

Nissim Francez

A POLY-CONNEXIVE LOGIC

Abstract. The paper introduces a variant of connexive logic in which connexivity is extended from the interaction of negation with implication to the interaction of negation also with conjunction and disjunction. The logic is presented by two deductively equivalent methods: an axiomatic one and a natural-deduction one. Both are shown to be complete for a four-valued model theory.

Keywords: connexive logic; connexive conjunction; connexive disjunction; falsification conditions

1. Introduction

As is well known, characteristics of *Connexive Logics* (see [14] for a general survey; see also [10]) are the following (formal) theorems, which are *not* theorems of classical logic. Let ‘ \rightarrow ’ denote implication and ‘ \neg ’ – negation. The following theorems:

$$\neg(\varphi \rightarrow \neg\varphi) \tag{A_1}$$

$$\neg(\neg\varphi \rightarrow \varphi) \tag{A_2}$$

are jointly known as *Aristotle’s thesis*.

In [5], the other characteristic relationships:

$$(\varphi \rightarrow \psi) \rightarrow \neg(\varphi \rightarrow \neg\psi) \tag{B_1}$$

$$(\varphi \rightarrow \neg\psi) \rightarrow \neg(\varphi \rightarrow \psi) \tag{B_2}$$

are attributed to the ancient philosopher and logician Boethius. For the history of connexive logics see [6]. It is generally agreed that connexive logics are not truth-functional, and arise from an attempt to capture

some *connection* between the antecedent and the consequent of a conditional.

In [8, 13, 15], a general method is proposed for obtaining connexive logics: *modifying the falsity conditions* of the conditional, leading to the interaction of the conditional and the negation generating the connexivity characteristics above.

In this paper, I apply the proposed methodology to other connectives, conjunction and disjunction too, endowing them too with the interaction with negation different than the classical (boolean) one. Such an interaction also captures connections between the arguments of the other binary connectives, transcending their truth. By this move, I expand the scope of connexivity from the conditional only to a more extensive signature of connectives. Therefore, I coin the resulting logic *PCON* — a *poly-connexive* logic.

Let me just mention that another family of logics, Relevance logics [1], also base their conditional on a connection (“relevance”) between the antecedent and consequent. Those logics were also expanded to include other connectives, in this case employing sub-structurality to capture the intended intensionality.

I leave the philosophical argument justifying such an extension of connexivity, as well as the motivation for the specific interactions embodied in *PCON*, to a separate discussion. Here, I only present and study the logic itself. *PCON* has an axiomatisation sound and complete for a relational model theory, as well as a deductively equivalent natural-deduction proof system \mathcal{N}_{PCON} .

The following are the characteristic interactions of the binary connectives with the negation:

$$\begin{aligned} (neg_i) \quad & \vdash_{PCON} \neg(\varphi \rightarrow \psi) \leftrightarrow [(\varphi \rightarrow \neg\psi) \vee (\neg\varphi \rightarrow \psi)] \\ (neg_c) \quad & \vdash_{PCON} \neg(\varphi \wedge \psi) \leftrightarrow [(\varphi \wedge \neg\psi) \vee (\neg\varphi \wedge \psi)] \\ (neg_d) \quad & \vdash_{PCON} \neg(\varphi \vee \psi) \leftrightarrow \neg\varphi \vee \neg\psi \end{aligned}$$

Clearly, none of the above is a thesis of classical logic.

The presentation of the axiomatic definition of *PCON*, its relational model theory and the soundness and completeness proofs are based on [8], with a certain change in notation and, of course, with the required modifications to fit the proposed logic.

2. Axiomatic definition of *PCON*

The object language \mathcal{L}_{PCON} consists of the usual closure of a countable set *Prop* of atomic propositions w.r.t. the traditional operators $\{\neg, \rightarrow, \wedge, \vee\}$. The meta-variables p and q range over atomic propositions, and φ, ψ – over arbitrary propositions. Also, meta-variables Σ and Δ denote multi-sets of object-language formulas. As usual, $\varphi \leftrightarrow \psi$ is defined by $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$.

DEFINITION 2.1 (Axiomatic definition of *PCON*). The Hilbert-style axiomatic definition of *PCON* is given by the following axiom schemes, divided into two groups, positive axioms (as for positive propositional intuitionistic logic), and negative axioms:

POSITIVE AXIOMS:

$$\begin{aligned} \varphi &\rightarrow (\psi \rightarrow \varphi) && \text{(Ax1)} \\ (\varphi \rightarrow (\psi \rightarrow \chi)) &\rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi)) && \text{(Ax2)} \\ \varphi \wedge \psi &\rightarrow \varphi && \text{(Ax3)} \\ \varphi \wedge \psi &\rightarrow \psi && \text{(Ax4)} \\ \varphi &\rightarrow (\psi \rightarrow \varphi \wedge \psi) && \text{(Ax5)} \\ \varphi &\rightarrow \varphi \vee \psi && \text{(Ax6)} \\ \psi &\rightarrow \varphi \vee \psi && \text{(Ax7)} \\ ((\varphi \rightarrow \chi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \vee \psi) \rightarrow \chi)) &&& \text{(Ax8)} \end{aligned}$$

NEGATIVE AXIOMS:

$$\begin{aligned} \neg\neg\varphi &\leftrightarrow \varphi && \text{(Ax9)} \\ \neg(\varphi \wedge \psi) &\leftrightarrow [(\varphi \wedge \neg\psi) \vee (\neg\varphi \wedge \psi)] && \text{(Ax10)} \\ \neg(\varphi \vee \psi) &\leftrightarrow (\neg\varphi \vee \neg\psi) && \text{(Ax11)} \\ \neg(\varphi \rightarrow \psi) &\leftrightarrow [(\varphi \rightarrow \neg\psi) \vee (\neg\varphi \rightarrow \psi)] && \text{(Ax12)} \end{aligned}$$

The single INFERENCE RULE:

$$\frac{\varphi \quad \varphi \rightarrow \psi}{\psi} \quad \text{(MP)}$$

Notably, *closure under uniform substitution* is not a rule in *PCON*.

For a finite set Γ of formulas, *derivability* in *PCON* of φ from Γ , denote by $\Gamma \vdash_{PCON} \varphi$, is defined as usual. The proof of the following proposition is standard and omitted.

PROPOSITION 2.1 (deduction theorem). $\Gamma, \varphi \vdash_{PCON} \psi$ iff $\Gamma \vdash_{PCON} \varphi \rightarrow \psi$.

3. Models for *PCON*

The following relational models for *PCON* are obtained by suitable modifications of those in Nelson's logic **N4** [see, e.g., 4]. The formulation follows [8].

DEFINITION 3.1 (models for *PCON*). A model for *PCON* is a triple $\langle W, \leq, V \rangle$, where:

- W is a non-empty set (of states),
- \leq is a partial-order on W ,
- $V: W \times Prop \longrightarrow \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$ is an assignment of truth values to pairs of states and atomic propositions with the condition that $i \in V(w_1, p)$ and $w_1 \leq w_2$ only if $i \in V(w_2, p)$ for all $p \in Prop$, $w_1, w_2 \in W$ and $i \in \{0, 1\}$.

Valuations V are then extended to interpretations I to state-formula pairs by the following conditions:

- $I(w, p) = V(w, p)$,
- $1 \in I(w, \neg\varphi)$ iff $0 \in I(w, \varphi)$,
- $0 \in I(w, \neg\varphi)$ iff $1 \in I(w, \varphi)$,
- $1 \in I(w, \varphi \wedge \psi)$ iff $1 \in I(w, \varphi)$ and $1 \in I(w, \psi)$,
- $0 \in I(w, \varphi \wedge \psi)$ iff either both $1 \in I(w, \varphi)$ and $0 \in I(w, \psi)$ or both $0 \in I(w, \varphi)$ and $1 \in I(w, \psi)$.
- $1 \in I(w, \varphi \vee \psi)$ iff $1 \in I(w, \varphi)$ or $1 \in I(w, \psi)$,
- $0 \in I(w, \varphi \vee \psi)$ iff $0 \in I(w, \varphi)$ or $0 \in I(w, \psi)$,
- $1 \in I(w, \varphi \rightarrow \psi)$ iff for all $x \in W$: if $w \leq x$ and $1 \in I(x, \varphi)$ then $1 \in I(x, \psi)$,
- $0 \in I(\varphi \rightarrow \psi)$ iff either for all $x \in W$ such that $w \leq x$: if $1 \in I(x, \varphi)$ then $0 \in I(x, \psi)$, or for all $x \in W$ such that $w \leq x$: if $0 \in I(x, \varphi)$ then $1 \in I(x, \psi)$.

DEFINITION 3.2 (consequence). $\Sigma \models_{PCON} \varphi$ iff for all models $\langle W, \leq, I \rangle$, and for all $w \in W$: $1 \in I(w, \varphi)$ if $1 \in I(w, \psi)$ for all $\psi \in \Sigma$.

Example 3.1 (counter model to de Morgan's laws). *PCON* invalidates de Morgan's laws. For example,

$$\not\models_{PCON} \neg(\varphi \wedge \psi) \leftrightarrow \neg\varphi \vee \neg\psi$$

A counter-model has one w , with $0 \in I(w, \varphi)$ and $0 \in I(w, \psi)$. Therefore, $1 \in I(w, \neg\varphi)$ and $1 \in I(w, \neg\psi)$, implying $1 \in I(w, \neg\varphi \vee \neg\psi)$. On the other hand, $1 \notin I(w, \neg(\varphi \wedge \psi))$, since $0 \in I(w, \neg(\varphi \wedge \psi))$.

4. Soundness and completeness

THEOREM 4.1 (soundness). *For all Γ, φ : if $\Gamma \vdash_{PCON} \varphi$ then $\Gamma \models_{PCON} \varphi$.*

The proof is standard, by induction on the length of the derivation in *PCON*.

For the completeness proof, first the following standard notions are introduced [cf. 8].

DEFINITION 4.1. Let Σ be a collection of formulas in the object language.

- Σ is *deductively closed* iff either $\Sigma \not\vdash \varphi$ or $\varphi \in \Sigma$.
- Σ is *prime* iff $\varphi \vee \psi \in \Sigma$ implies $\varphi \in \Sigma$ or $\psi \in \Sigma$.
- Σ is *prime deductively closed* (*pd*) iff Σ is both.
- Finally, Σ is *non-trivial* iff $\varphi \notin \Sigma$ for some φ .

The following two lemmas are well-known, and thus their proofs for *PCON* are omitted.

LEMMA 4.2. *If $\Sigma \not\vdash \varphi$ then there is a non-trivial *pd* Δ such that $\Sigma \subseteq \Delta$ and $\Delta \not\vdash \varphi$.*

LEMMA 4.3. *If Σ is *pd* and $\varphi \rightarrow \psi \notin \Sigma$, there is a non-trivial *pd* Θ such that $\Sigma \subseteq \Theta$, $\varphi \in \Theta$ and $\psi \notin \Theta$.*

THEOREM 4.4 (Completeness). *For all Γ, φ :*

$$\text{if } \Sigma \models_{PCON} \varphi \text{ then } \Sigma \vdash_{PCON} \varphi.$$

PROOF. The proof for the positive clauses is identical to that for the completeness proof in [8], but thereof for the negative clauses is modified to fit the new axioms (Ax10)–(Ax12).

One proves the contra-positive claim. Suppose that $\Sigma \not\vdash_{PCON} \varphi$. By Lemma 4.2, there is a *pd* Π such that $\Gamma \subseteq \Pi$ and $\varphi \notin \Pi$. Define the model $\mathcal{M} = \langle X, \leq, I \rangle$, where $X = \{\Delta : \Delta \text{ is a non-trivial } pd\}$, $\Delta \leq \Sigma$ iff $\Delta \subseteq \Sigma$ and I is defined as follows: For every state, Σ and atomic proposition, p

$$1 \in I(\Sigma, p) \text{ iff } p \in \Sigma \text{ and } 0 \in I(\Sigma, p) \text{ iff } \neg p \in \Sigma.$$

It is now shown that this condition extends to any formula φ :

$$1 \in I(\Sigma, \varphi) \text{ iff } \varphi \in \Sigma \text{ and } 0 \in I(\Sigma, \varphi) \text{ iff } \neg\varphi \in \Sigma. \quad (*)$$

The modified parts in the proof of (*) are those pertaining the negative clauses, that now goes as follows.

Conjunction: Assume $0 \in I(\Sigma, \chi \wedge \xi)$, and distinguish between two cases, in accordance with (Ax10).

(*1): $1 \in I(w, \varphi)$ and $0 \in I(w, \psi)$. By the induction hypothesis, (*1) holds iff $\chi \in \Delta$ and $\neg\xi \in \Delta$. Because Δ is deductively closed, $\chi \wedge \neg\xi \in \Delta$. By (Ax10), $\neg(\chi \wedge \xi) \in \Delta$.

(*2): $0 \in I(w, \varphi)$ and $1 \in I(w, \psi)$. Similar.

Disjunction: similar to conjunction (using (Ax11)) and omitted.

Implication: Assume $0 \in I(\Sigma, \chi \rightarrow \xi)$, and distinguish between two cases, in accordance with (Ax12).

(*1): For all Δ such that $\Sigma \subseteq \Delta$, if $1 \in I(\Delta, \chi)$, then $0 \in I(\Delta, \xi)$. By the induction hypothesis, (*1) holds iff For all Δ such that $\Sigma \subseteq \Delta$, if $\chi \in \Delta$, then $\neg\xi \in \Delta$. The latter holds iff $\chi \rightarrow \neg\xi \in \Delta$, and by (Ax12), iff $\neg(\chi \rightarrow \xi) \in \Delta$.

(*2): For all Δ such that $\Sigma \subseteq \Delta$, if $0 \in I(\Delta, \chi)$ then $1 \in I(\Delta, \xi)$. By the induction hypothesis, (*2) holds iff For all Δ such that $\Sigma \subseteq \Delta$, if $\neg\chi \in \Delta$, then $\xi \in \Delta$. The latter holds iff $\neg\chi \rightarrow \xi \in \Delta$, and by (Ax12), iff $\neg(\chi \rightarrow \xi) \in \Delta$. \square

5. A natural-deduction system for *PCON*

In this section, I introduce a natural-deduction proof-system \mathcal{N}_{PCON} for *PCON* and show its deductive equivalence to the axiomatic presentation of *PCON*. This establishes the soundness and completeness of \mathcal{N}_{PCON} .

5.1. The rules of \mathcal{N}_{PCON}

The system draws on ideas from [2], where negation was split. \mathcal{N}_{PCON} is presented in figures 1 and 2. The first figure presents the standard positive rules. The novel part, the negative rules (corresponding to falsification) are presented in Figure 2.

Remark (on the rules). *Conjunction*: The rules express the following equivalence, induced by the falsification condition of $\neg(\varphi \wedge \psi)$:

$$\neg(\varphi \wedge \psi) \equiv (\varphi \wedge \neg\psi) \vee (\neg\varphi \wedge \psi)$$

Disjunction: Here the equivalence induced by the falsification condition of $\neg(\varphi \vee \psi)$, reflected by the rules, is:

$$\neg(\varphi \vee \psi) \equiv \neg\varphi \vee \neg\psi$$

$$\begin{array}{c}
 [\varphi]_i \\
 \vdots \\
 \psi \\
 \hline
 \varphi \rightarrow \psi \quad (\rightarrow I^i)
 \end{array}
 \qquad
 \frac{\varphi \rightarrow \psi \quad \varphi}{\psi} \quad (\rightarrow E)$$

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi} \quad (\wedge I)
 \qquad
 \frac{\varphi \wedge \psi}{\varphi} \quad (\wedge E_1)
 \qquad
 \frac{\varphi \wedge \psi}{\psi} \quad (\wedge E_2)$$

$$\frac{\varphi}{\varphi \vee \psi} \quad (\vee I_1)
 \qquad
 \frac{\psi}{\varphi \vee \psi} \quad (\vee I_2)
 \qquad
 \frac{\varphi \vee \psi \quad \begin{array}{c} \vdots \\ \chi \\ \vdots \end{array} \quad \begin{array}{c} [\psi]_j \\ \vdots \\ \chi \\ \vdots \end{array}}{\chi} \quad (\vee E^{i,j})$$

Figure 1. The positive fragment of \mathcal{N}_{PCON}

$$\frac{\varphi \quad \neg \psi}{\neg(\varphi \wedge \psi)} \quad (\neg \wedge I_1)
 \qquad
 \frac{\neg \varphi \quad \psi}{\neg(\varphi \wedge \psi)} \quad (\neg \wedge I_2)
 \qquad
 \frac{\neg(\varphi \wedge \psi) \quad \begin{array}{c} \vdots \\ \chi \\ \vdots \end{array} \quad \begin{array}{c} [\varphi, \neg \psi]_i \\ \vdots \\ \chi \\ \vdots \end{array}}{\chi} \quad (\neg \wedge E^{i,j})$$

$$\frac{\neg \varphi}{\neg(\varphi \vee \psi)} \quad (\neg \vee I_1)
 \qquad
 \frac{\neg \psi}{\neg(\varphi \vee \psi)} \quad (\neg \vee I_2)
 \qquad
 \frac{\neg(\varphi \vee \psi) \quad \begin{array}{c} \vdots \\ \chi \\ \vdots \end{array} \quad \begin{array}{c} [\neg \varphi]_i \\ \vdots \\ \chi \\ \vdots \end{array}}{\chi} \quad (\neg \vee E^{i,j})$$

$$\frac{\begin{array}{c} [\varphi]_i \\ \vdots \\ \neg \psi \end{array}}{\neg(\varphi \rightarrow \psi)} \quad (\neg \rightarrow I_1^i)
 \qquad
 \frac{\begin{array}{c} [\neg \varphi]_i \\ \vdots \\ \psi \end{array}}{\neg(\varphi \rightarrow \psi)} \quad (\neg \rightarrow I_2^i)$$

$$\frac{\begin{array}{c} [\varphi] \\ \vdots \\ \neg \psi \end{array} \quad \begin{array}{c} [\neg \varphi] \\ \vdots \\ \psi \end{array}}{\chi} \quad (\neg \rightarrow E^{i,j})
 \qquad
 \frac{\neg \neg \varphi}{\varphi} \quad (\text{dne})
 \qquad
 \frac{\varphi}{\neg \neg \varphi} \quad (\text{dni})$$

Figure 2. The negative fragment of \mathcal{N}_{PCON}

Implication: The equivalence induced by the falsification condition of $\neg(\varphi \rightarrow \psi)$ is

$$\neg(\varphi \rightarrow \psi) \equiv (\varphi \rightarrow \neg\psi) \vee (\neg\varphi \rightarrow \psi)$$

There is a certain complication in the structure of the (E)-rule, that has premises of the form *discharged sub-derivations* (that can be thought of as *discharged rules*). This complication arises due to the fact that there are two (I)-rules, both of which discharge assumptions. See [7, 11, 12] for the need of such rules and the circumstances leading to them. \square

Tree-shaped *derivations* are defined recursively, almost as usual. The only difference is that an assumed rule may be applied, and discharged immediately after application, in addition to the applications of primitive rules. Examples of derivations are presented below, in the proof of deductive equivalence of the axiomatic presentation of *PCON* and its ND-presentation, \mathcal{N}_{PCON} .

Example 5.1 (Aristotle's theses).

$$\frac{\frac{[\varphi]_1}{\neg\neg\varphi} \text{ (dni)}}{\neg(\varphi \rightarrow \neg\varphi)} (\rightarrow\text{I}_1^1) \qquad \frac{\frac{[\neg\neg\varphi]_1}{\varphi} \text{ (dne)}}{\neg(\neg\varphi \rightarrow \varphi)} (\rightarrow\text{I}_2^1)$$

Example 5.2 (Boethius' theses).

$$\frac{\frac{\frac{[\varphi \rightarrow \psi]_1 \quad [\varphi]_2}{\psi} (\rightarrow\text{E})}{\neg\neg\psi} \text{ (dni)}}{\neg(\varphi \rightarrow \neg\psi)} (\rightarrow\text{I}_1^2)}{\frac{(\varphi \rightarrow \psi) \rightarrow \neg(\varphi \rightarrow \neg\psi)}{(\varphi \rightarrow \psi) \rightarrow \neg(\varphi \rightarrow \neg\psi)} (\rightarrow\text{I}^1)} \qquad \frac{\frac{\frac{[\varphi \rightarrow \neg\psi]_1 \quad [\varphi]_2}{\neg\psi} (\rightarrow\text{E})}{\neg(\varphi \rightarrow \psi)} (\rightarrow\text{I}_1^2)}{(\varphi \rightarrow \neg\psi) \rightarrow \neg(\varphi \rightarrow \psi)} (\rightarrow\text{I}^1)}{(1)}$$

Note that those are not theses in \mathcal{N} .

5.2. Deductive equivalence of *PCON* and \mathcal{N}_{PCON}

THEOREM 5.1. $\Gamma \vdash_{PCON} \varphi$ iff $\Gamma \vdash_{\mathcal{N}_{PCON}} \varphi$.

PROOF. The part of the proof concerning the positive fragment is standard and omitted. I show the proof for the negative fragment.

Assume $\Gamma \vdash_{\mathcal{N}_{PCON}} \varphi$. To show that $\Gamma \vdash_{PCON} \varphi$, it suffices to show the derivability in \mathcal{N}_{PCON} of the negative axioms.

For (Ax9): The derivations are:

$$\frac{[\varphi]_1}{\neg\neg\varphi} \text{ (dni)} \quad \frac{[\neg\neg\varphi]_1}{\varphi} \text{ (dne)}$$

$$\frac{}{\varphi \rightarrow \neg\neg\varphi} (\neg\rightarrow I_1^1) \quad \frac{}{\neg\neg\varphi \rightarrow \varphi} (\neg\rightarrow I_2^1)$$

For (Ax10): The derivations are in Figure 3.

For (Ax11): The derivations are

$$\frac{[\neg\varphi \vee \neg\psi]_1 \quad \frac{[\neg\varphi]_2}{\neg(\varphi \vee \psi)} (\neg\vee I_1) \quad \frac{[\neg\psi]_3}{\neg(\varphi \vee \psi)} (\neg\vee I_2)}{\neg(\varphi \vee \psi)} (\vee E^{2,3})$$

$$\frac{}{(\neg\varphi \vee \neg\psi) \rightarrow \neg(\varphi \vee \psi)} (\rightarrow I^1)$$

and

$$\frac{[\neg(\varphi \vee \psi)]_1 \quad \frac{[\neg\varphi]_2}{\neg\varphi \vee \neg\psi} (\vee I_1) \quad \frac{[\neg\psi]_3}{\neg\varphi \vee \neg\psi} (\vee I_2)}{\neg\varphi \vee \neg\psi} (\vee E^{2,3})$$

$$\frac{}{\neg(\varphi \vee \psi) \rightarrow (\neg\varphi \vee \neg\psi)} (\rightarrow I^1)$$

For (Ax12): The derivations are in Figure 4.

Assume $\Gamma \vdash_{\mathcal{N}_{PCON}} \varphi$. The proof that $\Gamma \vdash_{PCON} \varphi$ proceeds by induction on the \mathcal{N}_{PCON} -derivation, analysing the last \mathcal{N}_{PCON} -rule applied. Again, only the cases of negative rules are shown.

For $(\neg\wedge I_1)$: The premises for this last application $(\neg\wedge I)$ are (1) $\Gamma \vdash_{PCON} \varphi$ and (2) $\Gamma \vdash_{PCON} \neg\psi$. By the induction hypothesis, (3) $\Gamma \vdash_{PCON} \varphi$ and (4) $\Gamma \vdash_{PCON} \psi$. By (Ax5) and twice (MP) we get (6) $\Gamma \vdash_{PCON} \varphi \wedge \neg\psi$. By applying $(\vee I_1)$ to (6), we get (7) $\Gamma \vdash_{PCON} (\varphi \wedge \neg\psi) \vee (\neg\varphi \wedge \psi)$. By applying (MP) to (7) and (Ax10) we get the required $\Gamma \vdash_{PCON} \neg(\varphi \wedge \psi)$.

For $(\neg\wedge I_2)$: Similar.

For $(\neg\wedge E)$: The premises of the rule are $\neg(\varphi \wedge \psi)$, and two sub-derivations of χ , one from φ and $\neg\psi$ and the other from $\neg\varphi, \psi$. By the induction hypothesis, (1) $\Gamma \vdash_{PCON} \neg(\varphi \wedge \psi)$, (2) $\Gamma, \varphi, \neg\psi \vdash_{PCON} \chi$ and (3) $\Gamma, \neg\varphi, \psi \vdash_{PCON} \chi$. By applying (MP) to (1) and (Ax10), we get (4) $\Gamma \vdash_{PCON} (\varphi \wedge \neg\psi) \vee (\neg\varphi \wedge \psi)$. From (2) and (3) we get, using the deduction theorem, (5) $\Gamma \vdash_{PCON} \varphi \wedge \neg\psi \rightarrow \chi$ and (6) $\Gamma \vdash_{PCON} \neg\varphi \wedge \psi \rightarrow \chi$. Using (Ax8), we get $\Gamma \vdash_{PCON} ((\varphi \wedge \neg\psi) \vee (\neg\varphi \wedge \psi)) \rightarrow \chi$. By applying (MP) to (4) and (8), we get the required $\Gamma \vdash_{PCON} \chi$.

For $(\neg\rightarrow I_1)$: The premise of this rule is a sub-derivation of $\neg\psi$ from φ . By the induction hypothesis, (1) $\Gamma, \varphi \vdash_{PCON} \neg\psi$. By the deduction

$$\frac{(\varphi \wedge \neg\psi) \vee (\neg\varphi \wedge \psi)}{\frac{\frac{[\varphi \wedge \neg\psi]_1 \ (\wedge E_1) \quad [\varphi \wedge \neg\psi]_1 \ (\wedge E)}{\varphi} \quad \frac{[\neg\varphi \wedge \psi]_1 \ (\wedge E_1) \quad [\neg\varphi \wedge \psi]_2 \ (\wedge E_2)}{\psi}}{\neg(\varphi \wedge \psi)} \quad \frac{[\neg\varphi \wedge \psi]_1 \ (\wedge E_1) \quad [\neg\varphi \wedge \psi]_2 \ (\wedge E_2)}{\neg(\varphi \wedge \psi)} \quad (\vee I_1, 2)}{\neg(\varphi \wedge \psi)} \quad (\vee E^{1,2})} \quad (\rightarrow I_1^1)}{[(\varphi \wedge \neg\psi) \vee (\neg\varphi \wedge \psi)] \rightarrow \neg(\varphi \wedge \psi)} \quad (\rightarrow I_1^1)$$

and

$$\frac{[\neg(\varphi \wedge \psi)]_1 \quad \frac{[\varphi, \neg\psi]_2 \ (\wedge I)}{\varphi \wedge \neg\psi} \quad (\vee I_1) \quad \frac{[\neg\varphi, \psi]_3 \ (\wedge I)}{\neg\varphi \wedge \psi} \quad (\vee I_2)}{(\varphi \wedge \neg\psi) \vee (\neg\varphi \wedge \psi)} \quad (\vee I_1, 2)}{\neg(\varphi \wedge \psi) \rightarrow [(\varphi \wedge \neg\psi) \vee (\neg\varphi \wedge \psi)]} \quad (\rightarrow I^1)$$

Figure 3. The derivations for (Ax10)

$$\frac{\frac{[(\varphi \rightarrow \neg\psi) \vee (\neg\varphi \rightarrow \psi)]_1}{\neg(\varphi \rightarrow \psi)} \quad \frac{\frac{[\varphi \rightarrow \neg\psi]_2 \quad [\varphi]_3}{\neg\psi} (\rightarrow E) \quad \frac{[\neg\varphi \rightarrow \psi]_4 \quad [\neg\varphi]_5}{\psi} (\rightarrow E)}{\neg(\varphi \rightarrow \psi)} (\neg \rightarrow I_1^3) \quad \frac{\psi}{\neg(\varphi \rightarrow \psi)} (\neg \rightarrow I_2^5)}{\neg(\varphi \rightarrow \psi)} (\vee E^{2,4})}{\frac{\neg(\varphi \rightarrow \psi)}{[(\varphi \rightarrow \neg\psi) \vee (\neg\varphi \rightarrow \psi)] \rightarrow \neg(\varphi \rightarrow \psi)} (\rightarrow I^1)}$$

For the second derivation, note the applications of assumed rules, denoted as $*_1$ and $*_2$.

$$\frac{[\neg(\varphi \rightarrow \psi)]_1 \quad \frac{\frac{[\varphi]_2}{\neg\psi} (*_1)}{\varphi \rightarrow \neg\psi} (\rightarrow I^2)}{(\varphi \rightarrow \neg\psi) \vee (\neg\varphi \rightarrow \psi)} (\vee I_1) \quad \frac{\frac{[\neg\varphi]_3}{\psi} (*_2)}{\neg\varphi \rightarrow \psi} (\rightarrow I^3)}{(\varphi \rightarrow \neg\psi) \vee (\neg\varphi \rightarrow \psi)} (\vee I_2)}{\frac{[(\varphi \rightarrow \neg\psi) \vee (\neg\varphi \rightarrow \psi)]}{\neg(\varphi \rightarrow \psi) \rightarrow [(\varphi \rightarrow \neg\psi) \vee (\neg\varphi \rightarrow \psi)]} (\rightarrow I^1)}$$

Figure 4. The derivations for (Ax10)

theorem, (2) $\Gamma \vdash_{PCON} \varphi \rightarrow \neg\psi$. By (Ax6), we get (3) $\Gamma \vdash_{PCON} (\varphi \rightarrow \neg\psi) \vee (\neg\varphi \rightarrow \psi)$. By applying (MP) to (3) and (Ax12), we get the desired $\Gamma \vdash_{PCON} \neg(\varphi \rightarrow \psi)$.

For $(\neg\rightarrow I_2)$: Similar.

For $(\neg\rightarrow E)$: The premises of the rule are $\neg(\varphi \rightarrow \psi)$, and two sub-derivations of χ from the assumption-rules $\Gamma, \varphi \vdash_{\mathcal{N}_{PCON}} \neg\psi$ and $\Gamma, \neg\varphi \vdash_{\mathcal{N}_{PCON}} \psi$. By the induction hypothesis, (1) $\Gamma \vdash_{PCON} \neg(\varphi \rightarrow \psi)$, (2) $\Gamma \vdash_{PCON} (\varphi \rightarrow \neg\psi) \rightarrow \chi$ and (3) $\Gamma \vdash_{PCON} (\neg\varphi \rightarrow \psi) \rightarrow \chi$. By applying (MP) to (1) and (Ax12), we get (4) $\Gamma \vdash_{PCON} (\varphi \rightarrow \neg\psi) \vee (\neg\varphi \rightarrow \psi)$. By applying (MP) twice to (Ax8), (2) and (3), we get (5) $\Gamma \vdash_{PCON} ((\varphi \rightarrow \neg\psi) \vee (\neg\varphi \rightarrow \psi)) \rightarrow \chi$. By applying (MP) to (4) and (5) we get the required $\Gamma \vdash_{PCON} \chi$.

For (dni) and (dne): Obvious, by (Ax9). \square

6. Some additional properties of *PCON*

In this section I discuss properties of *PCON* that may look “strange” when viewed through the traditional bivalent view.

6.1. Inconsistency

Recall that a logic \mathcal{L} is inconsistent iff for some φ : $\vdash_{\mathcal{L}} \varphi$ and $\vdash_{\mathcal{L}} \neg\varphi$.

PROPOSITION 6.1 (inconsistency of *PCON*). *PCON is inconsistent.*

Thus, *PCON* shares inconsistency with Wansing’s *C* and Omori’s \mathcal{N} , a byproduct of Wansing’s falsity condition for the conditional, together with some other properties. I will consider several proofs of Proposition 6.1, each pointing to a different source of this inconsistency.

THE FIRST PROOF OF PROPOSITION 6.1.

$$\vdash_{PCON} \varphi \rightarrow (\varphi \rightarrow \varphi) \text{ and } \vdash_{PCON} \neg(\varphi \rightarrow (\varphi \rightarrow \varphi))$$

The derivations are:

$$\frac{\frac{[\varphi]_1}{\varphi \rightarrow \varphi} (\rightarrow I^1)}{\varphi \rightarrow (\varphi \rightarrow \varphi)} (\rightarrow I^2) \qquad \frac{\frac{[\varphi]_1}{\varphi \rightarrow \varphi} (\rightarrow I^1)}{\neg(\varphi \rightarrow (\varphi \rightarrow \varphi))} (\neg\rightarrow I^2_1)$$

Note that both derivations *vacuously* discharge assumptions (indexed by 2): φ in the first, $\neg\varphi$ in the second. \square

I have no firm opinion about the need to avoid inconsistency altogether in connexive logics. Still, a way to avoid these derivations is to abandon vacuous discharge in \mathcal{N}_{PCON} , in parallel of omitting (Ax1) in the axiomatic presentation. This is tantamount of taking as the base logic a variant of relevant logic instead of intuitionistic logic. This makes a lot of sense as far as the natural-language-driven motivation is concerned. I have not taken this move here merely to stay compatible (and comparable) to Wansing's **C** and Omori's \mathcal{N} . Indeed, there is a connexive extension of the relevance logic **BD** in [9]. For a connexive extension of a stronger relevance logic **R** [see, e.g., 1, 3].

THE SECOND PROOF OF PROPOSITION 6.1. $\vdash_{\mathcal{N}_{PCON}} \varphi \wedge \neg\varphi \rightarrow \varphi$ and $\vdash_{\mathcal{N}_{PCON}} \neg(\varphi \wedge \neg\varphi \rightarrow \varphi)$. The derivations are:

$$\frac{\frac{[\varphi \wedge \neg\varphi]_1}{\varphi} (\wedge E_1)}{\varphi \wedge \neg\varphi \rightarrow \varphi} (\rightarrow I^1) \qquad \frac{\frac{[\varphi \wedge \neg\varphi]_1}{\neg\varphi} (\wedge E_2)}{\neg(\varphi \wedge \neg\varphi \rightarrow \varphi)} (\neg\rightarrow I^1) \quad \square$$

THE THIRD PROOF OF PROPOSITION 6.1. Notice that $\vdash_{PCON} \varphi \rightarrow \varphi \vee \neg\varphi$ and $\vdash_{PCON} \neg(\varphi \rightarrow \varphi \vee \neg\varphi)$. The derivations are:

$$\frac{\frac{[\varphi]_1}{\varphi \vee \neg\varphi} (\vee I)}{\varphi \rightarrow \varphi \vee \neg\varphi} (\rightarrow I^1) \qquad \frac{\frac{[\neg\varphi]_1}{\varphi \vee \neg\varphi} (\vee I)}{\neg(\varphi \rightarrow \varphi \vee \neg\varphi)} (\neg\rightarrow I^2) \quad \square$$

6.2. Conclusion

In this paper, I have presented a variant, *PCON*, of connexive logic that extends the notion of connexive logic from its traditional confinement to implication (as related to negation) to other connectives, also relating them to negation in a non-traditional way. The presentation is in terms of two equivalent proof-systems: an axiomatic one; and a natural-deduction one, sound and complete w.r.t. a 4-valued model theory.

The issue of inconsistency, seemingly occurring in a variety of connexive logics, is still not fully understood. It certainly does not play a deductive role similar to its role in bivalent logics such as classical logic or intuitionistic logic. Some general strategies for avoiding inconsistency in connexive logics, if considered undesirable, are still lacking.

Acknowledgements. I thank Hitoshi Omori for several insightful discussion regarding *PCON* and related issues.

References

- [1] Anderson, A. R., and N. Belnap Jr., *Entailment*, vol. 1, Princeton University Press, N.J., 1975.
- [2] Francez, N., “Natural-deduction for two connexive logics”, *IfCoLog Journal of Logics and their Application* 3, 3 (2016): 479–504. Special issue on Connexive Logic.
- [3] Francez, N., “Relevant connexive logic”, *Logic and Logical Philosophy* 28, 3 (2019): 409–425. DOI: [10.12775/LLP.2019.007](https://doi.org/10.12775/LLP.2019.007)
- [4] Kamide, N., and H. Wansing, *Proof Theory of N_4 -related Paraconsistent Logics*, College Publications, London, 2015. Studies in Logic, vol. 54.
- [5] Kneale, W., and M. Kneale, *The Development of Logic*, Duckworth, London, 1962.
- [6] McCall, S., “A history of connexivity”, pages 415–449 in D. M. Gabbay, J. F. Pelletier and J. Woods (eds.), *Handbook of the History of Logic*, vol. 11, ‘Logic: a history of its central concepts’, Elsevier, Amsterdam, 2012. DOI: [10.1016/B978-0-444-52937-4.50008-3](https://doi.org/10.1016/B978-0-444-52937-4.50008-3)
- [7] Olkhovikov, G. K., and P. Schroeder-Heister, “On flattening general elimination rules”, *Review of Symbolic Logic* 7, 1 (2014). DOI: [10.1017/S1755020313000385](https://doi.org/10.1017/S1755020313000385)
- [8] Omori, H., “A note on francez’ half-connexive formula”, *IfCoLog Journal of Logic and their Applications* 3, 3 (2016): 505–512. Special issue on Connexive Logic.
- [9] Hitoshi., “A simple connexive extension of the basic relevant logic BD”, *IfCoLog Journal of Logic and their Applications* 3, 3 (2016): 467–b78. Special issue on Connexive Logic.
- [10] Priest, G., “Negation as cancellation, and connexive logic”, *Topoi* 18, 2 (1999): 141–148. DOI: [10.1023/A:1006294205280](https://doi.org/10.1023/A:1006294205280)
- [11] Schroeder-Heister, P., “A natural extension of natural deduction”, *Journal of Symbolic Logic* 49 (1984): 1284–1300. DOI: [10.2307/2274279](https://doi.org/10.2307/2274279)
- [12] Schroeder-Heister, P., “The calculus of higher-level rules, propositional quantification, and the foundational approach to proof-theoretic harmony”, *Studia Logica* 102, 6 (2014): 1185–1216. Special issue: “Gentzen’s and Jaśkowski’s heritage: 80 Years of natural deduction and sequent calculi”, A. Indrzejczak (ed.). DOI: [10.1007/s11225-014-9562-3](https://doi.org/10.1007/s11225-014-9562-3)
- [13] Wansing, H., “Connexive modal logic”, in R. Schmidt, I. Pratt-Hartmann, M. Reynolds and H. Wansing (eds.), *Advances in Modal Logic*, vol. 5, College Publications, King’s College, London, 2005.

- [14] Wansing, H., “Connexive logic”, in E. N. Zalta (ed.), *The Stanford Encyclopedia of Philosophy*. Fall 2014 edition, 2014. <http://plato.stanford.edu/archives/fall2014/entries/logic-connexive/>
- [15] Wansing, H., and M. Unterhuber, “Connexive conditional logic. Part I”, *Logic and Logical Philosophy* 28, 3 (2019): 567–610. DOI: [10.12775/LLP.2018.018](https://doi.org/10.12775/LLP.2018.018)

NISSIM FRANCEZ
the Technion-IIT
Computer Science Department
Haifa, Israel
`francez@cs.technion.ac.il`