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BOOLEAN CONNEXIVE LOGICS
Semantics and tableau approach

Abstract. In this paper we define a new type of connexive logics which we call Boolean connexive logics. In such logics negation, conjunction and disjunction behave in the classical, Boolean way. We determine these logics through application of the relating semantics. In the final section we present a tableau approach to the discussed logics.

Keywords: Boolean connexive logics; connexive logic; relating semantics; tableau approach; relatedness

1. Introduction

Connexive logic is based on the theses set forth by Aristotle and Boethius, which only use negation and implication connectives. What is more, these theses are inconsistent to the classical logic. Therefore, in connexive logic we must interpret at least one of these connectives in a non-classical manner.

In this study we shall only consider such connexive logics where the negation, conjunction and disjunction have the same meanings with those in the classical logic. Thus each logic of this type we shall refer to as *Boolean connexive logic* since they preserve the meanings of the basic Boolean connectives.

The study offers a new approach to the issue of connexivity. Rather than using for instance possible worlds or many logical values [McCall, 2012] — as the starting basis for the definition of connexive logic — we

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shall assume a certain type of intensional logic: relating logic. By combing the semantic structures for relating logics with a Boolean language we obtain several different logics. The strongest ones among them include Aristotle's and Boethius' connexive laws as their tautologies. Hence, they are connexive logics.

Further in the study we present the following issues. First, we bring back some basic issues involved in connexive logics. Further, we present the semantics of the relating logic which we shall use as grounds for specification of our Boolean connexive logics and other issues. By dint of the findings concerning relations between the Aristotle's and Boethius' theses and the conditions imposed on the relating relation, we can present a lattice of logics comprising the least Boolean connexive logic along with a natural extension. Lastly, as a decision-making procedure, we propose the tableau methods that we shall elaborate in Section 9 of the study.

2. Connexive logics — main ideas and definitions

There is an idea behind connexive logic that proposition A has nothing in common with proposition $\sim A$ in terms of the content. Similarly, if A has a common content with B, it cannot have any common content with $\sim B$, and vice versa, if A has a common content with $\sim B$, it cannot have any common content with B. A necessary condition for truthness of implication is a common content of premise and conclusion (the implication antecedent and consequent, respectively). Such intuitions form the motivation for connexive logics. The roots of connexive logic date back to the ancient times. In *Prior Analytics* 57b14 Aristotle writes: "It is impossible that if A then $\sim A$".

In the formal language which features at least two connectives, unary: $\sim$, referred to as *negation*, and binary: $\Rightarrow$, referred to as *implication*, the concept of connexivity is expressed by the requirement of occurrence of the following theses in a logic:

$$\sim(A \Rightarrow \sim A) \tag{A1}$$

$$\sim(\sim A \Rightarrow A) \tag{A2}$$

$$(A \Rightarrow B) \Rightarrow \sim(A \Rightarrow \sim B) \tag{B1}$$

$$(A \Rightarrow \sim B) \Rightarrow \sim(A \Rightarrow B) \tag{B2}$$

Theses (A1) and (A2) are referred to as *Aristotle's Theses*, while (B1) and (B2) as *Boethius' Theses*.

Since none of the propositions (A1), (A2), (B1), (B2) is a thesis of the classical logic, and at the same time the classical logic is Post-complete, then having attached any of those to the classical logic, we would produce an inconsistent logic. Thus, if we comprehend the negation and implication in the classical manner, we will produce an inconsistent logic. As a consequence, in any connexive logic the implication or the negation is non-classical. On the other hand if we do not assume the classical logic as a background propositions (A1), (A2), (B1), (B2) can be independent, which we show later.

Still, even if we adopt propositions (A1), (A2), (B1), (B2) as axioms, no common content is guaranteed. For propositions (A1), (A2), (B1), (B2) are valid in binary matrix $\{1, 0\}$ with distinguished value of 1, with classical material implication and negation defined as: $\sim 1 = \sim 0 = 1$. Similarly, these propositions are true in a binary matrix with classical negation and implication defined as: $x \Rightarrow y = 1$ iff $x = y$.

In order to express the desired intuitions concerning bearing the common content we need to make some assumptions regarding the comprehension of negation and implication. Andreas Kapsner [2012, p. 3] proposes some minimal semantic conditions for negation and implication:

(Ka) In no model proposition $A \Rightarrow \sim A$ is satisfiable.
(Kb) In no model propositions $A \Rightarrow B$ and $A \Rightarrow \sim B$ are both satisfiable.

By all means, if we treat the negation and implication connectives in the classical manner, then conditions (Ka) and (Kb), result from (A1) and (B1). On the other hand, these consequences do not have to happen when negation and implication are comprehended in a different manner.

Kapsner proposes a classification of connexive logics into *weakly* and *strongly* connexive. A logic is *weakly connexive*, if it comprises formulas (A1), (A2), (B1), (B2) as theses and it is closed under the rule *modus ponens*: $A, A \Rightarrow B \ / \ B$. If it additionally meets conditions (Ka) and (Kb), it is called a *strongly connexive* logic. Kapsner [2012, p. 4] provides additional conditions that make a logic *superconnexive* logic. However, these conditions lead either to the inconsistent logic, or to a logic that is not closed under uniform substitution. Therefore, we shall meanwhile put them aside.

It is worth to note that the conditions Kapsner sets forth for the strongly connexive logic do not remove all the non-intuitive proposals. Let us again consider the classical negation and implication defined by the conditions for the classical equivalence, meaning $x \Rightarrow y = 1$ iff $x = y$.

A logic with such defined conditions is strongly connexive, even though it has no virtues of reasonableness whatsoever. In order to eliminate symmetric connectives (like material equivalence)), we might apply the following additional condition:

(NS) For some two propositions A, B, in some model $A \Rightarrow B$ is, but $B \Rightarrow A$ is not satisfiable.

It should accompany any connexive logic, regardless of its strength. If a logic meets (NS) it can be called *properly connexive*.

There is also another way to accommodate the Aristotle's and Boethius' theses with the classical logic. It would take to disregard those inferences which feature tautologies and countertautologies as premises or conclusions. Thus, let us assume that the inference only invloves contingent propositions. If so, the intuitions related to bearing the common content are fulfilled in the meaning that $\sim A$ does not infer A (and vice versa) and if A infers B, then A does not infer $\sim B$. Moreover, if A infer B, then A and B comprise at least one common propositional variable. We must notice, however, that in this case we deal with a metalanguage entailment. In the subject language, the laws still do not hold. The property of sharing a variable leads to relevant logics, and specifically to the so-called containment logics [Ferguson, 2015; Fine, 1986].

The denomination of *connexive logic* is to promise some special connection, relation between formulas or premisses and conclusions [Wansing, 2014]. Therefore, in this paper we would like to apply relating semantics to define connexive logics. There are strong indications that by dint of its application we can accurately and directly express the relations between the propositions that lie in the heart of connexive logic. These relations do not have to follow from a common lingual form, but — as mentioned before — from a content similarity that is not always expressible in a logical form. So it can be for instance so that propositions A and B are related in terms of content, even if they do not include a single common propositional letter. The relating semantics makes this relation expressible.

3. Relating logic

Under the approach we propose, we shall define connexive logics using the relating semantics. Let us now remind a few basic facts pertaining to the relating logic and its relations to the Classical Propositional Logic.

3.1. Languages of CPL and RL

Let us consider the set of formulas $\mathbf{For_{CPL}}$ of Classical Propositional Logic (CPL), made up in a standard manner from: the set of variables $\mathbf{Var} = \{p, q, r, p_1, q_1, r_1, \ldots\}$, one unary connective: $\neg$, four binary connectives: $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, and brackets. Let $\models_{\mathbf{CPL}}$ be a consequence relation of **CPL** defined on $\mathbf{For_{CPL}}$ by the set of all classical valuations for $\mathbf{For_{CPL}}$.

Whereas the set of formulas of Relating Logic (**RL**) $\mathbf{For_{RL}}$ is generated with **Var**, negation $\neg$, four binary connectives: $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$ and four binary relating connectives that are relating counterparts of classical connectives: $\wedge^{\mathrm{w}}$, $\vee^{\mathrm{w}}$, $\rightarrow^{\mathrm{w}}$, $\leftrightarrow^{\mathrm{w}}$, and brackets.[1] Thus, $\mathbf{For_{CPL}} \subsetneq \mathbf{For_{RL}}$.

However, if we only take account of the relating part of **RL** formulas, meaning the smallest subset of $\mathbf{For_{RL}}$ closed to **Var**, $\neg$, $\wedge^{\mathrm{w}}$, $\vee^{\mathrm{w}}$, $\rightarrow^{\mathrm{w}}$, $\leftrightarrow^{\mathrm{w}}$, and brackets, we shall get the set $\mathbf{For^{w}_{RL}} \subsetneq \mathbf{For_{RL}}$ which is structurally identical to $\mathbf{For_{CPL}}$, because we have the following bijection between classical and relating formulas, $f_{\mathrm{w}} \colon \mathbf{For_{CPL}} \to \mathbf{For^{w}_{RL}}$ defined as follows:

(a) $f_{\mathrm{w}}(A) = A$, if $A \in \mathbf{Var}$;
(b) $f_{\mathrm{w}}(\neg A) = \neg f_{\mathrm{w}}(A)$;
(c) $f_{\mathrm{w}}(A * B) = f_{\mathrm{w}}(A) *^{\mathrm{w}} f_{\mathrm{w}}(B)$, for $* \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$.

3.2. Relating semantics

A model for the relating formulas is a pair $\langle v, R\rangle$, where $v \colon \mathbf{Var} \to \{0, 1\}$ and $R \subseteq \mathbf{For_{RL}} \times \mathbf{For_{RL}}$. The relation R is called *relating relation*. If two propositions A and B remain in the relation R: $R(A, B)$, then we state they are *related*. Being related may have various philosophical motivations and interpretations. Two formulas can be, for example, related by R: analytically, causally, thematically, temporally etc., or anywise we want [Jarmużek and Kaczkowski, 2014]. Clearly, v assigns just to any variable either truth or falsity. In a model $\langle v, R\rangle$ for formulas made up from the classical, logical constants, we use the well-known, classical truth conditions. For variables and relating formulas we have the following, general truth conditions:

- $\langle v, R\rangle \models A$ iff $v(A) = 1$, if $A \in \mathbf{Var}$;
- $\langle v, R\rangle \models A \wedge^{\mathrm{w}} B$ iff $\langle v, R\rangle \models A$ and $\langle v, R\rangle \models B$ and $R(A, B)$;

[1] The record of relating connectives as: $\wedge^{\mathrm{w}}$, $\vee^{\mathrm{w}}$, $\rightarrow^{\mathrm{w}}$, $\leftrightarrow^{\mathrm{w}}$ was developed in a collaboration with Mateusz Klonowski, a doctoral student, during a logic seminar held in Toruń and led by Tomasz Jarmużek. It also applies to other symbols we use in the context of relating logics.

- $\langle v, R\rangle \models A \vee^{\mathrm{w}} B$ iff $[\langle v, R\rangle \models A$ or $\langle v, R\rangle \models B]$ and $R(A, B)$;
- $\langle v, R\rangle \models A \rightarrow^{\mathrm{w}} B$ iff $[\langle v, R\rangle \not\models A$ or $\langle v, R\rangle \models B]$ and $R(A, B)$;
- $\langle v, R\rangle \models A \leftrightarrow^{\mathrm{w}} B$ iff $[\langle v, R\rangle \models A$ iff $\langle v, R\rangle \models B]$ and $R(A, B)$.

As we can see, the relating connectives have intensional character, since the Boolean conditions are not sufficient. For instance, a regular conjunction $A \wedge B$ is true in a model $\langle v, R\rangle$ iff both propositions: A and B are true in a model $\langle v, R\rangle$. However, the relating conjunction of two formulas $A \wedge^{\mathrm{w}} B$ is true in a model $\langle v, R\rangle$ not only when both propositions: A, B are true in a model $\langle v, R\rangle$, but also additionally proposition A must be related to proposition B, thus $R(A, B)$.

The set of all models for **RL** we denote by $\mathcal{M}_{\mathbf{RL}}$. By taking any subset $\boldsymbol{M}$ of $\mathcal{M}_{\mathbf{RL}}$ in a standard way we define a relating logic $\models_{\boldsymbol{M}}$:

$$X \models_{\boldsymbol{M}} A \text{ iff for all } \mathfrak{M} \in \boldsymbol{M}, \text{ if } \mathfrak{M} \models X, \text{ then } \mathfrak{M} \models A.$$

for any $X \cup \{A\} \subseteq \mathbf{For}_{\mathbf{RL}}$.[2]

3.3. Historical issues

The idea underlying the semantics based on the binary relation defined on a set of formulas probably has its origin in the works of Walton [1979] and Epstein [1979]. An example of its application may be the analysis of the content relationships which is the foundation of the so-called relatedness logics and dependence logics defined by Epstein [1990, pp. 61–84, 115–143] with some particular conditions imposed on models, and with very special intentions and limits, like only relating implication in the language. A more general approach — without assumptions imposed on the relating relation — was proposed in [Jarmużek and Kaczkowski, 2014]. The work also suggested some philosophical interpretations of relating relations, as for example causal, temporal or analytical interpretations [see Jarmużek and Kaczkowski, 2014, p. 54].[3] The aim of the papers was in turn to specify the various sentential logics taking account of the two properties of sentences: their logical value and their off-logical

[2] We can distinguish horizontal, vertical and diagonal conditions that may determine subclasses of $\mathcal{M}_{\mathbf{RL}}$, and consequently define specific relating logics. In this paper we shall introduce conditions tailored for the connexive context.

[3] It should be stressed that the idea of causal interpretation of the relating relation, as one of the many philosophical interpretations of such relation, was also indicated by Walton [1979, p. 131].

connections — taking account of the latter opens the ability to analyse the differently understood content relationships. The least relating logic is defined *modulo* all models [Jarmużek and Kaczkowski, 2014].

The classical logic of propositions might be treated as a special occurrence of the relating logic. For if we take account of the set of all relating models with the universal relation:

$$\mathcal{M}^{\mathrm{u}} = \{\langle v, R\rangle \in \mathcal{M}_{\mathbf{RL}} : R = \mathbf{For}_{\mathbf{RL}} \times \mathbf{For}_{\mathbf{RL}}\},$$

then the consequence relation $\models_{\mathcal{M}^{\mathrm{u}}}$ defined on the set of formulas $\mathbf{For}^{\mathrm{w}}_{\mathbf{RL}}$ is isomorphic to $\models_{\mathbf{CPL}}$ under function f_{w}, since all formulas from $\mathbf{For}^{\mathrm{w}}_{\mathbf{RL}}$ are related to each other and this way they behave classically.

4. Relating implication and semantics for connexive logics

In order to define connexive logic through the relating semantics, we define a combined set of formulas by the set of variables $\mathbf{Var} = \{p, q, r, p_1, q_1, r_1, \dots\}$, three Boolean connectives: $\neg$, $\wedge$, $\vee$, relating implication $\to^{\mathrm{w}}$, and brackets. Then we obtain the set of formulas $\mathbf{For}_{\mathbf{CF}}$, where $\mathbf{CF}$ is an abbreviation for *combined formulas*.

The set $\mathbf{For}_{\mathbf{CF}}$ was on purpose designed so as to include formulas made up from a single connective which behaves intensionally, whereas the other preserve their classical Boolean nature. Obviously, it is the relating implication $\to^{\mathrm{w}}$. The set of formulas $\mathbf{For}_{\mathbf{CF}}$ might be simply treated as a proper subset of formulas of relating logic $\mathbf{For}_{\mathbf{RL}}$.

A model for the combined formulas is a regular model for the relating logic: a pair $\langle v, R\rangle$, where the relation was reduced to the set $\mathbf{For}_{\mathbf{CF}}$, i.e., $R \subseteq \mathbf{For}_{\mathbf{CF}} \times \mathbf{For}_{\mathbf{CF}} =: \mathbf{For}^2_{\mathbf{CF}}$, while the valuation remained unchanged $v\colon \mathbf{Var} \to \{0, 1\}$. Let $\mathcal{M}_{\mathbf{CF}}$ be the set of all models for combined formulas.

Let us adopt some abbreviations. To simplify, instead of $R(A, B)$ we shall write $A\,R\,B$. Additionally we shall introduce some notation for the non-occurrence of relation R. Let $R \subseteq \mathbf{For}^2_{\mathbf{CF}}$. For all $A, B \in \mathbf{For}_{\mathbf{CF}}$ we put $A\,\widetilde{R}\,B$ iff $A\,R\,B$ does not hold.

As we shall find later, relation R will be this time useful to limit the truthness of some types of formulas through preclusion of the content relation between two propositions. We can even assume that the fact that in a given model $A\,R\,B$ we shall interpret as A *is connected to* B, whereas the fact that $A\,\widetilde{R}\,B$ can be interpreted as A *is not connected to* B. So far, in models from $\mathcal{M}_{\mathbf{CF}}$, the occurrence or non-occurrence of relation R is dependent on nothing. Further, we shall introduce some limitations. For the time being, let us, however, adopt the following truth conditions for combined formulas.

Definition 4.1 (Truth conditions for $\mathbf{For_{CF}}$). Let $\langle v, R\rangle \in \mathcal{M}_{\mathbf{CF}}$ and $A, B \in \mathbf{For_{CF}}$. Then:

- $\langle v, R\rangle \models A$ iff $v(A) = 1$, if $A \in \mathbf{Var}$;
- $\langle v, R\rangle \models \neg A$ iff $\langle v, R\rangle \not\models A$;
- $\langle v, R\rangle \models A \wedge B$ iff $\langle v, R\rangle \models A$ and $\langle v, R\rangle \models B$;
- $\langle v, R\rangle \models A \vee B$ iff $\langle v, R\rangle \models A$ or $\langle v, R\rangle \models B$;
- $\langle v, R\rangle \models A \rightarrow^{\mathrm{w}} B$ iff $[\langle v, R\rangle \not\models A$ or $\langle v, R\rangle \models B]$ and $R(A, B)$.

So, for a proposition $A \rightarrow^{\mathrm{w}} B$ to be true in a model, not only the truthness of proposition A must guarantee the truthness of a proposition B; a proposition A must also be connected to a proposition B. The internal structures of both proposition is obviously insignificant.

In order to facilitate seeking the relation between the semantic structures of the relating logic and the Aristotle's and Boethius' theses, we shall introduce a concept of model structure. Similarly, as in the modal logic, we will treat R as a structure of a given model. So we assume:

$$R \models A \text{ iff for all valuations of letters } v \text{ we have } \langle v, R\rangle \models A.$$

5. Models for quasi–connexive and connexive logics

In order to establish which of models $\mathcal{M}_{\mathbf{CF}}$ will be appropriate to define connexive logics, we shall specify the following conditions.

Definition 5.1. For any $R \subseteq \mathbf{For}^2_{\mathbf{CF}}$ we say that:

(a1) R is (a1) iff for any $A \in \mathbf{For_{CF}}$, $A \widetilde{R} \neg A$.
(a2) R is (a2) iff for any $A \in \mathbf{For_{CF}}$, $\neg A \widetilde{R} A$.
(b1) R is (b1) iff for arbitrary $A, B \in \mathbf{For_{CF}}$:
- if $A R B$ then $A \widetilde{R} \neg B$,
- $(A \rightarrow^{\mathrm{w}} B) R \neg(A \rightarrow^{\mathrm{w}} \neg B)$.

(b2) R is (b2) iff for arbitrary $A, B \in \mathbf{For_{CF}}$:
- if $A R B$ then $A \widetilde{R} \neg B$,
- $(A \rightarrow^{\mathrm{w}} \neg B) R \neg(A \rightarrow^{\mathrm{w}} B)$.

On a model level, the relevant conditions from Definition 5.1 exclude the connection of specific propositions. For instance, if relation R has a property (a1), then for none proposition A it is so that it is connected with its negation, i.e., a proposition $\neg A$. The same applies to the other conditions. In accordance with their notation, on the semantic level they conform to the theses of Aristotle and Boethius. From Definition 5.1, it almost directly follows that conditions (a1), (a2), (b1), (b2) imposed on models from class $\mathcal{M}_{\mathbf{CF}}$ are sufficient for occurrence of theses (A1), (A2), (B1), (B2), respectively. Below, we shall provide a proof for theses (A1) and (B2). The other proofs take the same courses.

Theorem 5.1. *For arbitrary $R \subseteq \mathbf{For}^2_{\mathbf{CF}}$ and $A, B \in \mathbf{For}_{\mathbf{CF}}$:*

1. *If R is* (a1) *then $R \models \neg(A \rightarrow^{\mathrm{w}} \neg A)$.*
2. *If R is* (a2) *then $R \models \neg(\neg A \rightarrow^{\mathrm{w}} A)$.*
3. *If R is* (b1) *then $R \models (A \rightarrow^{\mathrm{w}} B) \rightarrow^{\mathrm{w}} \neg(A \rightarrow^{\mathrm{w}} \neg B)$.*
4. *If R is* (b2) *then $R \models (A \rightarrow^{\mathrm{w}} \neg B) \rightarrow^{\mathrm{w}} \neg(A \rightarrow^{\mathrm{w}} B)$.*

Proof. *Ad* 1. Assume that R is (a1) and $A \in \mathbf{For}^2_{\mathbf{CF}}$. Then $A \widetilde{R} \neg A$, from Definition 5.1. Take any valuation of letters v. From Definition 4.1, $\langle v, R\rangle \not\models A \rightarrow^{\mathrm{w}} \neg A$. Thus, $\langle v, R\rangle \models \neg(A \rightarrow^{\mathrm{w}} \neg A)$. Since v was arbitrary, $R \models \neg(A \rightarrow^{\mathrm{w}} \neg A)$.

Ad 4. Assume that R is (b2) and $A, B \in \mathbf{For}^2_{\mathbf{CF}}$. Then, by Definition 5.1, both $(A \rightarrow^{\mathrm{w}} \neg B)\, R\, \neg(A \rightarrow^{\mathrm{w}} B)$ and either $A \widetilde{R} B$ or $A \widetilde{R} \neg B$. Take any valuation of letters v and assume that $\langle v, R\rangle \models A \rightarrow^{\mathrm{w}} \neg B$. From Definition 4.1, we have $A\, R\, \neg B$. Hence $A \widetilde{R} B$. From this and from Definition 4.1, $\langle v, R\rangle \not\models A \rightarrow^{\mathrm{w}} B$, i.e., $\langle v, R\rangle \models \neg(A \rightarrow^{\mathrm{w}} B)$. Thus, $\langle v, R\rangle \models (A \rightarrow^{\mathrm{w}} \neg B) \rightarrow^{\mathrm{w}} \neg(A \rightarrow^{\mathrm{w}} B)$. Since v was arbitrary, then $R \models (A \rightarrow^{\mathrm{w}} \neg B) \rightarrow^{\mathrm{w}} \neg(A \rightarrow^{\mathrm{w}} B)$. □

A theorem with implications the other way around is not true which is easy to demonstrate by providing relevant countermodels for each case. For instance, let us falsify a thesis that if $R \models \neg(\neg A \rightarrow^{\mathrm{w}} A)$, then R is (a2). Let us take such relation $R \subseteq \mathbf{For}^2_{\mathbf{CF}}$ that $\neg(p \wedge \neg p)\, R\, (p \wedge \neg p)$, while for the remaining formulas B: $\neg B \widetilde{R} B$. Thus for any $A \in \mathbf{For}_{\mathbf{CF}}$, $R \models \neg(\neg A \rightarrow^{w} A)$, by virtue of Definition 4.1, since $\langle v, R\rangle \models \neg(p \wedge \neg p)$ and $\langle v, R\rangle \not\models p \wedge \neg p$, for each valuation v. So the antecedent is fulfilled. However, from the construction of R, for some formula A, $\neg A\ R\ A$. Thus, (a2) does not hold. Similar countermodels, based on tautologies or countertautologies, may be presented for the remaining implications.

Still, we would also like to get converse implications. This would enable a transition from the syntactic formulation of connexive logic, i.e., adoption of (A1), (A2), (B1), (B2) as axioms, to the relevant classes of models within $\mathcal{M}_{\mathbf{CF}}$. There are probably various ways to receive a converse theorem. We offer, however, a rather intuitive way, which is probably also minimalistic. To this end, we will adopt one more property of a relating relation.

Definition 5.2 (Closure under negation). Let $R \subseteq \mathbf{For}^2_{\mathbf{CF}}$. Then:

(c1) R is *closed under negation* (or is just (c1)) iff for all A, $B \in \mathbf{For}_{\mathbf{CF}}$: if $A\,R\,B$ then $\neg A\,R\,\neg B$.

Let us note that the closure under negation is a minimal condition which preserves the connection of two propositions and their negations in terms of content. For, in accordance with (c1), it is so that if two propositions A and B are connected: $A\,R\,B$, then their negations are also connected $\neg A\,R\,\neg B$ which seems reasonable. We can also consider a stronger condition, reinforcing (c1) to a equivalence. But condition (c1) — as we will see in the subsequent part — will be sufficient to get a single equivalence among conditions (a1), (a2), (b1), (b2) with the theses of Aristotle and Boethius that are relevant. The examination of the reinforced (c1) as well as the other conditions producing a similar effect as (c1) we shall leave for further studies.

Condition (c1) features an interesting property. When imposed on models it will produce new theses.

Proposition 5.2. *If $R \subseteq \mathbf{For}^2_{\mathbf{CF}}$ is closed under negation then*

$$R \models \neg((A \rightarrow^{\mathrm{w}} B) \wedge \neg B \wedge \neg(\neg A \rightarrow^{\mathrm{w}} \neg B)).$$

Proof. Let $R \subseteq \mathbf{For}^2_{\mathbf{CF}}$ and R be closed under negation. Take any valuation of letters v. Assume that $\langle v, R\rangle \models (A \rightarrow^{\mathrm{w}} B) \wedge \neg B$. From Definition 4.1, $\langle v, R\rangle \models \neg B$ and $\langle v, R\rangle \models A \rightarrow^{\mathrm{w}} B$; and so $A\,R\,B$. The latter, due to condition (c1), results in $\neg A R \neg B$. Thus, by Definition 4.1, we have $\langle v, R\rangle \models \neg A \rightarrow^{\mathrm{w}} \neg B$, since $\langle v, R\rangle \models \neg B$. While from the arbitrariness of v, we get $R \models \neg A \rightarrow^{\mathrm{w}} \neg B$. □

In order to establish that the converse implication does not hold it is sufficient to take account of a model constructed in the following manner. Take such R that let for all formulas A, B a property from (c1) hold, except for the case when A and B form the formula $(p \vee \neg p)$.

Thus $(p \vee \neg p)\, R\, (p \vee \neg p)$, but $\neg(p \vee \neg p)\, \widetilde{R}\, \neg(p \vee \neg p)$. Let us now take any valuation v. For all $A, B \in \mathbf{For_{CF}}$ holds $\langle v, R\rangle \models \neg((A \to^{\mathrm{w}} B) \wedge \neg B \wedge \neg(\neg A \to^{\mathrm{w}} \neg B))$. For when A or B does not equal $(p \vee \neg p)$, then Proposition 5.2 works. In turn, when A and B are identical with the formula $(p \vee \neg p)$, the implication $\neg(p \vee \neg p) \to^{\mathrm{w}} \neg(p \vee \neg p)$ is false, thus $\langle v, R\rangle \models \neg(\neg(p\vee\neg p) \to^{\mathrm{w}} \neg(p\vee\neg p))$, the implication $(p\vee\neg p) \to^{\mathrm{w}} (p\vee\neg p)$ is true, but $\neg(p\vee\neg p)$ is false. Hence, $R \models \neg((A \to^{\mathrm{w}} B)\wedge\neg B\wedge\neg(\neg A \to^{\mathrm{w}} \neg B))$, although R is not closed under negation.

We must notice, however, that the adoption of conditions (a1), (a2), (b1), (b2) to a relating relation R does not warrant that the below formula occurs as a tautology: $R \models \neg((A \to^{\mathrm{w}} B) \wedge \neg B \wedge \neg(\neg A \to^{\mathrm{w}} \neg B))$.

Proposition 5.3. *Conditions* (a1), (a2), (b1), (b2) *imposed on R are not sufficient for $R \models \neg((A \to^{\mathrm{w}} B)\wedge\neg B\wedge\neg(\neg A \to^{\mathrm{w}} \neg B))$, i.e., for some $R \subseteq \mathbf{For}^2_{\mathbf{CF}}$ which is* (a1), (a2), (b1), (b2) *we have $R \not\models \neg((A \to^{\mathrm{w}} B) \wedge \neg B \wedge \neg(\neg A \to^{\mathrm{w}} \neg B))$.*

Proof. We define the relation R as the union of the following sets:

1. $\{\langle A \to^{\mathrm{w}} B, \neg(A \to^{\mathrm{w}} \neg B)\rangle : A, B \in \mathbf{For_{CF}}\}$,
2. $\{\langle A \to^{\mathrm{w}} \neg B, \neg(A \to^{\mathrm{w}} B)\rangle : A, B \in \mathbf{For_{CF}}\}$,
3. $\{\langle p, q\rangle\}$.

Obviously, relation R is (a1) and (a2). Moreover, it is also (b1) and (b2) since for all $A, B \in \mathbf{For_{CF}}$ it fulfils the conditions from Definition 5.1:

- if $A\, R\, B$ then $A\, \widetilde{R}\, \neg B$,
- $(A \to^{\mathrm{w}} B)\, R\, \neg(A \to^{\mathrm{w}} \neg B)$,
- $(A \to^{\mathrm{w}} \neg B)\, R\, \neg(A \to^{\mathrm{w}} B)$.

Now, we take such valuation v that for each variable $A \in \mathbf{Var}$: $v(A) = 0$. Then $\langle v, R\rangle \models (p \to^{\mathrm{w}} q) \wedge \neg q \wedge \neg(\neg p \to^{\mathrm{w}} \neg q)$, from Definition 4.1, because $\neg p\, \widetilde{R}\, \neg q$. Thus $R \not\models \neg((A \to^{\mathrm{w}} B) \wedge \neg B \wedge \neg(\neg A \to^{\mathrm{w}} \neg B))$. □

From propositions 5.2 and 5.3 two conclusions follow. Firstly, an addition of condition (c1) to define the class of the relating models brings new laws that are not generated by conditions (a1), (a2), (b1), (b2). Secondly, condition (c1) does not follow from those conditions.

In the subsequent section, we will show that the adoption of the condition of closure under negation produces a theorem on the correspondence for the Aristotle's and Boethius' theses. We will also demonstrate the all these conditions are independent.

6. Correspondence theorems for (A1), (A2), (B1), (B2)

We have the following theorem.

Theorem 6.1 (Correspondence theorem). *Let* $R \subseteq \mathbf{For_{CF}}$ *be* (c1). *Then*:

1. R *is* (a1) *iff* $R \models \neg(A \rightarrow^{\mathrm{w}} \neg A)$,
2. R *is* (a2) *iff* $R \models \neg(\neg A \rightarrow^{\mathrm{w}} A)$,
3. R *is* (b1) *iff* $R \models (A \rightarrow^{\mathrm{w}} B) \rightarrow^{\mathrm{w}} \neg(A \rightarrow^{\mathrm{w}} \neg B)$,
4. R *is* (b2) *iff* $R \models (A \rightarrow^{\mathrm{w}} \neg B) \rightarrow^{\mathrm{w}} \neg(A \rightarrow^{\mathrm{w}} B)$.

Proof. Let $R \subseteq \mathbf{For_{CF}}$ be (c1), i.e., for all $A, B \in \mathbf{For_{CF}}$: if $A\,R\,B$ then $\neg A\,R\,\neg B$. The implications from the left to the right in 1–4 hold, by virtue of Theorem 5.1. Let us remind that their proof does not require occurrence of condition (c1). We will demonstrate the converse implications one after another.

Ad 1. Assume that (i) $R \models \neg(A \rightarrow^{\mathrm{w}} \neg A)$ and suppose that there exists formula B such that $B\,R\,\neg B$. Let us now take any valuation v. Then either $\langle v, R\rangle \models B$ or $\langle v, R\rangle \not\models B$.

Let us consider the first case: $\langle v, R\rangle \models B$. Thus, $\langle v, R\rangle \not\models \neg B$. Hence $\langle v, R\rangle \models \neg\neg B$, by virtue of Definition 4.1. Since $B\,R\,\neg B$, from (c1) we have $\neg B\,R\,\neg\neg B$. Again, by virtue of Definition 4.1 we get, however, $\langle v, R\rangle \models \neg B \rightarrow^{\mathrm{w}} \neg\neg B$ which is inconsistent to (i).

Let us consider the second case: $\langle v, R\rangle \not\models B$. Having applied Definition 4.1, we get then $\langle v, R\rangle \models \neg B$. Hence $\langle v, R\rangle \models B \rightarrow^{\mathrm{w}} \neg B$ which is contradictive to (i).

Ad 2. Assume that (ii) $R \models \neg(\neg A \rightarrow^{\mathrm{w}} A)$ and suppose that there exists formula B such that $\neg B\,R\,B$. Let us take any valuation v. Then either $\langle v, R\rangle \models B$ or $\langle v, R\rangle \not\models B$.

Let us consider the first case: $\langle v, R\rangle \models B$. Again, having applied Definition 4.1. Thus, $\langle v, R\rangle \not\models \neg B$. Since $\neg B R B$, then $\langle v, R\rangle \models \neg B \rightarrow^{\mathrm{w}} B$ which is contradictive to (ii).

Let us consider the second case: $\langle v, R\rangle \not\models B$. Then $\langle v, R\rangle \models \neg B$ from Definition 4.1. Since $\neg B\,R\,B$, from (c1) we get $\neg\neg B\,R\,\neg B$. And from this and from Definition 4.1 we have $\langle v, R\rangle \models \neg\neg B \rightarrow^{\mathrm{w}} \neg B$ which is contradictive to (ii).

Ad 3. Assume that (iii) $R \models (A \rightarrow^{\mathrm{w}} B) \rightarrow^{\mathrm{w}} \neg(A \rightarrow^{\mathrm{w}} \neg B)$. Take any formulas $A, B \in \mathbf{For_{CF}}$. Then from Definition 4.1, for connective $\rightarrow^{\mathrm{w}}$ it follows that $(A \rightarrow^{\mathrm{w}} B)\,R\,\neg(A \rightarrow^{\mathrm{w}} \neg B)$. Therefore, still pending is the proof that: if $A\,R\,B$ then $A\,\widetilde{R}\,\neg B$.

Assume that $A\,R\,B$ and $A\,R\,\neg B$. Let us take any valuation v. Then either $\langle v, R\rangle \models A \rightarrow^{\mathrm{w}} B$ or $\langle v, R\rangle \not\models A \rightarrow^{\mathrm{w}} B$. Let us consider two cases.

Firstly: $\langle v, R\rangle \models A \rightarrow^{\mathrm{w}} B$. Then from (iii) and from Definition 4.1, we have $\langle v, R\rangle \models \neg(A \rightarrow^{\mathrm{w}} \neg B)$. Hence $\langle v, R\rangle \not\models A \rightarrow^{\mathrm{w}} \neg B$, and since $A\,R\,\neg B$, so $\langle v, R\rangle \models A$ and $\langle v, R\rangle \not\models \neg B$. Therefore, again from Definition 4.1, we have $\langle v, R\rangle \not\models \neg A$. Since $A\,R\,B$, then $\neg A\,R\,\neg B$, from (c1). Thus, from Definition 4.1, $\langle v, R\rangle \models \neg A \rightarrow^{\mathrm{w}} \neg B$, and then from (iii) we have $\langle v, R\rangle \models \neg(\neg A \rightarrow^{\mathrm{w}} \neg\neg B)$. On the other hand, since $A\,R\,\neg B$, then from (c1) we get $\neg A\,R\,\neg\neg B$, from this and from Definition 4.1, $\langle v, R\rangle \models \neg A$ and $\langle v, R\rangle \not\models \neg\neg B$, which contradicts that $\langle v, R\rangle \not\models \neg A$.

Secondly: $\langle v, R\rangle \not\models A \rightarrow^{\mathrm{w}} B$. Again, we will repeatedly apply Definition 4.1. Since $A\,R\,B$, then $\langle v, R\rangle \models A$ and $\langle v, R\rangle \not\models B$, hence $\langle v, R\rangle \not\models \neg A$. Since $A\,R\,B$, then from (c1) we have $\neg A\,R\,\neg B$. Thus, $\langle v, R\rangle \models \neg A \rightarrow^{\mathrm{w}} \neg B$. From ($\#_3$) we have $\langle v, R\rangle \models \neg(\neg A \rightarrow^{\mathrm{w}} \neg\neg B)$, thus $\langle v, R\rangle \not\models \neg A \rightarrow^{\mathrm{w}} \neg\neg B$. On the other hand, since $A\,R\,\neg B$, then from (c1) we have $\neg A\,R\,\neg\neg B$, thus $\langle v, R\rangle \models \neg A$ and $\langle v, R\rangle \not\models \neg\neg B$ which contradicts that $\langle v, R\rangle \not\models \neg A$.

Ad 4. Assume that (iv) $R \models (A \rightarrow^{\mathrm{w}} \neg B) \rightarrow^{\mathrm{w}} \neg(A \rightarrow^{\mathrm{w}} B)$. Take any formulas $A, B \in \mathbf{For_{CF}}$. Then, from Definition 4.1 for connective $\rightarrow^{\mathrm{w}}$ it follows that $(A \rightarrow^{\mathrm{w}} \neg B)\,R\,\neg(A \rightarrow^{\mathrm{w}} B)$. Similar to the previous point, still pending is the proof that if $A\,R\,B$, then $A\,\widetilde{R}\,\neg B$.

Assume that $A\,R\,B$ and $A\,R\,\neg B$. Let us take any valuation v. Then either $\langle v, R\rangle \models A \rightarrow^{\mathrm{w}} \neg B$ or $\langle v, R\rangle \not\models A \rightarrow^{\mathrm{w}} \neg B$.

The first case: $\langle v, R\rangle \models A \rightarrow^{\mathrm{w}} \neg B$. Then, from (iv) we have $\langle v, R\rangle \models \neg(A \rightarrow^{\mathrm{w}} B)$. Hence, from Definition 4.1, $\langle v, R\rangle \not\models A \rightarrow^{\mathrm{w}} B$. And since $A\,R\,B$, thus $\langle v, R\rangle \models A$ and $\langle v, R\rangle \not\models B$. Again, from Definition 4.1, $\langle v, R\rangle \not\models \neg A$ and $\langle v, R\rangle \not\models \neg\neg B$. From assumption $AR\neg B$ and from (c1) we get $\neg A\,R\,\neg\neg B$. Thus, from Definition 4.1, $\langle v, R\rangle \models \neg A \rightarrow^{\mathrm{w}} \neg\neg B$. In turn, from (iv) we get $\langle v, R\rangle \models \neg(\neg A \rightarrow^{\mathrm{w}} \neg B)$, and from this and from Definition 4.1, $\langle v, R\rangle \not\models \neg A \rightarrow^{\mathrm{w}} \neg B$. On the other hand, since $A\,R\,B$, so from (c1) we have $\neg A\,R\,\neg B$. Consequently, $\langle v, R\rangle \models \neg A$ and $\langle v, R\rangle \not\models \neg B$, by virtue of Definition 4.1. But it contradicts that $\langle v, R\rangle \not\models \neg\neg B$, as then $\langle v, R\rangle \models \neg B$.

The second case: $\langle v, R\rangle \not\models A \rightarrow^{\mathrm{w}} \neg B$. Since $A\,R\,\neg B$, then from Definition 4.1, $\langle v, R\rangle \models A$ and $\langle v, R\rangle \not\models \neg B$. Hence $\langle v, R\rangle \models \neg\neg B$. Since $A\,R\,\neg B$, from (c1) we get $\neg AR\neg\neg B$. From this and from Definition 4.1, $\langle v, R\rangle \models \neg A \rightarrow^{\mathrm{w}} \neg\neg B$. In turn, from (iv) we get: $\langle v, R\rangle \not\models \neg A \rightarrow^{\mathrm{w}} \neg B$. On the other hand, since $A\,R\,B$, then from (c1) we have $\neg A\,R\,\neg B$, thus $\langle v, R\rangle \models \neg A$ and $\langle v, R\rangle \not\models \neg B$ which contradicts that $\langle v, R\rangle \models A$. □

7. Independence of conditions

Theorem 6.1 allows — having assumed the condition of closure under negation (c1) — to demonstrate a single unambiguous connection between conditions (a1), (a2), (b1), (b2) and theses (A1), (A2), (B1), (B2), respectively. So, by dint of imposition of these conditions on models from the set $\mathcal{M}_{\mathbf{CF}}$ we can acquire logics satisfying the relevant laws.

But having additionally imposed conditions (c1) on models we find new laws (Proposition 5.2) that are not the laws of a model satisfying conditions (a1), (a2), (b1), (b2) (Proposition 5.3). These findings form some foundations of the independence issue. However, we should ultimately define the logical connections in which conditions (a1), (a2), (b1), (b2) and (c1) remain. If they were absolutely independent, then we could consider 2^5 of various logics among which two logics would be connexive and the remaining ones connexive to some extent. The issue of independence becomes solved by the subsequent theorem.

Theorem 7.1. *Conditions* (a1), (a2), (b1), (b2), (c1) *are independent.*

Proof. Condition (a1) does not follow from (a2), (b1), (b2), (c1). Let R be a binary relation in integral domain defined as $\{\langle n, n+1\rangle : n > 0\}$. Let also:

$$\neg a := \begin{cases} a+1 & \text{if } a \geqslant 1 \\ 1 & \text{otherwise.} \end{cases}$$

Next, let $a \rightarrow^{\mathrm{w}} b = 1$. In relation R, the left argument n is strictly less than the right one $n+1$. So, condition (a2) holds as: (i) if $a \geqslant 1$ then $\neg a > a$; (ii) whereas if $a < 1$ then also $\neg a > a$. Conditions (b1) and (b2) are fulfilled as well. For each a there exists at most one b such that $a\,R\,b$, furthermore $a \neq \neg a$, thus if $a\,R\,b$, then $a\,\widetilde{R}\,\neg b$. Moreover, $(a \rightarrow^{\mathrm{w}} b) = (a \rightarrow^{\mathrm{w}} \neg b) = 1\,R\,2 = \neg(a \rightarrow^{\mathrm{w}} \neg b) = \neg(a \rightarrow^{\mathrm{w}} b)$. Also condition (c1) is fulfilled since if $a\,R\,b$, then $0 < a < a+1 = b$, thus $\neg a = a+1 < (a+1)+1 = b+1 = \neg b$, hence $\neg a\,R\,\neg b$. In turn, $1\,R\,\neg 1$, thus condition (a1) does not hold.

Condition (a2) does not follow from (a1), (b1), (b2), (c1). Let for any natural number n, only $(n+1)Rn$ and $nR(n+3)$. Let also $\neg a = a+1$ and $a \rightarrow^{\mathrm{w}} b = a + 2b$. Assume that $a\,R\,b$. Then $a - b = 1$ or $b - a = 3$, while $a - \neg b = 0$ or $\neg b - a = 4$. Thus $a\widetilde{R}\neg a$, so condition (a1) holds. Since condition: if aRb, then $a\widetilde{R}\neg b$ is also fulfilled, so: (i) condition (b1) holds as $(a \rightarrow^{\mathrm{w}} b) = (a+2b)\,R\,(a+2b+3) = (a+2(b+1))+1 = (a+2\neg b)+1 =$

$(a \rightarrow^{\mathrm{w}} \neg b) + 1 = \neg(a \rightarrow^{\mathrm{w}} \neg b)$; (ii) condition (b2) holds as $(a \rightarrow^{\mathrm{w}} \neg b) = (a{+}2\neg b) = (a{+}2(b{+}1)) = (a{+}2b{+}2)R(a{+}2b{+}1) = \neg(a{+}2b) = \neg(a \rightarrow^{\mathrm{w}} b)$. Condition (c1) is obviously also fulfilled, since if aRb then $a{+}1Rb{+}1$. However $0 + 1\, R\, 0$. Thus condition (a2) does not hold as $\neg 0\, R\, 0$.

Condition (b1) does not follow from the set (a1), (a2), (b2), (c1). Let R be the equality relation in the set of integers: aRb iff $a = b$, $\neg a = a{+}1$ and let $a \rightarrow^{\mathrm{w}} b = b$. Then conditions (a1) and (a2) are fulfilled as $a \neq a{+}1$. In turn, (c1) is fulfilled since if $a = b$, then $a{+}1 = b{+}1$. Condition (b2) holds as $(a \rightarrow^{\mathrm{w}} \neg b) = \neg b = b{+}1 = (a \rightarrow^{\mathrm{w}} b){+}1 = \neg(a \rightarrow^{\mathrm{w}} b)$. Condition (b1) does not hold as $(a \rightarrow^{\mathrm{w}} b) = b \neq (b{+}1){+}1 = \neg(a \rightarrow^{\mathrm{w}} \neg b)$.

Condition (b2) does not follow from (a1), (a2), (b1), (c1). Let R be the equality relation in the set of integers: $a\, R\, b$ iff $a = b$. Let also $\neg a = a + 1$ and $a \rightarrow^{\mathrm{w}} b = a - b$. If $a = b$, then $a \neq b + 1$ and $a \rightarrow^{\mathrm{w}} b = a - b = (a - (b + 1)) + 1 = \neg(a \rightarrow^{\mathrm{w}} \neg b)$. Thus, condition (b1) is fulfilled. Whereas condition (b2) does not hold as $(a \rightarrow^{\mathrm{w}} \neg b) = a - (b + 1) \neq (a - b) + 1 = \neg(a \rightarrow^{\mathrm{w}} b)$.

Condition (c1) does not follow from (a1), (a2), (b1), (b2). Assume that only $2\, R\, 2$ and $1\, R\, 3$. Let also $\neg a = a + 1$ and

$$a \rightarrow^{\mathrm{w}} b := \begin{cases} 1 & \text{if } a + b \text{ is an even number} \\ 2 & \text{if } a + b \text{ is an odd number.} \end{cases}$$

Since $\neg a{-}a = 1$, conditions (a1), (a2) hold. In turn, neither $2R(2{+}1)$ nor $1R(3{+}1)$ hold so if aRb, then $a\widetilde{R}\neg b$. Assume that $a{+}b$ is an even number. Then $a \rightarrow^{\mathrm{w}} b = 1$, $a + \neg b$ is an odd number and we have: $\neg(a \rightarrow^{\mathrm{w}} \neg b) = (a \rightarrow^{\mathrm{w}} \neg b){+}1 = 2{+}1 = 3$. Consequently, $a \rightarrow^{\mathrm{w}} b = 1R3 = \neg(a \rightarrow^{\mathrm{w}} \neg b)$. While when $a + b$ is an odd number, then $a \rightarrow^{\mathrm{w}} b = 2$, $a + \neg b$ is an even number and we have: $\neg(a \rightarrow^{\mathrm{w}} \neg b) = (a \rightarrow^{\mathrm{w}} \neg b) + 1 = 1 + 1 = 2$. Consequently, $a \rightarrow^{\mathrm{w}} b = 2\, R\, 2 = \neg(a \rightarrow^{\mathrm{w}} \neg b)$ which proves condition (b1). Assume that $a + \neg b$ is an even number. Then $a \rightarrow^{\mathrm{w}} \neg b = 1$, $a + b$ is an odd number and we have: $\neg(a \rightarrow^{\mathrm{w}} b) = (a \rightarrow^{\mathrm{w}} b) + 1 = 2 + 1 = 3$. Consequently, $a \rightarrow^{\mathrm{w}} \neg b = 1\, R\, 3 = \neg(a \rightarrow^{\mathrm{w}} b)$. While when $a + \neg b$ is an odd number, then $a \rightarrow^{\mathrm{w}} \neg b = 2$, $a + b$ is an even number and we have: $\neg(a \rightarrow^{\mathrm{w}} b) = (a \rightarrow^{\mathrm{w}} b) + 1 = 1 + 1 = 2$. Consequently $(a \rightarrow^{\mathrm{w}} \neg b) = 2\, R\, 2 = \neg(a \rightarrow^{\mathrm{w}} b)$, which proves condition (b2). Still, since $2\, R\, 2$, but $(2 + 1)\, \widetilde{R}\, (2 + 1)$. So condition (c1) is not true. □

Let us note that there is also another way to prove this theorem. It consists in application of Theorem 6.1 and construction of relevant models on its basis. It allows to establish the independence of conditions (a1), (a2), (b1), (b2). In turn, from propositions 5.2 and 5.3, we know that (c1) does not follow from conditions (a1), (a2), (b1), (b2). In favour of the completeness of considerations, the knowledge should be complemented with the connection the other way around. But the issue is solved by the global Theorem 7.1. Thus we know that all five conditions are independent of each other.

8. Lattice of non-connexive, quasi-connexive and connexive logics

To begin with, let us propose certain terminology. A logic is *connexive* iff (A1), (A2), (B1), (B2) are its theses. A logic is *quasi-connexive* iff it is not connexive, but at least one of (A1), (A2), (B1), (B2) is its thesis.

From the independence Theorem 7.1 we know that there exist 32 (2^5) types of relations R, in terms of the contentions of our interest: (a1), (a2), (b1), (b2), (c1). In turn, each type of relations unambiguously defines a class of relating models, based on a given type of relation.

To begin with, let us consider conditions (a1), (a2), (b1), (b2). There are 16 classes (2^4) of models and each of them corresponds to one type of relating relation. Each of these classes designates one semantic relation of consequence on language $\mathbf{For_{CF}}$. Each of them is different which follows from Theorem 7.1. The weakest one does not require fulfilment of any condition (a1), (a2), (b1), (b2) through relation R in a model $\langle v, R\rangle$. It is not a connexive logic since none of laws (A1), (A2), (B1), (B2) is its thesis. Each of the remaining ones is significantly stronger. 14 of them are quasi-connexive having as a tautology at least one of laws (A1), (A2), (B1), (B2), and finally, one of these logics — the strongest among its tautologies has all substitutions of laws of Aristotle and Boethius. It is the smallest Boolean connexive logic. Beside laws (A1), (A2), (B1), (B2) and their classical consequences, it features none implication or negation-implication tautologies even though at the same time it comprises all classical tautologies expressed in a language including: $\neg$, $\wedge$, $\vee$.

We can expand this lattice through addition of condition (c1). For we are aware that the latter is independent on conditions (a1), (a2), (b1), (b2), and what is more — its addition introduces new laws. Accordingly, taking account of these conditions, we have a total of 32 logics; two of them are connexive, 28 are quasi-connexive, and 2 are non-connexive. At the bottom, it is worth noting that our two Boolean connexive logics:

1) one designated by conditions (a1), (a2), (b1), (b2), and 2) the other designated by conditions (a1), (a2), (b1), (b2), (c1), fulfil conditions (Ka), (Kb), but also (NS). Thus both are strong and properly connexive alike.

9. Tableau systems of Boolean connexive logics

Now, we shall outline the tableau approach to our logics. We will be governed here by a strategy adopted in [Jarmużek, 2013], which introduced a formalized tableau theory from some modal logics. Let us, however, disregard the formal concepts in favour of stressing the crucial points which determine the completeness of the tableau approach related to the semantically designated consequence relation.

To this end, we shall need a new language. A language of tableau proof. We extend the set of formulas $\mathbf{For_{CF}}$ with additional auxiliary expressions $A \,\mathsf{R}\, B$ and $A \,\widetilde{\mathsf{R}}\, B$, for arbitrary $A, B \in \mathbf{For_{CF}}$. We use a notation R instead of R on purpose, to differentiate the tableau language notation R from a relation R in a model. Intuitively, $A \,\mathsf{R}\, B$ means that the relating relation holds between A and B, while $A \,\widetilde{\mathsf{R}}\, B$ means that it does not. We denote the extended set by $\mathbf{Ex}$. Now, all tableau proofs are carried out in $\mathbf{Ex}$. As *a tableau inconsistent set of expressions* (that closes a given branch) we treat one comprising for some $A, B \in \mathbf{For_{CF}}$ at least one of the pairs: both A and $\neg A$ or both $A \,\mathsf{R}\, B$ and $A \,\widetilde{\mathsf{R}}\, B$.

Let us go to the tableau rules. For the formulas with main Boolean connectives, we shall assume the standard tableau rules. We do not need the enumerate or elaborate them as they have been thoroughly examined. Let us, however, bear in mind that the formulas do not include ones with a material implication. For the relating implication we assume the tableau rules introduced in [Jarmużek and Kaczkowski, 2014]:

$$(\to^{\mathrm{w}})\ \frac{A \to^{\mathrm{w}} B}{A \,\mathsf{R}\, B, \neg A \mid A \,\mathsf{R}\, B, B} \qquad\qquad (\neg\to^{\mathrm{w}})\ \frac{\neg(A \to^{\mathrm{w}} B)}{A, \neg B \mid A \,\widetilde{\mathsf{R}}\, B}$$

For the logics with conditions (a1) or (a2), we have rules:

$$(\mathrm{Ra1})\ \frac{A \,\mathsf{R} \neg A}{A \,\widetilde{\mathsf{R}} \neg A} \qquad\qquad (\mathrm{Ra2})\ \frac{\neg A \,\mathsf{R}\, A}{\neg A \,\widetilde{\mathsf{R}}\, A}$$

For the logics with condition (b1) we have two tableau rules:

$$(\mathrm{Rb1})\ \frac{A \,\mathsf{R}\, B}{A \,\widetilde{\mathsf{R}} \neg B} \qquad\qquad (\mathrm{Rb1'})\ \frac{(A \to^{\mathrm{w}} B) \,\widetilde{\mathsf{R}} \neg(A \to^{\mathrm{w}} \neg B)}{(A \to^{\mathrm{w}} B) \,\mathsf{R} \neg(A \to^{\mathrm{w}} \neg B)}$$

For the logics with condition (b2) we also have two rules:

$$\text{(Rb2)}\ \frac{A\,\mathsf{R}\,\neg B}{A\,\widetilde{\mathsf{R}}\,B} \qquad \text{(Rb2}'\text{)}\ \frac{(A \to^{\mathrm{w}} \neg B)\,\widetilde{\mathsf{R}}\,\neg(A \to^{\mathrm{w}} B)}{(A \to^{\mathrm{w}} \neg B)\,\mathsf{R}\,\neg(A \to^{\mathrm{w}} B)}$$

In fact, both (Rb1′) and (Rb2′) work in a similar way, since conditions (b1) and (b2) feature a common property: if $A\,R\,B$, then $A\,\widetilde{R}\,\neg B$ (see Definition 5.1). Hence, when dealing with a logic defined in this paper by conditions (b1) and (b2) we adopt it once.

Finally, we also have a rule for a logic defined by condition (c1):

$$\text{(Rc1)}\ \frac{\neg A\,\widetilde{\mathsf{R}}\,\neg B}{A\,\widetilde{\mathsf{R}}\,B}$$

For simplification, let us call the expressions in a tableau rule numerator *input*, while those in denominator *output*. Some rules, e.g., ($\to^{\mathrm{w}}$), ($\neg\!\to^{\mathrm{w}}$) and those for the Boolean connectives may have more than one output.

Let us now introduce a concept which is important for the tableau issues, which is in a certain sense extension of the concept of truthness in a model from the formulas on all expressions from **Ex**.

Definition 9.1 (Model suitable to a set of expressions). Let $\langle v, R\rangle$ be a model for $\mathbf{For_{CF}}$ and $X \subseteq \mathbf{Ex}$. Model $\langle v, R\rangle$ is *suitable to* X iff for all $A, B \in \mathbf{For_{CF}}$:

- if $A \in X$, then $\langle v, R\rangle \models A$
- if $A\,\mathsf{R}\,B \in X$, then $A\,R\,B$
- if $A\,\widetilde{\mathsf{R}}\,B \in X$, then $A\,\widetilde{R}\,B$.

Making use of the provided concept of a suitable model and conducting an inspection of the provided tableau rules, we are able to demonstrate that if a model $\langle v, R\rangle$ of given type, fulfilling some of conditions (a1), (a2), (b1), (b2), (c1), is suitable for a set of expressions $X \subseteq \mathbf{Ex}$, then application of a selected tableau rule relevant for conditions (a1), (a2), (b1), (b2), (c1) extends X to add expressions for which $\langle v, R\rangle$ is still suitable.

For convenience with formulation of the further theorems, let us introduce a function f from $\{(\mathrm{a1}), (\mathrm{a2}), (\mathrm{b1}), (\mathrm{b2}), (\mathrm{c1}), \emptyset\}$ into to the power set of $\{(\mathrm{Ra1}), (\mathrm{Ra2}), (\mathrm{Rb1}), (\mathrm{Rb1}'), (\mathrm{Rb2}), (\mathrm{Rb2}'), (\mathrm{Rc1})\}$, which to each condition assigns corresponding tableau rules: $f(\emptyset) := \emptyset$, $f(\mathrm{a1}) := \{(\mathrm{Ra1})\}$, $f(\mathrm{a2}) := \{(\mathrm{Ra2})\}$, $f(\mathrm{b1}) := \{(\mathrm{Rb1}), (\mathrm{Rb1}')\}$, $f(\mathrm{b2}) := \{(\mathrm{Rb2}), (\mathrm{Rb2}')\}$, $f(\mathrm{c1}) := \{(\mathrm{Rc1})\}$. Let us now phrase a proposition.

PROPOSITION 9.1 (Rules sound to model). *Let*:

- $\langle v, R\rangle$ *be a model for* $\mathbf{For_{CF}}$,
- $\langle v, R\rangle$ *be defined by a subset* W *of* $\{(\mathrm{a1}), (\mathrm{a2}), (\mathrm{b1}), (\mathrm{b2}), (\mathrm{c1})\}$,
- $X \subseteq \mathbf{Ex}$,
- $\langle v, R\rangle$ *be suitable to* X.

If some of the tableau rules that belong to:

1. *a set of tableau rules for the Boolean connectives,*
2. $\{(\rightarrow^{\mathrm{w}}), (\neg\!\rightarrow^{\mathrm{w}})\}$,
3. $\bigcup f(W)$

has been applied to X, *then* $\langle v, R\rangle$ *is suitable for at least one output obtained through application of this rule.*

PROOF. Let us make the above assumptions for a model. The proposition for the tableau rules for the Boolean connectives is true, thus for 1 [see Jarmużek, 2013].

In turn, the thesis for $\{(\rightarrow^{\mathrm{w}}), (\neg\!\rightarrow^{\mathrm{w}})\}$ follows from Definition 4.1 for $\mathbf{For_{CF}}$, which was also demonstrated in [Jarmużek and Kaczkowski, 2014], thus the proposition thesis occurs for 2 as well.

Finally, the thesis also occurs for specific rules (Ra1), (Ra2), (Rb1), (Rb1′), (Rb2), (Rb2′), (Rc1). Only (Rb1), (Rb2) and (Rc1) can be applied to X, if there exists for them a suitable model $\langle v, R\rangle$, and then this model fulfils conditions (b1), (b2) or (c1) respectively, hence also for the outputs the model is suitable. The other rules are inapplicable in these instances as they would contradict the assumption. For instance, if X comprised expression $A \,\mathsf{R}\, \neg A$, then the model could not be suitable for X, if it meets condition (a1). $\square$

The proof of completeness of our tableau methods in relation to the presented semantics still requires a converse proposition in a sense. Let us introduce a concept of model produced by a set of expressions.

DEFINITION 9.2 (Model generated by a branch). Let $X \subseteq \mathbf{Ex}$. The set $\mathrm{AT}(X)$ is defined as follows: $x \in \mathrm{AT}(X)$ iff either $x \in X \cap \{A \,\mathsf{R}\, B : A, B \in \mathbf{For_{CF}}\}$ or $x \in X \cap \mathbf{Var}$. Model $\langle v, R\rangle$ is *generated by* X iff

- for all $A, B \in \mathbf{For_{CF}}$: $A \,R\, B$ iff $A \,\mathsf{R}\, B \in \mathrm{AT}(X)$
- for any $x \in \mathbf{Var}$: $v(x) = 1$ iff $x \in \mathrm{AT}(X)$.

Assume we have a set of tableau rules that comprises:

1. a set of tableau rules for the Boolean connectives,
2. $\{(\rightarrow^{\mathrm{w}}), (\neg\rightarrow^{\mathrm{w}})\}$,
3. a set of tableau rules $\bigcup f(W)$ specified by a given subset W of $\{$(a1), (a2), (b1), (b2), (c1)$\}$.

If now we take a set of expressions $X \subseteq \mathbf{Ex}$ such that:

(i) it is closed under all of those rules — for each expression from X to which one of the rules is applicable, there exists at least one output in X,
(ii) X is not a tableau inconsistent a set of expressions.

then there exists a model $\langle v, R\rangle$ produced by that set. Therefore, we have one more proposition. Suppose that TR is a set of tableau rules exclusively comprised of tableau rules for the Boolean connectives and rules $(\rightarrow^{\mathrm{w}})$ and $(\neg\rightarrow^{\mathrm{w}})$.

Proposition 9.2 (Model sound to rules). *Let*:

- $X \subseteq \mathbf{Ex}$,
- *X not be a tableau inconsistent set of expressions,*
- *X be closed under* $\mathsf{TR} \cup \bigcup f(W)$, *for some subset W of* $\{$(a1), (a2), (b1), (b2), (c1)$\}$.

Then there exists a model $\langle v, R\rangle$ such that:

1. *for each formula $A \in X \cap \mathbf{For_{CF}}$, $\langle v, R\rangle \models A$,*
2. *$\langle v, R\rangle$ meets the set W.*

Proof. Let us make all the above assumptions. We know that X generates a model.

The first part of the proof is of the inductive nature due to the constitution of formulas and expressions through examination whether the tableau rules from the set $\mathsf{TR} \cup \bigcup f(W)$ introduce expressions that are sufficient for constitution of a model. For the rules from TR it is self-explanatory. The Boolean rules were examined in [Jarmużek, 2013], whereas the rules for the relating implication and its negation were examined in [Jarmużek and Kaczkowski, 2014], by virtue of Definition 4.1 they introduce elements that are sufficient for construction of a verification model. In turn, the remaining tableau rules are negative in nature. Rules (Ra1), (Ra2), (Rb1), (Rb1′), (Rb2), (Rb2′), (Rc1) are meant to close branches within proofs rather that to validate the verification formulas. Some of them: (Ra1), (Ra2), (Rb1′), (Rb2′), were not even applied to the expressions from X as X is not a tableau inconsistent set of expressions.

In the subsequent part, the proof relies on the fact that conditions (a1), (a2), (b1), (b2), (c1) are — as we well know — independent Theorem 7.1, thus we close the relation in a model $\langle v, R\rangle$ under conditions from W and obtain a model $\langle v, R'\rangle$ which meets W. In this model, all formulas that are true in $\langle v, R\rangle$ are true as well. □

Finally, we have a theorem on the completeness of tableaux and relating semantics for the discussed connexive models.

Theorem 9.3 (Completeness theorem). *Let* $W \subseteq \{(\mathrm{a1}), (\mathrm{a2}), (\mathrm{b1}), (\mathrm{b2}), (\mathrm{c1})\}$ *and* $\models\ \subseteq \wp(\mathbf{For_{CF}}) \times \mathbf{For_{CF}}$ *be the consequence relation defined by the set of models designated by* W*. Then for all* $X \subseteq \mathbf{For_{CF}}$ *and* $A \in \mathbf{For_{CF}}$ *the following facts are equivalent*:

(1) $X \models A$.
(2) *There exists a finite subset* $Y \subseteq X$ *such that each closure of* $Y \cup \{\neg A\}$ *under the set of tableau rules* $\mathsf{TR} \cup \bigcup f(W)$ *is a tableau inconsistent set of expressions.*

Proof. Let us adopt the assumptions. In the theorem proof, we make use of the prior propositions. For the implication “(1) ⇒ (2)” Proposition 9.2 is sufficient. In turn, for the implication “(2) ⇒ (1)” Proposition 9.1 is sufficient. □

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