Logic and **L**ogical **P**hilosophy
Volume 28 (2019), 481–511
DOI: 10.12775/LLP.2019.002

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BI-CLASSICAL CONNEXIVE LOGIC AND ITS MODAL EXTENSION: Cut-elimination, completeness and duality

Abstract. In this study, a new paraconsistent four-valued logic called bi-classical connexive logic (BCC) is introduced as a Gentzen-type sequent calculus. Cut-elimination and completeness theorems for BCC are proved, and it is shown to be decidable. Duality property for BCC is demonstrated as its characteristic property. This property does not hold for typical paraconsistent logics with an implication connective. The same results as those for BCC are also obtained for MBCC, a modal extension of BCC.

Keywords: Bi-classical connexive logic; sequent calculus; cut-elimination; duality; completeness

1. Introduction

The aim of this study is to develop a natural and technically advanced integration (or extension) of the following well-studied and philosophically plausible non-classical logics: *bi-intuitionistic logic* [24, 25, 26, 32], which is an extended intuitionistic logic with co-implication, and *connexive logics* [2, 17, 30, 33], which are extensions of *first-degree entailment* (FDE) or *Belnap and Dunn's useful four-valued logic* [3, 4, 6]. The development of such a natural integration is required to merge and integrate the research areas of these logics that have been studied independently till now. Such a technically advanced integration with some technically good properties, such as cut-elimination, decidability, and duality, is required to deeply analyze these logics and apply them to the field of computer science. For example, if such an integrated logic (or extended logic) has

Special Issue: Advances in Connexive Logic. Guest Editors: Hitoshi Omori and Heinrich Wansing
 Published online January 27, 2019

the properties as those mentioned above, then the logic would be useful for realizing and implementing an efficient reasoning mechanism with a theorem prover and/or a logic programming language.

However, integrating bi-intuitionistic and connexive logics implies serious problems for the cut-elimination and duality properties because constructing a cut-free Gentzen-type sequent calculus for bi-intuitionistic logic is known to be difficult and the duality property does not hold for the existing connexive logics. To address these problems, we adopt a classical version of the bi-intuitionistic logic called *bi-classical logic* (BC for short) as a base logic. We also introduce the characteristic logical inference rules that correspond to the connexive logic axioms $\sim(\alpha \to \beta) \leftrightarrow \alpha \to \sim\beta$ and $\sim(\alpha \leftarrow \beta) \leftrightarrow \sim\alpha \leftarrow \beta$, where $\to$, $\leftarrow$, and $\sim$ are implication, co-implication and paraconsistent negation connectives, respectively. On the basis of these settings, we can obtain a cut-free Gentzen-type sequent calculus for the natural integration (or extension) of bi-intuitionistic logic and basic connexive logic. For the calculus in this study, we show cut-elimination and completeness theorems along with other good properties, such as decidability and duality.

Thus, in this study, a new paraconsistent four-valued logic called *bi-classical connexive logic* (BCC for short) is introduced as a Gentzen-type sequent calculus with the settings explained above. The cut-elimination theorem for BCC is shown. The duality property for BCC is shown as a characteristic property of this logic. Moreover, the completeness theorem with respect to double valuation semantics is proved for BCC, and this logic is demonstrated to be decidable and paraconsistent. These results are proved using several theorems for syntactically and semantically embedding BCC into BC. The same results as those obtained for BCC are shown for the S4-type modal extension MBCC of BCC.

Some studies closely related to bi-intuitionistic logic, connexive logics and their neighbors and extensions are discussed below.

Bi-intuitionistic logic, also called *Heyting-Brouwer logic* was originally introduced by Rauszer [24, 25, 26]. The bi-intuitionistic logic has a faithful embedding in the future-past tense logic KtT4 [15]. A modal logic based on this logic was studied in [16]. The original Gentzen-type sequent calculus for the bi-intuitionistic logic by Rauszer [24] does not possess the cut-elimination property [22]. Some non-Gentzen-type sequent calculi for the bi-intuitionistic logic have been proposed by several researchers (e.g., [5, 31, 23, 22]). For a comparison of these sequent calculi, see [23, 22]. A restricted version RBL of a Gentzen-type sequent calculus for the bi-intuitionistic logic was introduced in [10], and the cut-elimination theorem for RBL was demonstrated. However, the Kripke completeness theorem for RBL has not yet been obtained. An alternative bi-intuitionistic logic, 2Int, was proposed in [32] to combine the notions of verification and its dual.

Connexive logics are considered to be philosophically plausible paraconsistent logics [2, 17, 30, 33, 21]. Although the origins of connexive logics came from Aristotle and Boethius, some modern perspectives have been given by Angell [2] and McCall [17]. A *material connexive logic*, MC, which is an extension of positive classical logic, was introduced in [33], and an extension of MC by adding classical negation, called a *dialetheic Belnap–Dunn logic* (dBD), was introduced in [19]. The logic BCC is regarded as an extension of MC by adding co-implication. An *intuitionistic connexive modal logic* was introduced in [30] to extend a certain basic intuitionistic connexive logic, which is considered a variant of *Nelson's paraconsistent four-valued logic* [1, 18, 13]. The allure of *connexive mathematics* was explained in [7] from the viewpoint of philosophy and history. A connexive extension of the basic relevant logic, BD, was studied in [20]. Natural deduction systems for two versions of connexive logics were studied in [8]. A survey on connexive logics can be found in [33]. Comprehensive information on connexive logics can be found on the internet [21]. Some recent results on connexive logics can be found in the special issue on connexive logics in the *IfCoLog Journal of Logics and their Applications*, 3 (3), 2016.

Bi-intuitionistic connexive logic, which is an integration of bi-intuitionistic logic and intuitionistic connexive logic, was originally introduced by Wansing [31] as a cut-free display calculus. *Bi-connexive logic* known as 2C, which is a connexive variant of 2Int with connexive co-implication, has been introduced [34], wherein a two-sorted typed-λ-calculus for 2C was studied. A version of the bi-intuitionistic connexive logic, *connexive Heyting-Brouwer logic*, was studied in [14]. The Kripke completeness theorem for this logic was proved, but a cut-free Gentzen-type sequent calculus for this logic was not constructed. The cut-elimination theorem and duality property were also shown for certain proper subsystems of this logic. A restricted version, RBCL, of a Gentzen-type sequent calculus for the bi-intuitionistic connexive logic was introduced in [10] and the cut-elimination theorem for RBCL was shown. However, the completeness theorem for RBCL has not yet been obtained. Therefore, the logic BCC developed in this study has some technical advantages over these bi-intuitionistic connexive logics because completeness and cut-elimination theorems hold for the proposed logic.

A *classical connexive modal logic* called CS4, which is based on the positive fragment of the normal modal logic S4, was introduced in [12] as a cut-free Gentzen-type sequent calculus. The Kripke-completeness theorem for CS4 was shown, and it was found to be embeddable into the positive fragment of S4 as well as decidable. Moreover, it was shown in [12] that the basic constructive connexive logic C can be faithfully embedded into CS4 and into a subsystem of CS4 lacking syntactic duality between necessity and possibility. The logic MBCC proposed in this study is also considered as a modified and plausible extension of CS4 with the addition of the co-implication connective. Furthermore, the logic MBCC has an advantage over CS4 because the aforementioned natural characteristic property of duality hold for this logic.

The rest of this paper is organized as follows.

In Section 2, the logics BCC and BC are introduced as Gentzen-type sequent calculi in the standard classical logic setting, and the cut-elimination and decidability theorems for BCC are proved using a theorem for syntactically embedding BCC into BC.

In Section 3, a self-translation of BCC is introduced and the duality property of BCC is shown using this translation. The duality property holds for the implication-free fragment of classical logic but does not hold for some typical paraconsistent logics with an implication connective. Another self-translation of BCC is also introduced, and by using this translation, a new property of BCC, called quasi-symmetry property, is shown as another characteristic property.

In Section 4, the completeness theorem with respect to double valuation semantics is proved using two theorems for semantically and syntactically embedding BCC into BC.

In Section 5, the same results as those obtained for BCC are shown for the S4-type modal extension MBCC of BCC (i.e., the cut-elimination, Kripke-completeness, decidability, paraconsistency, duality and quasi-symmetry properties are shown for MBCC).

2. Cut-elimination and decidability

Formulas of *bi-classical connexive logic* are constructed from countably many propositional variables, $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (implication), $\leftarrow$ (co-implication) and $\sim$ (paraconsistent negation). Small letters $p, q, ...$ are used to denote propositional variables, Greek small letters $\alpha, \beta, ...$ are used to denote formulas, and Greek capital letters $\Gamma, \Delta, ...$ are used to represent finite (possibly empty) sets of formulas. The symbol $\equiv$ is used to denote the equality of symbols. A *sequent* is an expression of the form $\Gamma \Rightarrow \Delta$. An expression $L \vdash \Gamma \Rightarrow \Delta$ means that $\Gamma \Rightarrow \Delta$ is provable in a sequent calculus L. If L of $L \vdash S$ is clear from the context, we omit L in it. Two sequent calculi L_1 and L_2 are said to be *theorem-equivalent* if $\{S \mid L_1 \vdash S\} = \{S \mid L_2 \vdash S\}$. A rule R of inference is said to be *admissible* in a sequent calculus L if the following condition is satisfied: For any instance

$$\frac{S_1 \dots S_n}{S}$$

of R, if $L \vdash S_i$ for all i, then $L \vdash S$. Moreover, R is said to be *derivable* in L if there is a derivation from $S_1, \dots, S_n$ to S in L.

Prior to define Gentzen-type sequent calculi BCC and BC, we define the languages of them. These languages will be required to define some translations.

Definition 2.1. We fix a set Φ of propositional variables and define the set $\Phi' := \{p' \mid p \in \Phi\}$ of propositional variables. The language $\mathcal{L}_{\mathrm{BCC}}$ of BCC is obtained from Φ by $\wedge$, $\vee$, $\rightarrow$, $\leftarrow$ and $\sim$. The language $\mathcal{L}_{\mathrm{BC}}$ of BC is obtained from Φ and Φ' by $\wedge$, $\vee$, $\rightarrow$ and $\leftarrow$.

A Gentzen-type sequent calculus BCC for the bi-classical connexive logic is introduced below.

Definition 2.2 (BCC). The initial sequents of BCC are of the following form, for any propositional variable p in Φ:

$$p \Rightarrow p \qquad\qquad {\sim}p \Rightarrow {\sim}p.$$

The structural inference rules of BCC are of the form:

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \alpha, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi} \text{ (cut)}$$

$$\frac{\Gamma \Rightarrow \Delta}{\alpha, \Gamma \Rightarrow \Delta} \text{ (we-left)} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \alpha} \text{ (we-right).}$$

The positive logical inference rules of BCC are of the form:

$$\frac{\alpha, \beta, \Gamma \Rightarrow \Delta}{\alpha \wedge \beta, \Gamma \Rightarrow \Delta}\ (\wedge\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha \quad \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \wedge \beta}\ (\wedge\text{right})$$

$$\frac{\alpha, \Gamma \Rightarrow \Delta \quad \beta, \Gamma \Rightarrow \Delta}{\alpha \vee \beta, \Gamma \Rightarrow \Delta}\ (\vee\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha, \beta}{\Gamma \Rightarrow \Delta, \alpha \vee \beta}\ (\vee\text{right})$$

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \beta, \Sigma \Rightarrow \Pi}{\alpha \rightarrow \beta, \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (\rightarrow\text{left}) \qquad \frac{\alpha, \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \rightarrow \beta}\ (\rightarrow\text{right})$$

$$\frac{\alpha, \Gamma \Rightarrow \Delta, \beta}{\alpha \leftarrow \beta, \Gamma \Rightarrow \Delta}\ (\leftarrow\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha \quad \beta, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi, \alpha \leftarrow \beta}\ (\leftarrow\text{right}).$$

The negative logical inference rules of BCC are of the form:

$$\frac{\alpha, \Gamma \Rightarrow \Delta}{\sim\sim\alpha, \Gamma \Rightarrow \Delta}\ (\sim\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma \Rightarrow \Delta, \sim\sim\alpha}\ (\sim\text{right})$$

$$\frac{\sim\alpha, \Gamma \Rightarrow \Delta \quad \sim\beta, \Gamma \Rightarrow \Delta}{\sim(\alpha \wedge \beta), \Gamma \Rightarrow \Delta}\ (\sim\wedge\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \sim\alpha, \sim\beta}{\Gamma \Rightarrow \Delta, \sim(\alpha \wedge \beta)}\ (\sim\wedge\text{right})$$

$$\frac{\sim\alpha, \sim\beta, \Gamma \Rightarrow \Delta}{\sim(\alpha \vee \beta), \Gamma \Rightarrow \Delta}\ (\sim\vee\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \sim\alpha \quad \Gamma \Rightarrow \Delta, \sim\beta}{\Gamma \Rightarrow \Delta, \sim(\alpha \vee \beta)}\ (\sim\vee\text{right})$$

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \sim\beta, \Sigma \Rightarrow \Pi}{\sim(\alpha \rightarrow \beta), \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (\sim\rightarrow\text{left}) \qquad \frac{\alpha, \Gamma \Rightarrow \Delta, \sim\beta}{\Gamma \Rightarrow \Delta, \sim(\alpha \rightarrow \beta)}\ (\sim\rightarrow\text{right})$$

$$\frac{\sim\alpha, \Gamma \Rightarrow \Delta, \beta}{\sim(\alpha \leftarrow \beta), \Gamma \Rightarrow \Delta}\ (\sim\leftarrow\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \sim\alpha \quad \beta, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi, \sim(\alpha \leftarrow \beta)}\ (\sim\leftarrow\text{right}).$$

A Gentzen-type sequent calculus BC for the bi-classical logic is defined below.

DEFINITION 2.3 (BC). BC is defined based on $\mathcal{L}_{\mathrm{BC}}$. BC is the $\sim$-free part of BCC (i.e., it is obtained from BCC by deleting the negated initial sequents and the negative logical inference rules, and we use $\Phi \cup \Phi'$ as the domain of propositional variables, instead of Φ).

Remark 2.4. We make the following remarks on BC and BCC.

1. Let L be BC or BCC. Sequents of the form $\alpha \Rightarrow \alpha$ for any formula α are provable in cut-free L. This fact can be shown by induction on α.

2. The logical inference rules ($\sim\rightarrow$left), ($\sim\rightarrow$right), ($\sim\leftarrow$left) and ($\sim\leftarrow$right) in BCC just correspond to the following characteristic axiom schemes for connexive logics:

(a) $\sim(\alpha \to \beta) \leftrightarrow \alpha \to \sim\beta$,
(b) $\sim(\alpha \leftarrow \beta) \leftrightarrow \sim\alpha \leftarrow \beta$.

3. BC is theorem-equivalent to Gentzen's sequent calculus LK for classical logic when the language includes the classical negation connective $\neg$, since $\leftarrow$ in BC can be defined by $\alpha \leftarrow \beta := \alpha \wedge \neg\beta$.

4. Let $\mathrm{BC}_{\neg}$ be the system which is obtained from BC by adding the following logical inference rules for $\neg$:

$$\frac{\Gamma \Rightarrow \Delta, \alpha}{\neg\alpha, \Gamma \Rightarrow \Delta}\ (\neg\text{left}) \qquad \frac{\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\alpha}\ (\neg\text{right}).$$

Then, we can prove the cut-free derivability of ($\leftarrow$left) and ($\leftarrow$right) with the definition $\alpha \leftarrow \beta := \alpha \wedge \neg\beta$ as follows:

$$\dfrac{\dfrac{\alpha, \Gamma \Rightarrow \Delta, \beta}{\alpha, \neg\beta, \Gamma \Rightarrow \Delta}\ (\neg\text{left})}{\alpha \wedge \neg\beta, \Gamma \Rightarrow \Delta}\ (\wedge\text{left})$$

$$\dfrac{\begin{array}{c}\Gamma \Rightarrow \Delta, \alpha \\ \vdots\ (\text{we-left}), (\text{we-right}) \\ \Gamma, \Sigma \Rightarrow \Delta, \Pi, \alpha\end{array} \qquad \dfrac{\begin{array}{c}\beta, \Sigma \Rightarrow \Pi \\ \vdots\ (\text{we-left}), (\text{we-right}) \\ \beta, \Gamma, \Sigma \Rightarrow \Delta, \Pi\end{array}}{\Gamma, \Sigma \Rightarrow \Delta, \Pi, \neg\beta}\ (\neg\text{right})}{\Gamma, \Sigma \Rightarrow \Delta, \Pi, \alpha \wedge \neg\beta}\ (\wedge\text{right}).$$

5. By the fact mentioned just above, we can obtain the fact that cut-free BC and cut-free LK are theorem-equivalent. By this fact and the cut-elimination theorem for LK, we can also obtain the cut-elimination theorem for $\mathrm{BC}_{\neg}$. By this cut-elimination theorem, we can obtain the fact that $\mathrm{BC}_{\neg}$ is a conservative extension of BC. By this conservative extension result and the cut-free equivalence between $\mathrm{BC}_{\neg}$ and LK, we can obtain the cut-elimination theorem for BC. We can also obtain the fact that BC is decidable.

6. As mentioned just above, the cut-elimination theorem for BC holds. But, the same theorem does not hold for the intuitionistic version of BC (i.e., a Gentzen-type sequent calculus for bi-intuitionistic logic) [22].

7. A counterexample of the failure of the cut-elimination theorem for a Gentzen-type sequent calculus for bi-intuitionistic logic was presented in [22] as

$$p \Rightarrow q, r \to ((p \leftarrow q) \wedge r)$$

where p, q, and r are distinct propositional variables. This sequent is provable in cut-free BC by:

$$
\dfrac{\dfrac{\dfrac{p \Rightarrow p \quad \dfrac{q \Rightarrow q}{q, r \Rightarrow q}\,(\text{we-left})}{p, r \Rightarrow q, p \leftarrow q}\,(\leftarrow \text{right}) \quad \begin{array}{c} r \Rightarrow r \\ \vdots\ (\text{we-left}), (\text{we-right}) \\ p, r \Rightarrow q, r \end{array}}{p, r \Rightarrow q, (p \leftarrow q) \wedge r}\,(\wedge\text{right})}{p \Rightarrow q, r \rightarrow ((p \leftarrow q) \wedge r)}\,(\rightarrow\text{right})
$$

where ($\rightarrow$right) cannot be applied when BC is replaced with a Gentzen-type sequent calculus for bi-intuitionistic logic.

8. Cut-elimination and decidability theorems for some extended versions of BC were shown in [11].

Next, we introduce a translation of BCC into BC, and by using this translation, we show a theorem for syntactically embedding BCC into BC.

DEFINITION 2.5. A mapping f from $\mathcal{L}_{\mathrm{BCC}}$ to $\mathcal{L}_{\mathrm{BC}}$ is defined inductively by:

1. for any $p \in \Phi$, $f(p) := p$ and $f({\sim} p) := p' \in \Phi'$,
2. $f(\alpha \sharp \beta) := f(\alpha) \sharp f(\beta)$ with $\sharp \in \{\wedge, \vee, \rightarrow, \leftarrow\}$,
3. $f({\sim}{\sim} \alpha) := f(\alpha)$,
4. $f({\sim}(\alpha \wedge \beta)) := f({\sim} \alpha) \vee f({\sim} \beta)$,
5. $f({\sim}(\alpha \vee \beta)) := f({\sim} \alpha) \wedge f({\sim} \beta)$,
6. $f({\sim}(\alpha \rightarrow \beta)) := f(\alpha) \rightarrow f({\sim} \beta)$,
7. $f({\sim}(\alpha \leftarrow \beta)) := f({\sim} \alpha) \leftarrow f(\beta)$.

An expression $f(\Gamma)$ denotes the result of replacing every occurrence of a formula α in Γ by an occurrence of $f(\alpha)$. Analogous notion is used for the other mappings discussed later.

Remark 2.6. A similar translation as defined in Definition 2.5 has been used by Gurevich [9], Rautenberg [27] and Vorob'ev [29] to embed Nelson's constructive logic [1, 18] into the positive intuitionistic logic.

THEOREM 2.7 (Syntactical embedding from BCC into BC). *Let Γ, Δ be sets of formulas in $\mathcal{L}_{\mathrm{BCC}}$, and f be the mapping defined in Definition 2.5.*

1. BCC $\vdash \Gamma \Rightarrow \Delta$ *iff* BC $\vdash f(\Gamma) \Rightarrow f(\Delta)$.
2. BCC $-$ (cut) $\vdash \Gamma \Rightarrow \Delta$ *iff* BC $-$ (cut) $\vdash f(\Gamma) \Rightarrow f(\Delta)$.

Proof. We show only 1.

"⇒" By induction on the proofs P of $\Gamma \Rightarrow \Delta$ in BCC. We distinguish the cases according to the last inference of P, and show some cases.

1. The case ($\sim p \Rightarrow \sim p$): The last inference of P is of the form: $\sim p \Rightarrow \sim p$ for any $p \in \Phi$. In this case, we obtain BC $\vdash f(\sim p) \Rightarrow f(\sim p)$, i.e., BC $\vdash p' \Rightarrow p'$ ($p' \in \Phi'$), by the definition of f.

2. The case ($\sim\!\to$left): The last inference of P is of the form:

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \sim \beta, \Sigma \Rightarrow \Pi}{\sim(\alpha \to \beta), \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (\sim\!\to\!\text{left}).$$

By induction hypothesis, we have BC $\vdash f(\Gamma) \Rightarrow f(\Delta), f(\alpha)$ and BC $\vdash$ $f(\sim \beta)$, $f(\Sigma) \Rightarrow f(\Pi)$. Then, we obtain the required fact:

$$\frac{\begin{array}{c}\vdots \\ f(\Gamma) \Rightarrow f(\Delta), f(\alpha)\end{array} \quad \begin{array}{c}\vdots \\ f(\sim \beta), f(\Sigma) \Rightarrow f(\Pi)\end{array}}{f(\alpha) \to f(\sim \beta), f(\Gamma), f(\Sigma) \Rightarrow f(\Delta), f(\Pi)}\ (\to\!\text{left})$$

where $f(\alpha) \to f(\sim \beta)$ coincides with $f(\sim(\alpha \to \beta))$ by the definition of f.

3. The case ($\sim\!\to$right): The last inference of P is of the form:

$$\frac{\alpha, \Gamma \Rightarrow \Delta, \sim \beta}{\Gamma \Rightarrow \Delta, \sim(\alpha \to \beta)}\ (\sim\!\to\!\text{right}).$$

By induction hypothesis: BC $\vdash f(\alpha), f(\Gamma) \Rightarrow f(\Delta), f(\sim \beta)$. Then, we obtain the required fact:

$$\frac{\begin{array}{c}\vdots \\ f(\alpha), f(\Gamma) \Rightarrow f(\Delta), f(\sim \beta)\end{array}}{f(\Gamma) \Rightarrow f(\Delta), f(\alpha) \to f(\sim \beta)}\ (\to\!\text{right})$$

where $f(\alpha) \to f(\sim \beta)$ coincides with $f(\sim(\alpha \to \beta))$ by the definition of f.

4. The case ($\sim\sim$left): The last inference of P is of the form:

$$\frac{\alpha, \Gamma \Rightarrow \Delta}{\sim \sim \alpha, \Gamma \Rightarrow \Delta}\ (\sim\sim\!\text{left}).$$

By induction hypothesis, we have BC $\vdash f(\alpha), f(\Gamma) \Rightarrow f(\Delta)$, where $f(\alpha)$ coincides with $f(\sim \sim \alpha)$ by the definition of f.

5. The case (cut): The last inference of P is of the form:

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \alpha, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (\text{cut}).$$

By induction hypothesis, we have BC $\vdash f(\Gamma) \Rightarrow f(\Delta), f(\alpha)$ and BC $\vdash f(\alpha), f(\Sigma) \Rightarrow f(\Pi)$. Then, we obtain the required fact:

$$\frac{\begin{matrix}\vdots \\ f(\Gamma) \Rightarrow f(\Delta), f(\alpha)\end{matrix} \quad \begin{matrix}\vdots \\ f(\alpha), f(\Sigma) \Rightarrow f(\Pi)\end{matrix}}{f(\Gamma), f(\Sigma) \Rightarrow f(\Delta), f(\Pi)}\ (\text{cut}).$$

"$\Leftarrow$" By induction on the proofs Q of $f(\Gamma) \Rightarrow f(\Delta)$ in BC. We distinguish the cases according to the last inference of Q.

In such cases, we must consider the cases concerning the double-negation condition $f({\sim}{\sim}\alpha) := f(\alpha)$. Indeed, there are infinitely many cases concerning the conditions, although these cases can be proved easily. Thus, we first explain some typical examples of such easily provable, but non-trivial cases.

The first example is the initial sequent cases. If Q is an initial sequent $p \Rightarrow p$, then we must consider the cases $f({\sim}^n p) \Rightarrow f({\sim}^m p)$ where n and m are even natural numbers and ${\sim}^n p$ represents $\overbrace{{\sim} \cdots {\sim}}^{n}\alpha$. These cases are, of course, easily provable, and the proof is almost the same as that of the case $f(p) \Rightarrow f(p)$ (i.e., in the case $f(p) \Rightarrow f(p)$, we do not have to use the double-negation condition on f, but in the cases $f({\sim}^n p) \Rightarrow f({\sim}^m p)$ $(0 < n, m)$, we need to use the double-negation condition on f). Thus, in what follows, we would like to focus only on the case $f(p) \Rightarrow f(p)$ as the most simplest case (but we will not show this case, since it is obvious).

The second example is the cases for the logical inference rules. For example, we can consider the following case. The last inference of Q is of the form:

$$\frac{f(\Gamma) \Rightarrow f(\Delta), f(\alpha) \quad f(\beta), f(\Sigma) \Rightarrow f(\Pi)}{f({\sim}{\sim}(\alpha \to \beta), f({\sim}{\sim}\Gamma), f(\Sigma)) \Rightarrow f(\Delta), f(\Pi)}\ ({\to}\text{left})$$

where $f({\sim}{\sim}(\alpha \to \beta))$ and $f({\sim}{\sim}\Gamma)$ respectively coincide with $f(\alpha \to \beta)$ and $f(\Gamma)$ by the double-negation condition on f. Of course, we can prove this case in a similar way as for the following most simplest case, which does not use the double-negation condition on f: The last inference of Q is of the form:

$$\frac{f(\Gamma) \Rightarrow f(\Delta), f(\alpha) \quad f(\beta), f(\Sigma) \Rightarrow f(\Pi)}{f(\alpha \to \beta), f(\Gamma), f(\Sigma) \Rightarrow f(\Delta), f(\Pi)} \ (\to\text{left}).$$

We would also like to focus only on the most simplest case (this case will be proved).

Therefore, from now on, we consider only the most simplest cases without using the double-negation condition on f.

We show some these cases in the following.

1. The case (cut): The last inference of Q is of the form:

$$\frac{\begin{matrix}\vdots\\ f(\Gamma) \Rightarrow f(\Delta), \beta\end{matrix} \quad \begin{matrix}\vdots\\ \beta, f(\Sigma) \Rightarrow f(\Pi)\end{matrix}}{f(\Gamma), f(\Sigma) \Rightarrow f(\Delta), f(\Pi)} \ (\text{cut}).$$

In this case, β is a formula of BC. We then have the fact $\gamma = f(\gamma)$ for any formula γ in BC. This can be shown by induction on γ. Thus, Q is of the form:

$$\frac{\begin{matrix}\vdots\\ f(\Gamma) \Rightarrow f(\Delta), f(\beta)\end{matrix} \quad \begin{matrix}\vdots\\ f(\beta), f(\Sigma) \Rightarrow f(\Pi)\end{matrix}}{f(\Gamma), f(\Sigma) \Rightarrow f(\Delta), f(\Pi)} \ (\text{cut}).$$

By induction hypothesis, we have BCC $\vdash \Gamma \Rightarrow \Delta, \beta$ and BCC $\vdash \beta, \Sigma \Rightarrow \Pi$. Then, we obtain the required fact:

$$\frac{\begin{matrix}\vdots\\ \Gamma \Rightarrow \Delta, \beta\end{matrix} \quad \begin{matrix}\vdots\\ \beta, \Sigma \Rightarrow \Pi\end{matrix}}{\Gamma, \Sigma \Rightarrow \Delta, \Pi} \ (\text{cut}).$$

2. The case ($\to$left): The last inference of Q is ($\wedge$left).

(a) The last inference of Q is of the form:

$$\frac{f(\Gamma) \Rightarrow f(\Delta), f(\alpha) \quad f(\beta), f(\Sigma) \Rightarrow f(\Pi)}{f(\alpha \to \beta), f(\Gamma), f(\Sigma) \Rightarrow f(\Delta), f(\Pi)} \ (\to\text{left})$$

where $f(\alpha \to \beta)$ coincides with $f(\alpha) \to f(\beta)$ by the definition of f. By induction hypothesis, we have BCC $\vdash \Gamma \Rightarrow \Delta, \alpha$ and BCC $\vdash \beta, \Sigma \Rightarrow \Pi$. We thus obtain the required fact:

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \Delta, \alpha\end{array} \quad \begin{array}{c}\vdots\\ \beta, \Sigma \Rightarrow \Pi\end{array}}{\alpha \to \beta, \Gamma, \Sigma \Rightarrow \Delta, \Pi} \ (\to\text{left}).$$

(b) The last inference of Q is of the form:

$$\dfrac{f(\Gamma) \Rightarrow f(\Delta), f(\alpha) \quad f({\sim}\beta), f(\Sigma) \Rightarrow f(\Pi)}{f({\sim}(\alpha \to \beta)), f(\Gamma), f(\Sigma) \Rightarrow f(\Delta), f(\Pi)} \ (\to\text{left})$$

where $f({\sim}(\alpha \to \beta))$ coincides with $f(\alpha) \to f({\sim}\beta)$ by the definition of f. By induction hypothesis, we have BCC $\vdash \Gamma \Rightarrow \Delta, \alpha$ and BCC $\vdash {\sim}\beta, \Sigma \Rightarrow \Pi$. We thus obtain the required fact:

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \Delta, \alpha\end{array} \quad \begin{array}{c}\vdots\\ {\sim}\beta, \Sigma \Rightarrow \Pi\end{array}}{{\sim}(\alpha \to \beta), \Gamma, \Sigma \Rightarrow \Delta, \Pi} \ ({\sim}{\to}\text{left})$$ □

Using Theorem 2.7, we can obtain the cut-elimination theorem for BCC.

THEOREM 2.8 (Cut-elimination for BCC). *The rule* (cut) *is admissible in cut-free* BCC.

PROOF. Suppose that BCC $\vdash \Gamma \Rightarrow \Delta$. Then BC $\vdash f(\Gamma) \Rightarrow f(\Delta)$, by Theorem 2.7(1), and hence BC $-$ (cut) $\vdash f(\Gamma) \Rightarrow f(\Delta)$, by the cut-elimination theorem for BC. Then, by Theorem 2.7(2), we obtain BCC $-$ (cut) $\vdash \Gamma \Rightarrow \Delta$. □

Using Theorem 2.7, we can also obtain the decidability of BCC.

THEOREM 2.9 (Decidability for BCC). BCC *is decidable.*

PROOF. By decidability of BC (i.e., LK) for each α, it is possible to decide if $\Rightarrow f(\alpha)$ is provable in BC. Then, by Theorem 2.7, BCC is also decidable. □

Using Theorem 2.8, we can show the paraconsistency of BCC with respect to $\sim$.

Definition 2.10. Let $\sharp$ be a negation connective. A sequent calculus L is called *explosive* with respect to $\sharp$ if for any formulas α and β, the sequent $\alpha, \sharp\alpha \Rightarrow \beta$ is provable in L. It is called *paraconsistent* with respect to $\sharp$ if it is not explosive with respect to $\sharp$.

Theorem 2.11 (Paraconsistency for BCC). BCC *is paraconsistent with respect to* $\sim$.

Proof. Consider a sequent $p, {\sim} p \Rightarrow q$ where p and q are distinct propositional variables. Then, the unprovability of this sequent is guaranteed by Theorem 2.8, since there is no cut-free proof of it in BCC. □

3. Duality and quasi-symmetry

First, we introduce a self-translation of BCC, and by using this translation, we show the duality property of BCC.

Definition 3.1. A mapping f from $\mathcal{L}_{\mathrm{BCC}}$ to $\mathcal{L}_{\mathrm{BCC}}$ is defined inductively by:

1. $f(p) := p$ for any $p \in \Phi$,
2. $f(\alpha \wedge \beta) := f(\alpha) \vee f(\beta)$,
3. $f(\alpha \vee \beta) := f(\alpha) \wedge f(\beta)$,
4. $f(\alpha \to \beta) := f(\beta) \leftarrow f(\alpha)$,
5. $f(\alpha \leftarrow \beta) := f(\beta) \to f(\alpha)$,
6. $f({\sim} \alpha) := {\sim} f(\alpha)$.

Proposition 3.2. *Let f be the mapping defined in Definition 3.1. Then, we have: $ff(\alpha) = \alpha$ for any formula α in $\mathcal{L}_{\mathrm{BCC}}$.*

Proof. By induction on α. We show only the case for $\alpha \equiv \beta \to \gamma$ as follows. $ff(\beta \to \gamma) = f(f(\gamma) \leftarrow f(\beta)) = ff(\beta) \to ff(\gamma) = \beta \to \gamma$ (by induction hypothesis). □

Remark 3.3. We note that we have a more general result by using Proposition 3.2. Suppose that f is the mapping defined in Definition 3.1. Then, we have: BCC $\vdash \Gamma \Rightarrow \Delta$ iff BCC $\vdash ff(\Gamma) \Rightarrow ff(\Delta)$.

The following theorem shows the duality property for BCC.

Theorem 3.4 (Duality for BCC). *Let Γ and Δ be (possibly empty) sets of formulas in $\mathcal{L}_{\mathrm{BCC}}$, and f be the mapping defined in Definition 3.1.*

1. BCC $\vdash \Gamma \Rightarrow \Delta$ *iff* BCC $\vdash f(\Delta) \Rightarrow f(\Gamma)$,
2. BCC $-$ (cut) $\vdash \Gamma \Rightarrow \Delta$ *iff* BCC $-$ (cut) $\vdash f(\Delta) \Rightarrow f(\Gamma)$.

PROOF. We show only 1.

"$\Rightarrow$" By induction on the proofs P of $\Gamma \Rightarrow \Delta$ in BCC. We distinguish the cases according to the last inference of P, and show some cases.

1. The case ($\rightarrow$right): The last inference of P is of the form:

$$\frac{\alpha, \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \rightarrow \beta}\ (\rightarrow\text{right}).$$

By induction hypothesis, BCC $\vdash f(\beta), f(\Delta) \Rightarrow f(\Gamma), f(\alpha)$. Then, we obtain the required fact:

$$\frac{\begin{array}{c}\vdots\\ f(\beta), f(\Delta) \Rightarrow f(\Gamma), f(\alpha)\end{array}}{f(\beta) \leftarrow f(\alpha), f(\Delta) \Rightarrow f(\Gamma)}\ (\leftarrow\text{left})$$

where $f(\beta) \leftarrow f(\alpha)$ coincides with $f(\alpha \rightarrow \beta)$ by the definition of f.

2. The case ($\rightarrow$left): The last inference of P is of the form:

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \beta, \Sigma \Rightarrow \Pi}{\alpha \rightarrow \beta, \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (\rightarrow\text{left}).$$

By induction hypothesis, we have BCC $\vdash f(\alpha), f(\Delta) \Rightarrow f(\Gamma)$ and BCC $\vdash f(\Pi) \Rightarrow f(\Sigma), f(\beta)$. Then, we obtain the required fact:

$$\frac{\begin{array}{c}\vdots\\ f(\Pi) \Rightarrow f(\Sigma), f(\beta)\end{array} \quad \begin{array}{c}\vdots\\ f(\alpha), f(\Delta) \Rightarrow f(\Gamma)\end{array}}{f(\Pi), f(\Delta) \Rightarrow f(\Sigma), f(\Gamma), f(\beta) \leftarrow f(\alpha)}\ (\leftarrow\text{right})$$

where $f(\beta) \leftarrow f(\alpha)$ coincides with $f(\alpha \rightarrow \beta)$ by the definition of f.

3. The case ($\sim\rightarrow$right): The last inference of P is of the form:

$$\frac{\alpha, \Gamma \Rightarrow \Delta, \sim\beta}{\Gamma \Rightarrow \Delta, \sim(\alpha \rightarrow \beta)}\ (\sim\rightarrow\text{right}).$$

By induction hypothesis, we have BCC $\vdash f(\sim\beta), f(\Delta) \Rightarrow f(\Gamma), f(\alpha)$ where $f(\sim\beta)$ coincides with $\sim f(\beta)$ by the definition of f. Then, we obtain the required fact:

$$\frac{\begin{array}{c}\vdots\\ \sim f(\beta), f(\Delta) \Rightarrow f(\Gamma), f(\alpha)\end{array}}{\sim(f(\beta) \leftarrow f(\alpha)), f(\Delta) \Rightarrow f(\Gamma)}\ (\sim\leftarrow\text{left})$$

where $\sim(f(\beta) \leftarrow f(\alpha))$ coincides with $f(\sim(\alpha \rightarrow \beta))$ by the definition of f.

4. The case (∼→left): The last inference of P is of the form:

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad {\sim}\beta, \Sigma \Rightarrow \Pi}{{\sim}(\alpha \to \beta), \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ ({\sim}{\to}\text{left}).$$

By induction hypothesis, we have BCC $\vdash f(\alpha), f(\Delta) \Rightarrow f(\Gamma)$ and BCC $\vdash f(\Pi) \Rightarrow f(\Sigma), f({\sim}\beta)$ where $f({\sim}\beta)$ coincides with ${\sim}f(\beta)$ by the definition of f. Then, we obtain the required fact:

$$\frac{\begin{array}{c}\vdots\\ f(\Pi) \Rightarrow f(\Sigma), {\sim}f(\beta)\end{array} \quad \begin{array}{c}\vdots\\ f(\alpha), f(\Delta) \Rightarrow f(\Gamma)\end{array}}{f(\Pi), f(\Delta) \Rightarrow f(\Sigma), f(\Gamma), {\sim}(f(\beta) \leftarrow f(\alpha))}\ ({\sim}{\leftarrow}\text{right})$$

where ${\sim}(f(\beta) \leftarrow f(\alpha))$ coincides with $f({\sim}(\alpha \to \beta))$ by the definition of f.

"⇐" By induction on the proofs Q of $f(\Delta) \Rightarrow f(\Gamma)$ in BCC. We distinguish the cases according to the last inference of Q, and show only the following case.

1. The case (cut): The last inference of Q is of the form:

$$\frac{f(\Gamma) \Rightarrow f(\Delta), \alpha \quad \alpha, f(\Sigma) \Rightarrow f(\Pi)}{f(\Gamma), f(\Sigma) \Rightarrow f(\Delta), f(\Pi)}\ (\text{cut})$$

where α is equivalent to $ff(\alpha)$ by Proposition 3.2. By induction hypothesis, we obtain BCC $\vdash \Pi \Rightarrow \Sigma, f(\alpha)$ and BCC $\vdash f(\alpha), \Pi \Rightarrow \Sigma$. Then, we obtain the required fact:

$$\frac{\Pi \Rightarrow \Sigma, f(\alpha) \quad f(\alpha), \Delta \Rightarrow \Gamma}{\Delta, \Pi \Rightarrow \Gamma, \Sigma}\ (\text{cut})$$ □

Next, we introduce another self-translation of BCC, and by using this translation, we show the quasi-symmetry property of BCC.

DEFINITION 3.5. A mapping f from $\mathcal{L}_{\text{BCC}}$ to $\mathcal{L}_{\text{BCC}}$ is defined inductively by:

1. $f(p) := p$ and $f({\sim}p) := {\sim}p$ for any $p \in \Phi$,
2. $f(\alpha \circ \beta) := f(\alpha) \circ f(\beta)$ where $\circ \in \{\wedge, \vee\}$,
3. $f(\alpha \to \beta) := f(\beta) \leftarrow {\sim}f(\alpha)$,
4. $f(\alpha \leftarrow \beta) := {\sim}f(\beta) \to f(\alpha)$,
5. $f({\sim}{\sim}\alpha) := f(\alpha)$,
6. $f({\sim}(\alpha \wedge \beta)) := f({\sim}\alpha) \vee f({\sim}\beta)$,

7. $f({\sim}(\alpha \vee \beta)) := f({\sim}\alpha) \wedge f({\sim}\beta)$,
8. $f({\sim}(\alpha \to \beta)) := f({\sim}\beta) \leftarrow {\sim} f(\alpha)$,
9. $f({\sim}(\alpha \leftarrow \beta)) := {\sim} f(\beta) \to f({\sim}\alpha)$.

The following theorem shows the quasi-symmetry property for BCC.

THEOREM 3.6 (Quasi-symmetry for BCC). *Let Γ and Δ be (possibly empty) sets of formulas in $\mathcal{L}_{\mathrm{BCC}}$, and f be the mapping defined in Definition 3.5.*

1. *If* $\mathrm{BCC} \vdash \Gamma \Rightarrow \Delta$, *then* $\mathrm{BCC} \vdash {\sim} f(\Delta) \Rightarrow {\sim} f(\Gamma)$.
2. *If* $\mathrm{BCC} - (\mathrm{cut}) \vdash \Gamma \Rightarrow \Delta$, *then* $\mathrm{BCC} - (\mathrm{cut}) \vdash {\sim} f(\Delta) \Rightarrow {\sim} f(\Gamma)$.

PROOF. We show only 1 by induction on the proofs P of $\Gamma \Rightarrow \Delta$ in BCC. We distinguish the cases according to the last inference of P, and show some cases.

1. The case ($\wedge$left): The last inference of P is of the form:

$$\frac{\alpha, \beta, \Gamma \Rightarrow \Delta}{\alpha \wedge \beta, \Gamma \Rightarrow \Delta} \ (\wedge\text{left}).$$

By induction hypothesis, $\mathrm{BCC} \vdash {\sim} f(\Delta) \Rightarrow {\sim} f(\Gamma), {\sim} f(\alpha), {\sim} f(\beta)$. So we obtain the required fact:

$$\frac{\begin{array}{c}\vdots\\ {\sim} f(\Delta) \Rightarrow {\sim} f(\Gamma), {\sim} f(\alpha), {\sim} f(\beta)\end{array}}{{\sim} f(\Delta) \Rightarrow {\sim} f(\Gamma), {\sim}(f(\alpha) \wedge f(\beta))} \ ({\sim}\wedge\text{right})$$

where ${\sim}(f(\alpha) \wedge f(\beta))$ coincides with ${\sim} f(\alpha \wedge \beta)$ by the definition of f.

2. The case ($\wedge$right): The last inference of P is of the form:

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \wedge \beta} \ (\wedge\text{right}).$$

By induction hypothesis, we have $\mathrm{BCC} \vdash {\sim} f(\alpha), {\sim} f(\Delta) \Rightarrow {\sim} f(\Gamma)$ and $\mathrm{BCC} \vdash {\sim} f(\beta), {\sim} f(\Delta) \Rightarrow {\sim} f(\Gamma)$. Then, we obtain the required fact:

$$\frac{\begin{array}{c}\vdots\\ {\sim} f(\alpha), {\sim} f(\Delta) \Rightarrow {\sim} f(\Gamma)\end{array} \quad \begin{array}{c}\vdots\\ {\sim} f(\beta), {\sim} f(\Delta) \Rightarrow {\sim} f(\Gamma)\end{array}}{{\sim}(f(\alpha) \wedge f(\beta)), {\sim} f(\Delta) \Rightarrow {\sim} f(\Gamma)} \ ({\sim}\wedge\text{left})$$

where ${\sim}(f(\alpha) \wedge f(\beta))$ coincides with ${\sim} f(\alpha \wedge \beta)$ by the definition of f.

3. The case ($\leftarrow$left): The last inference of P is of the form:

$$\frac{\alpha, \Gamma \Rightarrow \Delta, \beta}{\alpha \leftarrow \beta, \Gamma \Rightarrow \Delta} \ (\leftarrow\text{left}).$$

By induction hypothesis, BCC $\vdash \sim f(\beta), \sim f(\Delta) \Rightarrow \sim f(\Gamma), \sim f(\alpha)$. We obtain the required fact:

$$\frac{\begin{array}{c}\vdots\\ \sim f(\beta), \sim f(\Delta) \Rightarrow \sim f(\Gamma), \sim f(\alpha)\end{array}}{\sim f(\Delta) \Rightarrow \sim f(\Gamma), \sim(\sim f(\beta) \to f(\alpha))} \ (\sim\to\text{right})$$

where $\sim(\sim f(\beta) \to f(\alpha))$ coincides with $\sim f(\alpha \leftarrow \beta)$ by the definition of f.

4. The case ($\leftarrow$right): The last inference of P is of the form:

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \beta, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi, \alpha \leftarrow \beta} \ (\leftarrow\text{right}).$$

By induction hypothesis, we have BCC $\vdash \sim f(\alpha), \sim f(\Delta) \Rightarrow \sim f(\Gamma)$ and BCC $\vdash \sim f(\Pi) \Rightarrow \sim f(\Sigma), \sim f(\beta)$. Then, we obtain the required fact:

$$\frac{\begin{array}{c}\vdots\\ \sim f(\Pi) \Rightarrow \sim f(\Sigma), \sim f(\beta)\end{array} \quad \begin{array}{c}\vdots\\ \sim f(\alpha), \sim f(\Delta) \Rightarrow \sim f(\Gamma)\end{array}}{\sim(\sim f(\beta) \to f(\alpha)), \sim f(\Pi), \sim f(\Delta) \Rightarrow \sim f(\Sigma), \sim f(\Gamma)} \ (\sim\to\text{left})$$

where $\sim(\sim f(\beta) \to f(\alpha))$ coincides with $\sim f(\alpha \leftarrow \beta)$ by the definition of f.

5. The case ($\sim\wedge$left): The last inference of P is of the form:

$$\frac{\sim\alpha, \Gamma \Rightarrow \Delta \quad \sim\beta, \Gamma \Rightarrow \Delta}{\sim(\alpha \wedge \beta), \Gamma \Rightarrow \Delta} \ (\sim\wedge\text{left}).$$

By induction hypothesis, we have BCC $\vdash \sim f(\Delta) \Rightarrow \sim f(\Gamma), \sim f(\sim\alpha)$ and BCC $\vdash \sim f(\Delta) \Rightarrow \sim f(\Gamma), \sim f(\sim\beta)$. Then, we obtain the required fact:

$$\frac{\begin{array}{c}\vdots\\ \sim f(\Delta) \Rightarrow \sim f(\Gamma), \sim f(\sim\alpha)\end{array} \quad \begin{array}{c}\vdots\\ \sim f(\Delta) \Rightarrow \sim f(\Gamma), \sim f(\sim\beta)\end{array}}{\sim f(\Delta) \Rightarrow \sim f(\Gamma), \sim(f(\sim\alpha) \vee f(\sim\beta))} \ (\sim\vee\text{right})$$

where $\sim(f(\sim\alpha) \vee f(\sim\beta))$ coincides with $\sim f(\sim(\alpha \wedge \beta))$ by the definition of f.

6. The case ($\sim$left): The last inference of P is of the form:

$$\frac{\alpha, \Gamma \Rightarrow \Delta}{\sim\sim\alpha, \Gamma \Rightarrow \Delta}\ (\sim\text{left}).$$

By induction hypothesis, we have BCC $\vdash\ \sim f(\Delta) \Rightarrow\ \sim f(\Gamma), \sim f(\alpha)$ where $\sim f(\alpha)$ coincides with $\sim f(\sim\sim\alpha)$ by the definition of f.

7. The case ($\sim\rightarrow$left): The last inference of P is of the form:

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \sim\beta, \Sigma \Rightarrow \Pi}{\sim(\alpha \rightarrow \beta), \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (\sim\rightarrow\text{left}).$$

By induction hypothesis, BCC $\vdash\ \sim f(\Pi) \Rightarrow\ \sim f(\Sigma), \sim f(\sim\beta)$ and BCC $\vdash\ \sim f(\alpha), \sim f(\Delta) \Rightarrow\ \sim f(\Gamma)$. Then, we obtain the required fact:

$$\frac{\begin{matrix}\vdots \\ \sim f(\Pi) \Rightarrow\ \sim f(\Sigma), \sim f(\sim\beta)\end{matrix} \quad \begin{matrix}\vdots \\ \sim f(\alpha), \sim f(\Delta) \Rightarrow\ \sim f(\Gamma)\end{matrix}}{\sim f(\Pi), \sim f(\Delta) \Rightarrow\ \sim f(\Sigma), \sim f(\Gamma), \sim(f(\sim\beta) \leftarrow\ \sim f(\alpha))}\ (\sim\leftarrow\text{right})$$

where $\sim(f(\sim\beta) \leftarrow\ \sim f(\sim\alpha))$ coincides with $\sim f(\sim(\alpha \rightarrow \beta))$ by the definition of f. ☐

Remark 3.7. We make the following remarks on Theorem 3.6.

1. If the converses of Theorem 3.6 hold or not is left as an open problem.

2. To explain about this fact, we consider to show the converse of (1) in Theorem 3.6 by induction on the proofs Q of $\sim f(\Delta) \Rightarrow\ \sim f(\Gamma)$ in BCC. We distinguish the cases according to the last inference of Q. Then, the following cases cannot be shown (i.e., the induction hypothesis cannot be used).

(a) The case ($\sim$right): The last inference of Q is ($\sim$right):

$$\frac{\sim f(\Delta) \Rightarrow\ \sim f(\Gamma), p}{\sim f(\Delta) \Rightarrow\ \sim f(\Gamma), \sim f(\sim p)}\ (\sim\text{right}).$$

where p is a propositional variable.

(b) The case (cut): The last inference of Q is (cut):

$$\frac{\sim f(\Delta) \Rightarrow\ \sim f(\Gamma), \alpha \quad \alpha, \sim f(\Pi) \Rightarrow\ \sim f(\Sigma)}{\sim f(\Delta), \sim f(\Pi) \Rightarrow\ \sim f(\Gamma), \sim f(\Sigma)}\ (\text{cut})$$

where α is not a $\sim$-formula.

3. A counterexample of the converses of Theorem 3.6 has not yet been found.

4. Completeness

First, we introduce a dual valuation semantics for BCC.

Definition 4.1 (Semantics for BCC). Double valuations v^+ and v^- are mappings from Φ to the set $\{t, f\}$ of truth values. The valuations v^+ and v^- are extended to mappings from the set of all formulas to $\{t, f\}$ by:

1. $v^+(\alpha \wedge \beta) = t$ iff $v^+(\alpha) = t$ and $v^+(\beta) = t$,
2. $v^+(\alpha \vee \beta) = t$ iff $v^+(\alpha) = t$ or $v^+(\beta) = t$,
3. $v^+(\alpha \rightarrow \beta) = t$ iff $v^+(\alpha) = f$ or $v^+(\beta) = t$,
4. $v^+(\alpha \leftarrow \beta) = t$ iff $v^+(\alpha) = t$ and $v^+(\beta) = f$,
5. $v^+({\sim}\alpha) = t$ iff $v^-(\alpha) = t$,
6. $v^-(\alpha \wedge \beta) = t$ iff $v^-(\alpha) = t$ or $v^-(\beta) = t$,
7. $v^-(\alpha \vee \beta) = t$ iff $v^-(\alpha) = t$ and $v^-(\beta) = t$,
8. $v^-(\alpha \rightarrow \beta) = t$ iff $v^+(\alpha) = f$ or $v^-(\beta) = t$,
9. $v^-(\alpha \leftarrow \beta) = t$ iff $v^-(\alpha) = t$ and $v^+(\beta) = f$,
10. $v^-({\sim}\alpha) = t$ iff $v^+(\alpha) = t$.

A formula α is called *BCC-valid* if $v^+(\alpha) = t$ holds for any double valuations v^+ and v^-.

Remark 4.2. The semantics which is defined in Definition 4.1 is regarded as a four-valued semantics, since the following four cases can be considered for the double valuations v^+ and v^-: For any formula α,

1. $v^+(\alpha) = t$ and $v^-(\alpha) = t$,
2. $v^+(\alpha) = t$ and $v^-(\alpha) = f$,
3. $v^+(\alpha) = f$ and $v^-(\alpha) = t$,
4. $v^+(\alpha) = f$ and $v^-(\alpha) = f$.

In order to show a theorem for semantically embedding BCC into BC, we present the standard semantics for BC.

Definition 4.3 (Semantics for BC). A valuation v is a mapping from $\Phi \cup \Phi'$ to the set $\{t, f\}$ of truth values. The valuation v is extended to the mapping from the set of all formulas to $\{t, f\}$ by:

1. $v(\alpha \wedge \beta) = t$ iff $v(\alpha) = t$ and $v(\beta) = t$,
2. $v(\alpha \vee \beta) = t$ iff $v(\alpha) = t$ or $v(\beta) = t$,
3. $v(\alpha \rightarrow \beta) = t$ iff $v(\alpha) = f$ or $v(\beta) = t$,
4. $v(\alpha \leftarrow \beta) = t$ iff $v(\alpha) = t$ and $v(\beta) = f$.

A formula α is called *BC-valid* if $v(\alpha) = t$ holds for any valuations v.

Remark 4.4. The following completeness theorem for BC (i.e., essentially the completeness theorem for classical logic) holds: For any formula α in $\mathcal{L}_{\mathrm{BC}}$: $\mathrm{BC} \vdash \Rightarrow \alpha$ iff α is BC-valid.

Lemma 4.5. *Let f be the mapping defined in Definition 2.5. For any double valuations v^+ and v^-, we can construct a valuation v such that for any formula α in $\mathcal{L}_{\mathrm{BCC}}$,*

1. $v^+(\alpha) = t$ *iff* $v(f(\alpha)) = t$,
2. $v^-(\alpha) = t$ *iff* $v(f(\sim\alpha)) = t$.

Proof. We define a valuation v by: $v(p) := v^+(p)$ if $p \in \Phi$; $v(p') := v^-(p)$ if $p' \in \Phi'$. This lemma is then proved by (simultaneous) induction on α.

Base step: The case $\alpha \equiv p$, where $p \in \Phi$. For 1: $v^+(p) = t$ iff $v(p) = t$ (by the assumption) iff $v(f(p)) = t$ (by the definition of f). For 2: $v^-(p) = t$ iff $v(p') = t$ (by the assumption) iff $v(f(\sim p)) = t$ (by the definition of f).

Induction step: The case $\alpha \equiv \beta\wedge\gamma$: For 1: $v^+(\beta\wedge\gamma) = t$ iff $v^+(\beta) = t$ and $v^+(\gamma) = t$ iff $v(f(\beta)) = t$ and $v(f(\gamma)) = t$ (by induction hypothesis) iff $v(f(\beta) \wedge f(\gamma)) = t$ iff $v(f(\beta \wedge \gamma))) = t$ (by the definition of f). For 2: $v^-(\beta\wedge\gamma) = t$ iff $v^-(\beta) = t$ or $v^-(\gamma) = t$ iff $v(f(\sim\beta)) = t$ or $v(f(\sim\gamma)) = t$ (by induction hypothesis) iff $v(f(\sim\beta)\vee f(\sim\gamma)) = t$ iff $v(f(\sim(\beta\wedge\gamma))) = t$ (by the definition of f).

The case $\alpha \equiv \beta \vee \gamma$: Similar to Case $\alpha \equiv \beta \wedge \gamma$.

The case $\alpha \equiv \beta \to \gamma$: For 1: $v^+(\beta \to \gamma) = t$ iff $v^+(\beta) = f$ or $v^n(\gamma) = t$ iff $v(f(\beta)) = f$ or $v(f(\gamma)) = t$ (by induction hypothesis) iff $v(f(\beta) \to f(\gamma)) = t$ iff $v(f(\beta \to \gamma)) = t$ (by the definition of f). For 2: $v^-(\beta \to \gamma) = t$ iff $v^+(\beta) = f$ or $v^-(\gamma) = t$ iff $v(f(\beta)) = f$ or $v(f(\sim\gamma)) = t$ (by induction hypothesis) iff $v(f(\beta) \to f(\sim\gamma)) = t$ iff $v(f(\sim(\beta \to \gamma))) = t$ (by the definition of f).

The case $\alpha \equiv \beta \leftarrow \gamma$: For 1, $v^+(\beta \leftarrow \gamma) = t$ iff $v^+(\beta) = t$ and $v^+(\gamma) = f$ iff $v(f(\beta)) = t$ and $v(f(\gamma)) = f$ (by induction hypothesis) iff $v(f(\beta) \leftarrow f(\gamma)) = t$ iff $v(f(\beta \leftarrow \gamma)) = t$ (by the definition of f). For 2, $v^-(\beta \leftarrow \gamma) = t$ iff $v^-(\beta) = t$ and $v^+(\gamma) = f$ iff $v(f(\sim\beta)) = t$ and $v(f(\gamma)) = f$ (by induction hypothesis) iff $v(f(\sim\beta) \leftarrow f(\gamma)) = t$ iff $v(f(\sim(\beta \leftarrow \gamma))) = t$ (by the definition of f).

The case $\alpha \equiv \sim\beta$: For 1, $v^+(\sim\beta) = t$ iff $v^-(\beta) = t$ iff $v(f(\sim\beta)) = t$ (by induction hypothesis). For 2, $v^-(\sim\beta) = t$ iff $v^+(\beta) = t$ iff $v(f(\beta)) = t$ (by induction hypothesis) iff $v(f(\sim\sim\beta)) = t$ (by the definition of f). $\square$

Similar to Lemma 4.5 we obtain:

LEMMA 4.6. *Let f be the mapping defined in Definition 2.5. For any valuations v, we can construct double valuations v^+ and v^- such that for any formula α in $\mathcal{L}_{\mathrm{BCC}}$,*

1. $v^+(\alpha) = t$ *iff* $v(f(\alpha)) = t$,
2. $v^-(\alpha) = t$ *iff* $v(f({\sim}\alpha)) = t$.

By Lemmas 4.5 and 4.6 we obtain:

THEOREM 4.7 (Semantical embedding from BCC into BC). *Let f be the mapping defined in Definition 2.5. For any formula α in $\mathcal{L}_{\mathrm{BCC}}$,*

$$\alpha \text{ is BCC-valid iff } f(\alpha) \text{ is BC-valid.}$$

THEOREM 4.8 (Completeness for BCC). *For any formula α in $\mathcal{L}_{\mathrm{BCC}}$,*

$$\mathrm{BCC} \vdash \Rightarrow \alpha \text{ iff } \alpha \text{ is BCC-valid.}$$

PROOF. We have: $\mathrm{BCC} \vdash \Rightarrow \alpha$ iff $\mathrm{BC} \vdash \Rightarrow f(\alpha)$ (by Theorem 2.7) iff $f(\alpha)$ is BC-valid (by the completeness theorem for BC) iff α is BCC-valid (by Theorem 4.7). □

5. Modal extension

Formulas of modal bi-classical connexive logic are constructed from countably many propositional variables, $\wedge$, $\vee$, $\rightarrow$, $\leftarrow$, $\sim$, $\Box$ (necessity) and $\Diamond$ (possibility). An expression $\sharp\Gamma$ ($\sharp \in \{\Box, \Diamond\}$) is used to represent the set $\{\sharp\gamma \mid \gamma \in \Gamma\}$.

Prior to define Gentzen-type sequent calculi MBCC and MBC, we define the languages of them.

DEFINITION 5.1. We fix a set Φ of propositional variables and define the set $\Phi' := \{p' \mid p \in \Phi\}$ of propositional variables. The language $\mathcal{L}_{\mathrm{MBCC}}$ of MBCC is obtained from Φ by $\wedge$, $\vee$, $\rightarrow$, $\leftarrow$, $\Box$, $\Diamond$ and $\sim$. The language $\mathcal{L}_{\mathrm{MBC}}$ of MBC is obtained from Φ and Φ' by $\wedge$, $\vee$, $\rightarrow$, $\leftarrow$, $\Box$ and $\Diamond$.

A Gentzen-type sequent calculus MBCC for the modal bi-classical connexive logic is introduced below.

DEFINITION 5.2 (MBCC). MBCC is defined based on $\mathcal{L}_{\mathrm{MBCC}}$. MBCC is obtained from BCC by adding the logical inference rules of the form:

$$\frac{\alpha, \Gamma \Rightarrow \Delta}{\Box\alpha, \Gamma \Rightarrow \Delta}\ (\Box\text{left}) \qquad \frac{\Box\Gamma, \sim\Diamond\Sigma \Rightarrow \alpha}{\Box\Gamma, \sim\Diamond\Sigma \Rightarrow \Box\alpha}\ (\Box\text{right}^*)$$

$$\frac{\alpha \Rightarrow \Diamond\Gamma, \sim\Box\Sigma}{\Diamond\alpha \Rightarrow \Diamond\Gamma, \sim\Box\Sigma}\ (\Diamond\text{left}^*) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma \Rightarrow \Delta, \Diamond\alpha}\ (\Diamond\text{right})$$

$$\frac{\sim\alpha \Rightarrow \Diamond\Gamma, \sim\Box\Sigma}{\sim\Box\alpha \Rightarrow \Diamond\Gamma, \sim\Box\Sigma}\ (\sim\Box\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \sim\alpha}{\Gamma \Rightarrow \Delta, \sim\Box\alpha}\ (\sim\Box\text{right})$$

$$\frac{\sim\alpha, \Gamma \Rightarrow \Delta}{\sim\Diamond\alpha, \Gamma \Rightarrow \Delta}\ (\sim\Diamond\text{left}) \qquad \frac{\Box\Gamma, \sim\Diamond\Sigma \Rightarrow \sim\alpha}{\Box\Gamma, \sim\Diamond\Sigma \Rightarrow \sim\Diamond\alpha}\ (\sim\Diamond\text{right})$$

A Gentzen-type sequent calculus MBC for the modal bi-classical logic is defined below.

DEFINITION 5.3 (MBC). MBC is defined based on $\mathcal{L}_{\mathrm{MBC}}$. MBC is obtained from BC by adding ($\Box$left), ($\Diamond$right), and the logical inference rules of the form:

$$\frac{\Box\Gamma \Rightarrow \alpha}{\Box\Gamma \Rightarrow \Box\alpha}\ (\Box\text{right}) \qquad \frac{\alpha \Rightarrow \Diamond\Gamma}{\Diamond\alpha \Rightarrow \Diamond\Gamma}\ (\Diamond\text{left}).$$

Remark 5.4. We make the the following remarks on MBC and MBCC.

1. Let L be MBC or MBCC. Sequents of the form $\alpha \Rightarrow \alpha$ for any formula α are provable in L. This fact can be shown by induction on α.

2. ($\sim\Box$left), ($\sim\Box$right), ($\sim\Diamond$left) and ($\sim\Diamond$right) correspond to the following axiom schemes:

(a) $\sim\Box\alpha \leftrightarrow \Diamond\sim\alpha$,
(b) $\sim\Diamond\alpha \leftrightarrow \Box\sim\alpha$.

3. MBC is regarded as a Gentzen-type sequent calculus for the extended positive fragment of the modal logic S4 with co-implication.

4. Several modifications of MBC and MBCC without co-implication, called S4 and CS4, were studied in [11].

5. MBC is logically equivalent to S4 when the language includes the classical negation connective. The cut-elimination theorem for MBC holds, and MBC is decidable. These facts can be obtained in a similar manner as shown in Remark 2.4. For more information on Gentzen-type sequent calculi for S4, see, e.g., [28].

Definition 5.5. A mapping f from $\mathcal{L}_{\mathrm{MBCC}}$ to $\mathcal{L}_{\mathrm{MBC}}$ is defined inductively by the same conditions as in Definition 2.5 and the following new conditions:

1. $f(\Box\alpha) := \Box f(\alpha)$,
2. $f(\Diamond\alpha) := \Diamond f(\alpha)$,
3. $f({\sim}\Box\alpha) := \Diamond f({\sim}\alpha)$,
4. $f({\sim}\Diamond\alpha) := \Box f({\sim}\alpha)$.

Theorem 5.6 (Syntactical embedding from MBCC into MBC). *Let* Γ, Δ *be sets of formulas in* $\mathcal{L}_{\mathrm{MBCC}}$, *and* f *be the mapping defined in Definition 5.5. Then*:

1. $\mathrm{MBCC} \vdash \Gamma \Rightarrow \Delta$ *iff* $\mathrm{MBC} \vdash f(\Gamma) \Rightarrow f(\Delta)$.
2. $\mathrm{MBCC} - (\mathrm{cut}) \vdash \Gamma \Rightarrow \Delta$ *iff* $\mathrm{MBC} - (\mathrm{cut}) \vdash f(\Gamma) \Rightarrow f(\Delta)$.

Proof. We show only 1.

"$\Rightarrow$" By induction on the proofs P of $\Gamma \Rightarrow \Delta$ in MBCC. We distinguish the cases according to the last inference of P, and show some cases for the modal extension. The cases for the non-modal parts including (cut) are the same as those of BCC.

1. The case (${\sim}\Diamond$right*): The last inference of P is of the form:

$$\frac{\Box\Gamma, {\sim}\Diamond\Sigma \Rightarrow {\sim}\alpha}{\Box\Gamma, {\sim}\Diamond\Sigma \Rightarrow {\sim}\Diamond\alpha}\ ({\sim}\Diamond\mathrm{right}^*).$$

By induction hypothesis: $\mathrm{MBC} \vdash f(\Box\Gamma), f({\sim}\Diamond\Sigma) \Rightarrow f({\sim}\alpha)$, where $f(\Box\Gamma)$ and $f({\sim}\Diamond\Sigma)$ respectively coincide with $\Box f(\Gamma)$ and $\Box f({\sim}\Sigma)$ by the definition of f. Then, we obtain:

$$\frac{\begin{array}{c}\vdots\\ \Box f(\Gamma), \Box f({\sim}\Sigma) \Rightarrow f({\sim}\alpha)\end{array}}{\Box f(\Gamma), \Box f({\sim}\Sigma) \Rightarrow \Box f({\sim}\alpha)}\ (\Box\mathrm{right})$$

where $\Box f({\sim}\alpha)$ coincides with $f({\sim}\Diamond\alpha)$ by the definition of f. Therefore we have the required fact: $\mathrm{MBC} \vdash f(\Box\Gamma), f({\sim}\Diamond\Sigma) \Rightarrow f({\sim}\Diamond\alpha)$.

2. The case (${\sim}\Box$left*): The last inference of P is of the form:

$$\frac{{\sim}\alpha \Rightarrow \Diamond\Gamma, {\sim}\Box\Sigma}{{\sim}\Box\alpha \Rightarrow \Diamond\Gamma, {\sim}\Box\Sigma}\ ({\sim}\Box\mathrm{left}^*).$$

By induction hypothesis: MBC $\vdash f({\sim}\alpha) \Rightarrow f(\Diamond\Gamma), f({\sim}\Box\Sigma)$, where $f(\Diamond\Gamma)$ and $f({\sim}\Box\Sigma)$ respectively coincide with $\Diamond f(\Gamma)$ and $\Diamond f({\sim}\Sigma)$ by the definition of f. Then, we obtain:

$$\dfrac{\begin{array}{c}\vdots\\ f({\sim}\alpha) \Rightarrow \Diamond f(\Gamma), \Diamond f({\sim}\Sigma)\end{array}}{\Diamond f({\sim}\alpha) \Rightarrow \Diamond f(\Gamma), \Diamond f({\sim}\Sigma)}\ (\Diamond\text{left})$$

where $\Diamond f({\sim}\alpha)$ coincides with $f({\sim}\Box\alpha)$ by the definition of f. Therefore we have the required fact: MBC $\vdash f({\sim}\Box\alpha) \Rightarrow f(\Diamond\Gamma), f({\sim}\Box\alpha)$.

"$\Leftarrow$" By induction on the proofs Q of $f(\Gamma) \Rightarrow f(\Delta)$ in MBC. We distinguish the cases according to the last inference of Q, and show some cases for the modal extension. The cases for the non-modal parts including (cut) are the same as those of BCC. Similar to the proof of Theorem 2.7, we omit the cases according to the condition $f({\sim}{\sim}\alpha) := f(\alpha)$.

The Case ($\Box$right): The last inference of Q is ($\Box$right).

1. The last inference of P is of the form:

$$\dfrac{f(\Box\Gamma), f({\sim}\Diamond\Sigma) \Rightarrow f({\sim}\alpha)}{f(\Box\Gamma), f({\sim}\Diamond\Sigma) \Rightarrow f({\sim}\Diamond\alpha)}\ (\Box\text{right})$$

where $f(\Box\Gamma)$, $f({\sim}\Diamond\Sigma)$ and $f({\sim}\Diamond\alpha)$ respectively coincide with $\Box f(\Gamma)$, $\Box f({\sim}\Sigma)$ and $\Box f({\sim}\alpha)$ by the definition of f. By induction hypothesis, we have: MBCC $\vdash \Box\Gamma, {\sim}\Diamond\Sigma \Rightarrow {\sim}\alpha$. Hence, we obtain the required fact:

$$\dfrac{\begin{array}{c}\vdots\\ \Box\Gamma, {\sim}\Diamond\Sigma \Rightarrow {\sim}\alpha\end{array}}{\Box\Gamma, {\sim}\Diamond\Sigma \Rightarrow {\sim}\Diamond\alpha}\ ({\sim}\Diamond\text{right}^*)$$

2. The last inference of P is of the form:

$$\dfrac{f(\Box\Gamma), f({\sim}\Diamond\Sigma) \Rightarrow f(\alpha)}{f(\Box\Gamma), f({\sim}\Diamond\Sigma) \Rightarrow f(\Box\alpha)}\ (\Box\text{right})$$

where $f(\Box\Gamma)$, $f({\sim}\Diamond\Sigma)$ and $f(\Box\alpha)$ respectively coincide with $\Box f(\Gamma)$, $\Box f({\sim}\Sigma)$ and $\Box f(\alpha)$ by the definition of f. This case can be shown in a similar way as in 1. $\Box$

We can obtain the following theorems for MBCC in a similar way as for BCC.

Theorem 5.7. 1. *The rule* (cut) *is admissible in cut-free* MBCC.
2. MBCC *is decidable.*
3. MBCC *is paraconsistent with respect to* $\sim$.

The duality and quasi-symmetry properties can also be shown for MBCC.

Definition 5.8. A mapping f from $\mathcal{L}_{\mathrm{MBCC}}$ to $\mathcal{L}_{\mathrm{MBCC}}$ is defined inductively by the same conditions as in Definition 3.1 and the following new conditions:

1. $f(\Box\alpha) := \Diamond f(\alpha)$,
2. $f(\Diamond\alpha) := \Box f(\alpha)$.

Proposition 5.9. *Let f be the mapping defined in Definition 5.8. Then, we have: $ff(\alpha) = \alpha$ for any formula α in $\mathcal{L}_{\mathrm{MBCC}}$.*

Theorem 5.10 (Duality for MBCC). *Let* Γ, Δ *be* (*possibly empty*) *sets of formulas in* $\mathcal{L}_{\mathrm{MBCC}}$, *and* f *be the mapping defined in Definition 5.8.*

1. MBCC $\vdash \Gamma \Rightarrow \Delta$ *iff* MBCC $\vdash f(\Delta) \Rightarrow f(\Gamma)$.
2. MBCC $-$ (cut) $\vdash \Gamma \Rightarrow \Delta$ *iff* MBCC $-$ (cut) $\vdash f(\Delta) \Rightarrow f(\Gamma)$.

Definition 5.11. A mapping f from $\mathcal{L}_{\mathrm{MBCC}}$ to $\mathcal{L}_{\mathrm{MBCC}}$ is defined inductively by the same conditions as in Definition 3.5 and the following new conditions:

1. $f(\Box\alpha) := \Box f(\alpha)$,
2. $f(\Diamond\alpha) := \Diamond f(\alpha)$,
3. $f({\sim}\Box\alpha) := \Diamond f({\sim}\beta)$,
4. $f({\sim}\Diamond\alpha) := \Box f({\sim}\beta)$.

Theorem 5.12 (Quasi-symmetry for MBCC). *Let* Γ *and* Δ *be* (*possibly empty*) *sets of formulas in* $\mathcal{L}_{\mathrm{MBCC}}$, *and* f *be the mapping defined in Definition 5.11.*

1. *If* MBCC $\vdash \Gamma \Rightarrow \Delta$, *then* MBCC $\vdash {\sim} f(\Delta) \Rightarrow {\sim} f(\Gamma)$.
2. *If* MBCC $-$ (cut) $\vdash \Gamma \Rightarrow \Delta$, *then* MBCC $-$ (cut) $\vdash {\sim} f(\Delta) \Rightarrow {\sim} f(\Gamma)$.

Next, we introduce the Kripke semantics for MBCC and MBC, and prove the Kripke completeness theorem for MBCC.

Definition 5.13. A structure $\langle M, R\rangle$ is called a *Kripke frame* if M is a non-empty set and R is a transitive and reflexive binary relation on M.

DEFINITION 5.14. *Double valuations* $\models^+$ and $\models^-$ on a Kripke frame $\langle M, R\rangle$ are mappings from Φ to the power set 2^M of M. We will write $x \models^\star p$ for $x \in \models^\star (p)$ with $\star \in \{+, -\}$. These double valuations $\models^+$ and $\models^-$ are extended to mappings from the set of all formulas to 2^M by:

1. $x \models^+ \alpha \wedge \beta$ iff $x \models^+ \alpha$ and $x \models^+ \beta$,
2. $x \models^+ \alpha \vee \beta$ iff $x \models^+ \alpha$ or $x \models^+ \beta$,
3. $x \models^+ \alpha \rightarrow \beta$ iff $x \models^+ \alpha$ implies $x \models^+ \beta$,
4. $x \models^+ \alpha \leftarrow \beta$ iff $x \models^+ \alpha$ and $x \not\models^+ \beta$,
5. $x \models^+ \Box\alpha$ iff $\forall y \in M$ $[xRy$ implies $y \models^+ \alpha]$,
6. $x \models^+ \Diamond\alpha$ iff $\exists y \in M$ $[xRy$ and $y \models^+ \alpha]$,
7. $x \models^+ {\sim}\alpha$ iff $x \models^- \alpha$,
8. $x \models^- \alpha \wedge \beta$ iff $x \models^- \alpha$ or $x \models^- \beta$,
9. $x \models^- \alpha \vee \beta$ iff $x \models^- \alpha$ and $x \models^- \beta$,
10. $x \models^- \alpha \rightarrow \beta$ iff $x \models^+ \alpha$ implies $x \models^- \beta$,
11. $x \models^- \alpha \leftarrow \beta$ iff $x \models^- \alpha$ and $x \not\models^+ \beta$,
12. $x \models^- \Box\alpha$ iff $\exists y \in M$ $[xRy$ and $y \models^- \alpha]$,
13. $x \models^- \Diamond\alpha$ iff $\forall y \in M$ $[xRy$ implies $y \models^- \alpha]$,
14. $x \models^- {\sim}\alpha$ iff $x \models^+ \alpha$.

DEFINITION 5.15. A *MBCC-Kripke model* is a structure $\langle M, R, \models^+, \models^-\rangle$ such that $\langle M, R\rangle$ is a Kripke frame and $\models^+$ and $\models^-$ are double valuations on $\langle M, R\rangle$. A formula α is *true* in a *MBCC*-Kripke model $\langle M, R, \models^+, \models^-\rangle$ iff $x \models^+ \alpha$ for any $x \in M$, and is *MBCC-valid* in a Kripke frame $\langle M, R\rangle$ iff it is true for every double valuations $\models^+$ and $\models^-$ on the Kripke frame.

DEFINITION 5.16. A *valuation* $\models$ on a Kripke frame $\langle M, R\rangle$ is a mapping from $\Phi \cup \Phi'$ to the power set 2^M of M. We will write $x \models p$ for $x \in \models (p)$. The valuation $\models$ is extended to a mapping from the set of all formulas to 2^M by:

1. $x \models \alpha \wedge \beta$ iff $x \models \alpha$ and $x \models \beta$,
2. $x \models \alpha \vee \beta$ iff $x \models \alpha$ or $x \models \beta$,
3. $x \models \alpha \rightarrow \beta$ iff $x \models \alpha$ implies $x \models \beta$,
4. $x \models \alpha \leftarrow \beta$ iff $x \models \alpha$ and $x \not\models \beta$,
5. $x \models \Box\alpha$ iff $\forall y \in M$ $[xRy$ implies $y \models \alpha]$,
6. $x \models \Diamond\alpha$ iff $\exists y \in M$ $[xRy$ and $y \models \alpha]$.

DEFINITION 5.17. A *MBC-Kripke model* is a structure $\langle M, R, \models\rangle$ such that $\langle M, R\rangle$ is a Kripke frame and $\models$ is a valuation on $\langle M, R\rangle$. A formula α is *true* in a *MBC*-Kripke model $\langle M, R, \models\rangle$ iff $x \models \alpha$ for any $x \in M$, and is *MBC-valid* in a Kripke frame $\langle M, R\rangle$ iff it is true for every valuation $\models$ on the Kripke frame.

Remark 5.18. The following completeness theorem w.r.t. MBC-Kripke models holds: For any formula α in $\mathcal{L}_{\text{MBC}}$,

$$\text{MBC} \vdash \Rightarrow \alpha \text{ iff } \alpha \text{ is MBC-valid.}$$

Lemma 5.19. *Let f be the mapping defined in Definition 5.5. For any* MBCC-*Kripke model $\langle M, R, \models^+, \models^-\rangle$, we can construct a* MBC-*Kripke model $\langle M, R, \models\rangle$ such that for any formula α in $\mathcal{L}_{\text{MBCC}}$ and any $x \in M$,*

1. $x \models^+ \alpha$ *iff* $x \models f(\alpha)$,
2. $x \models^- \alpha$ *iff* $x \models f({\sim}\alpha)$.

Proof. Suppose that $\langle M, R, \models^+, \models^-\rangle$ is a MBCC-Kripke model where $\models^+$ and $\models^-$ are mappings from Φ to 2^M. Suppose that $\langle M, R, \models\rangle$ is a MBC-Kripke model where $\models$ is a mapping from $\Phi \cup \Phi'$ to 2^M. Suppose moreover that these models satisfy the following conditions: for any $x \in M$ and any $p \in \Phi$,

1. $x \models^+ p$ iff $x \models p$,
2. $x \models^- p$ iff $x \models p'$.

Then, the claim of the lemma is proved by (simultaneous) induction on the complexity of α.

Base step: The case $\alpha \equiv p$, where $p \in \Phi$: For 1: $x \models^+ p$ iff $x \models p$ iff $x \models f(p)$ (by the definition of f). For 2: $x \models^- p$ iff $x \models p'$ iff $x \models f({\sim}p)$ (by the definition of f).

Induction step: We show some cases.

The case $\alpha \equiv {\sim}\beta$: For 1: $x \models^+ {\sim}\beta$ iff $x \models^- \beta$ iff $x \models f({\sim}\beta)$ (by induction hypothesis for 2). For 2: $x \models^- {\sim}\beta$ iff $x \models^+ \beta$ iff $x \models f(\beta)$ (by induction hypothesis for 1) iff $x \models f({\sim}{\sim}\beta)$ (by the definition of f).

The case $\alpha \equiv \beta \wedge \gamma$: For 1, we obtain: $x \models^+ \beta \wedge \gamma$ iff $x \models^+ \beta$ and $x \models^+ \gamma$ iff $x \models f(\beta)$ and $x \models f(\gamma)$ (by induction hypothesis for 1) iff $x \models f(\beta) \wedge f(\gamma)$ iff $x \models f(\beta \wedge \gamma)$ (by the definition of f). For 2: $x \models^- \beta \wedge \gamma$ iff $x \models^- \beta$ or $x \models^- \gamma$ iff $x \models f({\sim}\beta)$ or $x \models f({\sim}\gamma)$ (by induction hypothesis for 2) iff $x \models f({\sim}\beta) \vee f({\sim}\gamma)$ iff $x \models f({\sim}(\beta \wedge \gamma))$ (by the definition of f).

The case $\alpha \equiv \beta \to \gamma$: For 1: $x \models^+ \beta \to \gamma$ iff $x \models^+ \beta$ implies $x \models^+ \gamma$ iff $x \models f(\beta)$ implies $x \models f(\gamma)$ (by induction hypothesis for 1) iff $x \models f(\beta) \to f(\gamma)$ iff $x \models f(\beta \to \gamma)$ (by the definition of f).

For 2: $x \models^{-} \beta \to \gamma$ iff $x \models^{+} \beta$ implies $x \models^{-} \gamma$ iff $x \models f(\beta)$ implies $x \models f({\sim}\gamma)$ (by induction hypotheses for 1 and 2) iff $x \models f(\beta) \to f({\sim}\gamma)$ iff $x \models f({\sim}(\beta \to \gamma))$ (by the definition of f).

The case $\alpha \equiv \beta \leftarrow \gamma$: For 1: $x \models^{+} \beta \leftarrow \gamma$ iff $x \models^{+} \beta$ and $x \not\models^{+} \gamma$ iff $x \models f(\beta)$ and $x \not\models f(\gamma)$ (by induction hypothesis for 1) iff $x \models f(\beta) \leftarrow f(\gamma)$ iff $x \models f(\beta \leftarrow \gamma)$ (by the definition of f). For 2: $x \models^{-} \beta \to \gamma$ iff $x \models^{-} \beta$ and $x \not\models^{+} \gamma$ iff $x \models f({\sim}\beta)$ and $x \not\models f(\gamma)$ (by induction hypotheses for 1 and 2) iff $x \models f({\sim}\beta) \leftarrow f(\gamma)$ iff $x \models f({\sim}(\beta \leftarrow \gamma))$ (by the definition of f).

The case $\alpha \equiv \Box\beta$: For 1: $x \models^{+} \Box\beta$ iff $\forall y \in M[xRy$ implies $y \models^{+} \beta]$ iff $\forall y \in M[xRy$ implies $y \models f(\beta)]$ (by induction hypothesis for 1) iff $x \models \Box f(\beta)$ iff $x \models f(\Box\beta)$ (by the definition of f). For 2: $x \models^{-} \Box\beta$ iff $\exists y \in M[xRy$ and $y \models^{-} \beta]$ iff $\exists y \in M[xRy$ and $y \models f({\sim}\beta)]$ (by induction hypothesis for 2) iff $x \models \Diamond f({\sim}\beta)$ iff $x \models f({\sim}\Box\beta)$ (by the definition of f). □

Similar to Lemma 5.19 we obtain:

LEMMA 5.20. *Let f be the mapping defined in Definition 5.5. For any* MBC-*Kripke model $\langle M, R, \models\rangle$, we can construct a* MBCC-*Kripke model $\langle M, R, \models^{+}, \models^{-}\rangle$ such that for any formula α in $\mathcal{L}_{\mathrm{MBCC}}$ and any $x \in M$,*

1. *$x \models f(\alpha)$ iff $x \models^{+} \alpha$,*
2. *$x \models f({\sim}\alpha)$ iff $x \models^{-} \alpha$.*

By lemmas 5.19 and 5.20 we obtain:

THEOREM 5.21 (Semantical embedding from MBCC into BCC). *Let f be the mapping defined in Definition 5.5. For any formula α in $\mathcal{L}_{\mathrm{MBCC}}$,*

$$\alpha \textit{ is } \mathrm{MBCC}\textit{-valid iff } f(\alpha) \textit{ is } \mathrm{MBC}\textit{-valid.}$$

By using Theorems 5.6 and 5.21 we have:

THEOREM 5.22 (Completeness for MBCC). *For any formula α in $\mathcal{L}_{\mathrm{MBCC}}$,*

$$\mathrm{MBCC} \vdash \Rightarrow \alpha \textit{ iff } \alpha \textit{ is } \mathrm{MBCC}\textit{-valid.}$$

Acknowledgments. We would like to thank the anonymous referees for their valuable comments and suggestions. This work was supported by JSPS KAKENHI Grant Numbers JP18K11171, JP16KK0007 and JSPS Core-to-Core Program (A. Advanced Research Networks).

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