L} \in \{\mathbf{MML}_n^{\mathbf{K}}, \mathbf{MML}_n^{G}\}$. For each $\alpha \in \mathscr{F}_{\mathrm{M}}$, it holds that if α is $\boldsymbol{L}$-valid, then α is provable in the Hilbert-style calculus for $\boldsymbol{L}$.*

Proof. Analogous to the completeness proof for $\mathbf{K}$ and $G^{a,b,c,d}$ (see, e.g., [10]). We emphasize three points only. First, in the definition of a canonical model we put $R_j(x,y)$ iff for each $\alpha \in \mathscr{F}_{\mathrm{M}}$, $\Box_j \alpha \in x$ implies $\alpha \in y$. Clearly, for $a \geqslant 0$ it holds that $R_j^a(x,y)$ iff for each $\alpha \in \mathscr{F}_{\mathrm{M}}$, $\Box_j^a \alpha \in x$ implies $\alpha \in y$. Second, due to the axioms (Ax4)–(Ax8) the connectives (groups of connectives) $\Diamond_j$, $\neg_j \Diamond_j$, $\neg_k \Diamond_j$, $\neg_j \Box_j$, and $\neg_k \Box_j$ are expressed via $\Box_j$ and the connectives of $\mathbf{ML}_n$. Third, the role of classical negation in the definition of maximal consistent sets the double negation $\neg_k \neg_j$ plays here. $\dashv$

Immediately from Lemmas 7.7 and 7.8 we have:

Theorem 7.9. *Let* $\boldsymbol{L} \in \{\mathbf{MML}_n^{\mathbf{K}}, \mathbf{MML}_n^{\mathbf{G}}\}$*. For each* $\alpha \in \mathscr{F}_{\mathrm{M}}$*, it holds that* α *is* $\boldsymbol{L}$*-valid iff* α *is provable in the Hilbert-style calculus for* $\boldsymbol{L}$*.*

Corollary 7.10. *Let* $\boldsymbol{L} \in \{\mathbf{MML}_n^{\mathbf{S4}}, \mathbf{MML}_n^{\mathbf{S5}}\}$*. For each* $\alpha \in \mathscr{F}_{\mathrm{M}}$*, it holds that* α *is* $\boldsymbol{L}$*-valid iff* α *is provable in the Hilbert-style calculus for* $\boldsymbol{L}$*.*

8. Conclusion

An obvious task for the future work is the presentation of multilattice analogues of the other modal logics. As Takano pointed out in his paper [58], modal logics might be divided into three groups depending on the properties of their sequent calculi. The first group consists of **K**, **D**, **T**, **K4**, **KD4**, **S4**, **K45** and **KD45**. These logics "have sequent calculi with the cut-elimination property (and so the subformula property)" [58, p. 116]. The second group contains the logics **KB**, **KDB**, **B**, **K4B**, and **S5** which "have sequent calculi with the subformula property but without the cut-elimination property" [58, p. 116]. The third group consists of two logics: **K5** and **KD5**. Takano formulates sequent calculi for these logics, but they have neither cut elimination nor the subformula property.[5] Clearly, the existence of sequent calculi for all these logics is a bit of a help for the presentation of their multilattice analogues. For the second and the third groups of logics the hypersequent framework may be useful. Let us mention logics **K4.3** and **KD4.3** (cut-free hypersequent calculi for them were presented by Indrzejczak [23]) as well as the logic **S4.3** which has two various hypersequent formulations (by Indrzejczak [22] and Kurokawa [32]) and the logic **S4.2** which also has a hypersequent formulation due to Kurokawa [32]. Besides, Lahav [33] presents a method for the transformation of Kripke frame properties into the hypersequent rules. The adaptation of this method for modal multilattice logic seems to be a promising task. Yet another area of further research is the presentation of a temporal multilattice logic which might be based on the temporal logic **Kt4.3** (see [25, 24] for hypersequent calculi for this logic) or the temporal logic **KtT4** which has already had a four-valued modification (see [11]) based on Belnap and Dunn's logic which is the first step towards the presentation of its multilattice version.

[5] However, Takano presented a modified subformula property for these logics.

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