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CONNEXIVE CONDITIONAL LOGIC. Part I

Abstract. In this paper, first some propositional conditional logics based on Belnap and Dunn's useful four-valued logic of first-degree entailment are introduced semantically, which are then turned into systems of weakly and unrestrictedly connexive conditional logic. The general frame semantics for these logics makes use of a set of allowable (or admissible) extension/anti-extension pairs. Next, sound and complete tableau calculi for these logics are presented. Moreover, an expansion of the basic conditional connexive logics by a constructive implication is considered, which gives an opportunity to discuss recent related work, motivated by the combination of indicative and counterfactual conditionals. Tableau calculi for the basic constructive connexive conditional logics are defined and shown to be sound and complete with respect to their semantics. This semantics has to ensure a persistence property with respect to the preorder that is used to interpret the constructive implication.

Keywords: conditional logic; connexive logic; paraconsistent logic; Chellas frames; Segerberg frames; general frames; first-degree entailment logic; Aristotle's theses; Boethius' theses; extension/anti-extension pairs; tableaux

1. Introduction

In this paper we consider a weak conditional, $\Box\!\!\rightarrow$, that satisfies unrestrictedly Aristotle's Theses and Boethius' Theses if a very natural condition is imposed on suitable semantical models (see below):

(AT) $\sim(\sim A \Box\!\!\rightarrow A)$,
(AT)′ $\sim(A \Box\!\!\rightarrow \sim A)$,
(BT) $(A \Box\!\!\rightarrow B) \Box\!\!\rightarrow \sim(A \Box\!\!\rightarrow \sim B)$, and
(BT)′ $(A \Box\!\!\rightarrow \sim B) \Box\!\!\rightarrow \sim(A \Box\!\!\rightarrow B)$.

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Logics validating these theses are called "connexive logics" and turn out to be contra-classical in the sense that some classically invalid formulas can be proven.[1] One can construct a connexive logic from scratch or obtain one by expanding a given non-connexive logic, such as classical logic, by a connexive implication. We take first-degree entailment logic as our starting point for obtaining a connexive logic and then expand our language by a further constructive conditional.[2]

There are several ways of defining systems of connexive logic by making use of various semantical constructions and proof-theoretical frameworks, for surveys and some recent contributions see [McCall, 2012; Wansing, 2014; Wansing et al., 2016]. A straightforward and conceptually clear road to connexivity consists of requiring suitable falsity conditions for implications. This approach is particularly natural for expansions of first-degree entailment logic, **FDE**, because **FDE** is a basic and simple four-valued logic which clearly separates truth and falsity from each other as two independent semantical dimensions. The connexive logic **C** from [Wansing, 2005] imposes such falsity conditions on the constructive implication in a certain expansion of **FDE**, namely David Nelson's paraconsistent logic **N4** [Almukdad and Nelson, 1984; Kamide and Wansing, 2015; Odintsov, 2008]. In this paper we consider weakly and unrestrictedly connexive expansions of **FDE** starting from a different and very weak implication. The perspective for this endeavor is that of *conditional logic*, see, for example, [Nute, 1984], a setting in which the conditional is usually denoted by '$\Box\!\!\rightarrow$'. Another starting point has been taken by Hitoshi Omori [2016], who considers a connexive variant of implication in relevance logic. One could also apply the modification that leads from **N4** to **C** to the substructural subsystems of **N4** introduced in [Wansing, 1993a,b]. We, however, do not pursue these approaches here.

Our point of departure for obtaining systems of connexive conditional logic is the logic **CK** introduced by Brian Chellas [1975] as a basic system of conditional logic. The semantics in [Chellas, 1975] employs what are now called "Chellas frames" ([Unterhuber, 2013, Ch. 4.3]; see Section 2). In addition to a non-empty set W of indices ("possible worlds"), a Chel-

[1] Note that there are some conditional logics [e.g., Adams, 1975] which validate versions of Aristotle's and Boethius' Theses by using a restriction that ensures that these logics do not conflict with classical logic [cf. Unterhuber, 2016]. These logics are in our terminology not connexive.

[2] Whereas it is possible to expand classical logic by a connexive conditional, this strategy may lead to counter-intuitive properties [Unterhuber, 2016].

las frame contains a ternary relation $R \subseteq W \times W \times \mathrm{Pow}(W)$, where $\mathrm{Pow}(W)$ is the powerset of W. A conditional $A \mathrel{\Box\!\!\rightarrow} B$ is then treated as a necessity statement $[A]B$ for a normal necessity operator $[A]$ indexed by A. In a model based on a Chellas frame, the relation R can be seen as comprising a collection of binary accessibility relations $R_{[\![A]\!]}$ on W, indexed by the truth set $[\![A]\!]$ of A in that model, for every formula A. Later Krister Segerberg [1989] considered general Chellas frames — which we call *Segerberg frames* [see also Unterhuber, 2013, Ch. 4.3; Unterhuber and Schurz, 2014].[3]

Since our base logic is the paraconsistent and paracomplete logic **FDE**, the set W is now to be understood as a set of states that may fail to support the truth or the falsity of a formula or support both the truth and the falsity of a formula. In order to obtain correspondences between schematic formulas and conditions on *frames*, we consider a suitable modification of Segerberg frames (i.e., a modification of general Chellas frames). A modified Segerberg frame is a triple $\langle W, R, P\rangle$, where $\langle W, R\rangle$ is a Chellas frame and $P \subseteq \mathrm{Pow}(W) \times \mathrm{Pow}(W)$. The set P may be seen as a set of allowable (or admissible) extension/anti-extension pairs.[4]

In Section 2, after introducing the syntax and semantics for the basic **FDE**-based conditional logic $\mathbf{CK}_{\mathrm{FDE}}$, we define a variant of it, the system **cCL**, which is obtained from $\mathbf{CK}_{\mathrm{FDE}}$ by changing the falsity conditions of implications. The system **cCL** validates Boethius' theses in rule form:

$$(A \mathrel{\Box\!\!\rightarrow} B) \vdash {\sim}(A \mathrel{\Box\!\!\rightarrow} {\sim}B), \qquad (A \mathrel{\Box\!\!\rightarrow} {\sim}B) \vdash {\sim}(A \mathrel{\Box\!\!\rightarrow} B),$$

and in this sense system **cCL** is a *weakly connexive conditional logic*, where the lower-case '**c**' stands for "weakly connexive."

In Section 3 we shall first define a sound and complete tableau calculus for $\mathbf{CK}_{\mathrm{FDE}}$ and then modify this calculus to obtain a tableau calculus for **cCL**. Whereas in the constructive connexive logic **C** the falsity

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[3] The semantics for conditional logics in [Priest, 2008, Chapter 5] uses binary relations R_A on W, for every *formula* A instead of relations R_X for sets of worlds X. This version of the semantics can be equipped with sound and complete tableau calculi, but the formula-annotated relations make the semantics dependent on the language, so that it is not suitable for developing a purely structural correspondence theory.

[4] Extensions and anti-extensions are nothing but truth sets and falsity sets, where the latter are sets of possible worlds that support the falsity of some formula. Extension/anti-extension pairs can be considered intensions in the sense that they determine for each possible world whether a certain formula is true, false, both, or neither.

conditions for the connexive implication lead to the validation of both Aristotle's and Boethius' theses, in **cCL**, Aristotle's and Boethius' theses are true in a model for **cCL** iff for all $w, w' \in W$ and all $X \subseteq W$, wR_Xw' implies $w' \in X$, which corresponds to $A \Box\!\!\rightarrow A$. The connexive logic characterized by the class of all models for **cCL** satisfying this natural condition will be referred to as **CCL**. Some properties of **cCL** and **CCL** are pointed out in Section 4.

It is not uncommon to consider logical systems with more than one implication, integrating different types of implication in one system. In conditional logic, the connective $\Box\!\!\rightarrow$ is usually added to a language of classical logic containing material implication, $\supset$. Moreover, there are, for example, the Lambek Calculus [Lambek, 1958] with its two directional implications and the systems of consequential implication [Pizzi and Williamson, 1997], which comprise, in addition to Boolean implication, a conditional that, notation adjusted, satisfies a "weak" Boethius' thesis, namely $(A \Box\!\!\rightarrow B) \supset {\sim}(A \Box\!\!\rightarrow {\sim}B)$. Andreas Kapsner and Hitoshi Omori [2017] define a logic with two implications, to be seen as representing an indicative and a counterfactual natural language conditional, respectively. Whereas their counterfactual conditional satisfies a weak version of Boethius' thesis, the indicative one is constructive (see Section 5).[5] Mathieu Vidal [2017b] takes a different approach to validating restrictedly connexive principles, based on a conditional logic. To this effect, he uses the composition of two functions — so-called *neutralization* and *expansion functions* (see Section 5).

We shall discuss Kapsner and Omori's system as well as a semantics due to Vidal [2017b] in Section 5 before defining two constructive conditional logics, **cCCL** and **CCCL**, in Section 6. Whilst **cCCL** is weakly connexive, **CCCL** is connexive. Our semantics expands the systems **cCL** and **CCL** by adding support of truth and support of falsity conditions for a constructive conditional, mirroring the approach of Kaspner and Omori. However, with respect to $\Box\!\!\rightarrow$, **cCCL** and **CCCL** do not go beyond the postulates validated in systems **cCL** and **CCL**. This contrasts with Kaspner and Omori's setting, who take the much stronger semantics of Lewis [1973] as a starting point for obtaining restrictedly connexive conditionals. The logics **cCCL** and **CCCL** satisfy the persistence properties from Nelson's constructive logics with strong negation. We shall define

[5] Note that a conditional can be both, constructive and connexive. For example, the conditional of the system **C** of Wansing [2005] satisfies both properties.

From the literature	**FDE**	first-degree entailment logic [Belnap, 1977; Dunn, 1976]
	N4	David Nelson's paraconsistent constructive logic [Almukdad and Nelson, 1984]
	C	paraconsistent constructive connexive logic [Wansing, 2005]
	CK	basic conditional logic [Chellas, 1975; Segerberg, 1989; Unterhuber, 2013; Unterhuber and Schurz, 2014]
	CKR	conditional logic **CK** extended by $A \boxright A$ (reflexivity) ([Unterhuber, 2013, Ch. 7.1]; see also [Chellas, 1975; Segerberg, 1989; Unterhuber and Schurz, 2014])
This paper	$\mathbf{CK}_{\text{FDE}}$	basic conditional logic **CK** based on **FDE** [6]
	$\mathbf{CKR}_{\text{FDE}}$	conditional logic **CKR** based on **FDE**
	cCL	basic weakly connexive conditional logic
	CCL	basic connexive conditional logic, **cCL** extended by $A \boxright A$
	cCCL	basic weakly connexive constructive conditional logic
	CCCL	basic connexive constructive conditional logic, **cCCL** extended by $A \boxright A$

Table 1. Overview of the systems investigated in the paper and relevant systems in the literature

tableau proof systems for **cCCL** and **CCCL** and show these calculi to be sound and complete. In Section 7, we conclude the paper with some brief remarks on future work.

Since we will refer to and introduce a number of logical systems, Table 1 may be helpful. The central logics of the present paper are **cCL** and **CCL** and their expansion by a second constructive implication, **cCCL** and **CCCL**.

2. $\mathbf{CK}_{\text{FDE}}$, $\mathbf{CKR}_{\text{FDE}}$, cCL, and CCL: syntax and semantics

The language $\mathfrak{L}$ of connexive conditional logic is based on a denumerable set PV of propositional variables $p_1, p_2, p_3, \ldots$. We use lower case letter

[6] Priest [2008, Ch. 8] describes a system that is a fragment of our system $\mathbf{CK}_{\text{FDE}}$. It lacks the rules that correspond to Left Logical Equivalence in system **CK** (see Section 3).

p, q, r etc. to stand for propositional variables and upper-case letters A, B, C, etc. to stand for formulas. The formulas of $\mathfrak{L}$ are given by the following grammar:

$$A := p \mid {\sim}A \mid (A \wedge A) \mid (A \vee A) \mid (A \mathrel{\Box\!\!\rightarrow} A) \mid (A \mathrel{\Diamond\!\!\rightarrow} A)$$

We shall sometimes omit outermost brackets of formulas.[7]

As in [Unterhuber, 2013; Unterhuber and Schurz, 2014], in addition to $\mathfrak{L}$ we shall use a language $\mathfrak{L}_{\mathbf{FC}}$ for talking about (general) frame conditions. The language $\mathfrak{L}_{\mathbf{FC}}$ is a two-sorted, set-theoretic language which contains (i) variables $w, w', w'', \ldots$ for states and X, Y, X', Y', X_1, X_2, … for sets of states, (ii) the connectives $\neg$ ("negation"), $\curlywedge$ ("conjunction"), $\curlyvee$ ("disjunction"), and $\Rightarrow$ ("material implication"), (iii) the quantifiers $\forall$ ("the universal quantifier"), and $\exists$ ("the existential quantifier"), (iv) the non-logical ternary predicate R (denoting "the accessibility relation"), (v) the non-logical binary predicate P (denoting the set of "admissible extension/anti-extension pairs"), and (vi) the constant W. We shall abbreviate '$R(w, w', X)$' by '$wR_X w'$', instead of '$\neg\, x \in X$', we shall write '$x \notin X$', and later we shall introduce another binary predicate, $\leq$, (denoting the set of state pairs $\langle w, w' \rangle$ such that w' is a "possible expansion of" w). Moreover, we shall also sometimes use the connectives and quantifiers of $\mathfrak{L}_{\mathbf{FC}}$ in our metalanguage (and sometimes will not pay attention to the use-mention distinction).

Definition 1. A pair $\langle W, R \rangle$ is a Chellas frame (or just a frame) iff

- W is a non-empty set, intuitively understood as a set of information states, and
- $R \subseteq W \times W \times \mathrm{Pow}(W)$, where $\mathrm{Pow}(W)$ is the power set of W.

If $\langle W, R \rangle$ is a frame, then $\mathfrak{M} = \langle W, R, v^+, v^- \rangle$ is a model iff v^+ and v^- are valuation functions $v^+ \colon \mathrm{PV} \longrightarrow \mathrm{Pow}(W)$ and $v^- \colon \mathrm{PV} \longrightarrow \mathrm{Pow}(W)$.

Since we want to define expansions of the paraconsistent logic **FDE**, we shall draw a distinction between the support of truth and the support of falsity in models.

Definition 2. A model $\mathfrak{M} = \langle W, R, v^+, v^- \rangle$ is a model for $\mathbf{CK}_{\mathrm{FDE}}$ iff support of truth and support of falsity relations $\models^+$ and $\models^-$ between $\mathfrak{M}$, states $w \in W$, and formulas from $\mathfrak{L}$ are inductively defined as follows:

[7] The connective $\mathrel{\Diamond\!\!\rightarrow}$ is sometimes called a "might conditional".

$\mathfrak{M}, w \models^+ p$	iff	$w \in v^+(p)$ for $p \in \mathrm{PV}$
$\mathfrak{M}, w \models^- p$	iff	$w \in v^-(p)$ for $p \in \mathrm{PV}$
$\mathfrak{M}, w \models^+ {\sim}A$	iff	$\mathfrak{M}, w \models^- A$
$\mathfrak{M}, w \models^- {\sim}A$	iff	$\mathfrak{M}, w \models^+ A$
$\mathfrak{M}, w \models^+ A \wedge B$	iff	$\mathfrak{M}, w \models^+ A$ and $\mathfrak{M}, w \models^+ B$
$\mathfrak{M}, w \models^- A \wedge B$	iff	$\mathfrak{M}, w \models^- A$ or $\mathfrak{M}, w \models^- B$
$\mathfrak{M}, w \models^+ A \vee B$	iff	$\mathfrak{M}, w \models^+ A$ or $\mathfrak{M}, w \models^+ B$
$\mathfrak{M}, w \models^- A \vee B$	iff	$\mathfrak{M}, w \models^- A$ and $\mathfrak{M}, w \models^- B$
$\mathfrak{M}, w \models^+ A \Box\!\!\rightarrow B$	iff	for all $w' \in W$ such that $wR_{[\![A]\!]^{\mathfrak{M}}}w'$ it holds that $\mathfrak{M}, w' \models^+ B$
$\mathfrak{M}, w \models^- A \Box\!\!\rightarrow B$	iff	there is a $w' \in W$ such that $wR_{[\![A]\!]^{\mathfrak{M}}}w'$ and $\mathfrak{M}, w' \models^- B$
$\mathfrak{M}, w \models^+ A \Diamond\!\!\rightarrow B$	iff	there is a $w' \in W$ such that $wR_{[\![A]\!]^{\mathfrak{M}}}w'$ and $\mathfrak{M}, w' \models^+ B$
$\mathfrak{M}, w \models^- A \Diamond\!\!\rightarrow B$	iff	for all $w' \in W$ such that $wR_{[\![A]\!]^{\mathfrak{M}}}w'$ it holds that $\mathfrak{M}, w' \models^- B$

where the set $[\![A]\!]^{\mathfrak{M}}$ is defined as $\{w \mid \mathfrak{M}, w \models^+ A\}$, i.e., as the set of all states supporting the truth of A in $\mathfrak{M}$. When the context is clear, we sometimes write $[\![A]\!]$ instead of $[\![A]\!]^{\mathfrak{M}}$.

Remark 1. The support of truth and support of falsity conditions for formulas $A \Box\!\!\rightarrow B$ and $A \Diamond\!\!\rightarrow B$ privilege the set $\{w \mid \mathfrak{M}, w \models^+ A\}$ over the set $\{w \mid \mathfrak{M}, w \models^- A\}$, i.e. over $[\![{\sim}A]\!]$. We may think of $\{w \mid \mathfrak{M}, w \models^+ A\}$ as the extension of A in $\mathfrak{M}$ and of $\{w \mid \mathfrak{M}, w \models^- A\}$ as the anti-extension of A in $\mathfrak{M}$. If one wants to simultaneously impose conditions on the relation R related to both extensions and anti-extensions, one would have to use a four-place relation $R \subseteq W \times W \times \mathrm{Pow}(W) \times \mathrm{Pow}(W)$. This will give us the following conditions:

$\mathfrak{M}, w \models^+ A \Box\!\!\rightarrow B$	iff	for all $w' \in W$ such that $wR_{\langle [\![A]\!]^{\mathfrak{M}}, [\![{\sim}A]\!]^{\mathfrak{M}} \rangle}w'$ it holds that $\mathfrak{M}, w' \models^+ B$
$\mathfrak{M}, w \models^- A \Box\!\!\rightarrow B$	iff	there is a $w' \in W$ such that $wR_{\langle [\![A]\!]^{\mathfrak{M}}, [\![{\sim}A]\!]^{\mathfrak{M}} \rangle}w'$ and $\mathfrak{M}, w' \models^- B$
$\mathfrak{M}, w \models^+ A \Diamond\!\!\rightarrow B$	iff	there is a $w' \in W$ such that $wR_{\langle [\![A]\!]^{\mathfrak{M}}, [\![{\sim}A]\!]^{\mathfrak{M}} \rangle}w'$ and $\mathfrak{M}, w' \models^+ B$
$\mathfrak{M}, w \models^- A \Diamond\!\!\rightarrow B$	iff	for all $w' \in W$ such that $wR_{\langle [\![A]\!]^{\mathfrak{M}}, [\![{\sim}A]\!]^{\mathfrak{M}} \rangle}w'$ it holds that $\mathfrak{M}, w' \models^- B$.

This modification is especially reasonable if one wants to expand the object language by a constructive implication, $\rightarrow$, and a constructive equivalence connective, $\leftrightarrow$, i.e, if one wants to add the conditional $\Box\!\!\rightarrow$ not to **FDE**, but to Nelson's constructive paraconsistent logic **N4**. Whereas in **N4** provable equivalence fails to be a congruence relation, provable strong equivalence *is* a congruence relation, where the strong equivalence of A and B, $A \Leftrightarrow B$, is defined as $(A \leftrightarrow B) \wedge (\sim A \leftrightarrow \sim B)$. This provides a strong reason for identifying a proposition not just with a set of states, understood as an extension, but as a pair of sets of states, understood as an extension/anti-extension pair. In this paper, however, we shall follow the standard approach that makes use of a ternary accessibility relation, i.e., a binary relation between worlds, indexed by a set of states (see, however, Remark 3).

DEFINITION 3. A triple $\langle W, R, P\rangle$ is a general frame (or Segerberg frame) for $\mathbf{CK}_{\mathrm{FDE}}$ iff

- $\langle W, R\rangle$ is a frame,
- $R \subseteq W \times W \times \mathrm{l}(P)$, where $\mathrm{l}(P) = \{X \mid \langle X, Y\rangle \in P\}$ and
- $P \subseteq (\mathrm{Pow}(W) \times \mathrm{Pow}(W))$, where P satisfies the following conditions:
  1. if $\langle X, Y\rangle \in P$, then $\langle Y, X\rangle \in P$,[8]
  2. if $\langle X, Y\rangle, \langle X', Y'\rangle \in P$, then $\langle X \cap X', Y \cup Y'\rangle \in P$ and $\langle X \cup X', Y \cap Y'\rangle \in P$,
  3. if $\langle X, Y\rangle, \langle X', Y'\rangle \in P$, then $\langle\{w \in W \mid \forall w' \in W(wR_X w' \Rightarrow w' \in X')\} \in P$ and $\{w \in W \mid \exists w' \in W(wR_X w' \curlywedge w' \in Y')\} \in P$,
  4. if $\langle X, Y\rangle, \langle X', Y'\rangle \in P$, then $\langle\{w \in W \mid \exists w' \in W(wR_X w' \curlywedge w' \in X')\} \in P$ and $\{w \in W \mid \forall w' \in W(wR_X w' \Rightarrow w' \in Y')\} \in P$.

The set P is a set of pairs of sets of states; intuitively P contains the admissible extension/anti-extension pairs.

DEFINITION 4. Let $\langle W, R, P\rangle$ be a general frame for $\mathbf{CK}_{\mathrm{FDE}}$. The tuple $\langle W, R, P, v^+, v^-\rangle$ is a general model for $\mathbf{CK}_{\mathrm{FDE}}$ iff $\langle W, R, v^+, v^-\rangle$ is a model and $\langle [\![p]\!], [\![\sim p]\!]\rangle \in P$ for every $p \in \mathrm{PV}$. Support of truth and support of falsity relations $\models^+$ and $\models^-$ are defined as in the case of models for $\mathbf{CK}_{\mathrm{FDE}}$.

[8] In view of 1, instead of $\mathrm{l}(P)$ we could have used $\mathrm{r}(P) := \{Y \mid \langle X, Y\rangle \in P\}$.

LEMMA 1. *Let* $\langle W, R, P, v^+, v^-\rangle$ *be a general model for* $\mathbf{CK}_{\mathrm{FDE}}$. *Then for every* $\mathfrak{L}$*-formula* A, $\langle [\![A]\!], [\![\sim A]\!]\rangle \in P$.

PROOF. By induction on the complexity of A. If A is a propositional variable, the claim holds by definition. Let A be a formula $\sim B$ and assume that $\langle [\![B]\!], [\![\sim B]\!]\rangle \in P$. Then, by 1, $\langle [\![\sim B]\!], [\![B]\!]\rangle \in P$ and by the support of truth and support of falsity conditions for negated formulas, $\langle [\![\sim B]\!], [\![\sim\sim B]\!]\rangle \in P$. Let A be a conjunction $B \wedge C$ and assume that $\langle [\![B]\!], [\![\sim B]\!]\rangle \in P$ and $\langle [\![C]\!], [\![\sim C]\!]\rangle \in P$. By 2, $\langle [\![B]\!] \cap [\![C]\!], [\![\sim B]\!] \cup [\![\sim C]\!]\rangle \in P$ and by Definition 2, $\langle [\![B \wedge C]\!], [\![\sim(B \wedge C)]\!]\rangle \in P$. The case that A is a disjunction $B \vee C$ is dual. Let A be a formula $B \mathrel{\Box\!\!\rightarrow} C$ and assume that $\langle [\![B]\!], [\![\sim B]\!]\rangle \in P$ and $\langle [\![C]\!], [\![\sim C]\!]\rangle \in P$. Then, by 3,

$$\langle \{w \in W \mid \forall w' \in W(wR_{[\![B]\!]}w' \Rightarrow w' \in [\![C]\!])\} \in P,$$
$$\{w \in W \mid \exists w' \in W(wR_{[\![B]\!]}w' \curlywedge w' \in [\![\sim C]\!])\}\rangle \in P.$$

By Definition 2, $\langle [\![B \mathrel{\Box\!\!\rightarrow} C]\!], [\![\sim(B \mathrel{\Box\!\!\rightarrow} C)]\!]\rangle \in P$. The case that A is a formula $B \mathrel{\Diamond\!\!\rightarrow} C$ is analogous and makes use of condition 4. □

We now modify the support of falsity conditions for conditionals $A \mathrel{\Box\!\!\rightarrow} B$ in analogy to the modification that leads from **N4** to **C**, and we modify the support of falsity conditions for strongly negated formulas $A \mathrel{\Diamond\!\!\rightarrow} B$ accordingly, so as to obtain models for a weakly connexive variant of $\mathbf{CK}_{\mathrm{FDE}}$, system **cCL**.

DEFINITION 5. A model $\mathfrak{M} = \langle W, R, v^+, v^-\rangle$ is a model for **cCL** iff support of truth and support of falsity relations $\models^+$ and $\models^-$ between $\mathfrak{M}$, states $w \in W$, and formulas from $\mathfrak{L}$ are inductively defined as in Definition 2 except that:

$\mathfrak{M}, w \models^+ A \mathrel{\Box\!\!\rightarrow} B$	iff	for all $w' \in W$ such that $wR_{[\![A]\!]}w'$ it holds that $\mathfrak{M}, w' \models^+ B$
$\mathfrak{M}, w \models^- A \mathrel{\Box\!\!\rightarrow} B$	iff	for all $w' \in W$ such that $wR_{[\![A]\!]}w'$ it holds that $\mathfrak{M}, w' \models^- B$
$\mathfrak{M}, w \models^+ A \mathrel{\Diamond\!\!\rightarrow} B$	iff	there is a $w' \in W$ such that $wR_{[\![A]\!]}w'$ and $\mathfrak{M}, w' \models^+ B$
$\mathfrak{M}, w \models^- A \mathrel{\Diamond\!\!\rightarrow} B$	iff	there is a $w' \in W$ such that $wR_{[\![A]\!]}w'$ and $\mathfrak{M}, w' \models^- B$.

Remark 2. The falsity conditions for $A \mathrel{\Box\!\!\rightarrow} B$ in models for **cCL** coincide with the falsity conditions for $A \mathrel{\Diamond\!\!\rightarrow} B$ in models for $\mathbf{CK}_{\mathrm{FDE}}$. Whereas in $\mathbf{CK}_{\mathrm{FDE}}$ the connectives $\mathrel{\Box\!\!\rightarrow}$ and $\mathrel{\Diamond\!\!\rightarrow}$ are dual to each other in the sense that (i) a state supports the truth (falsity) of $A \mathrel{\Box\!\!\rightarrow} B$ if and only if it supports the falsity (truth) of $A \mathrel{\Diamond\!\!\rightarrow} \sim B$ and (ii) a state supports the truth (falsity) of $A \mathrel{\Diamond\!\!\rightarrow} B$ if and only if it supports the falsity (truth) of $A \mathrel{\Box\!\!\rightarrow} \sim B$, this kind of duality fails in **cCL**.

Definition 6. A triple $\langle W, R, P\rangle$ is a general frame (or Segerberg frame) for **cCL** iff

- $\langle W, R\rangle$ is a frame,
- $R \subseteq W \times W \times \mathrm{l}(P)$, where $\mathrm{l}(P) = \{X \mid \langle X, Y\rangle \in P\}$ and
- $P \subseteq (\,\mathrm{Pow}(W) \times \mathrm{Pow}(W))$, where P satisfies the following conditions:
  1. if $\langle X, Y\rangle \in P$, then $\langle Y, X\rangle \in P$,
  2. if $\langle X, Y\rangle, \langle X', Y'\rangle \in P$, then $\langle X \cap X', Y \cup Y'\rangle \in P$ and $\langle X \cup X', Y \cap Y'\rangle \in P$,
  3. if $\langle X, Y\rangle, \langle X', Y'\rangle \in P$, then $\langle\{w \in W \mid \forall w' \in W(wR_X w' \Rightarrow w' \in X')\} \in P$ and $\{w \in W \mid \forall w' \in W(wR_X w' \Rightarrow w' \in Y')\} \in P$,
  4. if $\langle X, Y\rangle, \langle X', Y'\rangle \in P$, then $\langle\{w \in W \mid \exists w' \in W(wR_X w' \curlywedge w' \in X')\} \in P$ and $\{w \in W \mid \exists w' \in W(wR_X w' \curlywedge w' \in Y')\} \in P$.

Definition 7. Let $\langle W, R, P\rangle$ be a general frame for **cCL**. The tuple $\langle W, R, P, v^+, v^-\rangle$ is a general model for **cCL** iff $\langle W, R, v^+, v^-\rangle$ is a model and $\langle [\![p]\!], [\![\sim p]\!]\rangle \in P$ for every $p \in \mathrm{PV}$. Support of truth and support of falsity relations $\models^+$ and $\models^-$ are defined as in the case of models for **cCL**.

Lemma 2. *Let $\langle W, R, P, v^+, v^-\rangle$ be a general model for* **cCL**. *Then for every $\mathfrak{L}$-formula A, $\langle [\![A]\!], [\![\sim A]\!]\rangle \in P$.*

Proof. By induction on the complexity of A. □

Definition 8. We say that a formula A is valid in a model $\mathfrak{M} = \langle W, R, v^+, v^-\rangle$ or a general model $\mathfrak{M} = \langle W, R, P, v^+, v^-\rangle$ iff $\mathfrak{M}, w \models^+ A$ for all $w \in W$. If A is valid in $\mathfrak{M}$, we write $\mathfrak{M} \models A$. We say that A is valid on a frame $\mathfrak{F} = \langle W, R\rangle$ or a general frame $\mathfrak{F} = \langle W, R, P\rangle$, and write $\mathfrak{F} \models A$, iff $\mathfrak{M} \models A$ for all models $\mathfrak{M} = \langle W, R, v^+, v^-\rangle$, respectively general models $\mathfrak{M} = \langle W, R, P, v^+, v^-\rangle$, based on $\mathfrak{F}$. A formula is valid with respect to a class of (general) models or (general) frames iff it is valid in every (general) model, respectively on every (general) frame from that class.

It is very natural to require that if $wR_{[\![A]\!]}w'$, then $w' \in [\![A]\!]$, or, in purely structural terms, that if $wR_X w'$, then $w' \in X$. We now semantically define four basic conditional logics.

DEFINITION 9. The logic $\mathbf{CK}_{\mathrm{FDE}}$ ($\mathbf{cCL}$) is the set of all $\mathfrak{L}$-formulas valid with respect to the class of all models for $\mathbf{CK}_{\mathrm{FDE}}$ ($\mathbf{cCL}$).

The logic $\mathbf{CKR}_{\mathrm{FDE}}$ ($\mathbf{CCL}$) is the set of all $\mathfrak{L}$-formulas valid with respect to the class of all models for $\mathbf{CK}_{\mathrm{FDE}}$ ($\mathbf{cCL}$) satisfying the following condition on frames:

$$\mathbb{C}_{A\Box\!\!\to A} \quad (\forall X \subseteq W)(\forall w, w' \in W) wR_X w' \Rightarrow w' \in X.$$

If a model, general frame, or general model for $\mathbf{CK}_{\mathrm{FDE}}$ ($\mathbf{cCL}$) satisfies $\mathbb{C}_{A\Box\!\!\to A}$ it will be called a model, general frame, or general model, respectively, for $\mathbf{CKR}_{\mathrm{FDE}}$ ($\mathbf{CCL}$).

DEFINITION 10. Let $\Gamma \cup \{A\}$ be a set of $\mathfrak{L}$-formulas, and let $\mathbf{L}$ be a logic. We say that Γ entails A in $\mathbf{L}$ ($\Gamma \models_{\mathbf{L}} A$) iff for every model $\mathfrak{M}$ for $\mathbf{L}$ and state w of $\mathfrak{M}$ it holds that if $\mathfrak{M}, w \models^+ B$ for every $B \in \Gamma$, then $\mathfrak{M}, w \models^+ A$.

DEFINITION 11. A formula A C-corresponds to a frame condition $\mathbb{C}$ (and vice versa) iff for all frames $\mathfrak{F}$ it holds that $\mathfrak{F}$ satisfies $\mathbb{C}$ iff $\mathfrak{F} \models A$. A formula A S-corresponds to a frame condition $\mathbb{C}$ (and vice versa) iff for all general frames $\mathfrak{F}$ it holds that $\mathfrak{F}$ satisfies $\mathbb{C}$ iff $\mathfrak{F} \models A$.

DEFINITION 12. Let A, B_1, ..., B_n be $\mathfrak{L}$-formulas. A derivability statement $\{B_1, \ldots, B_n\} \vdash A$ C-corresponds to a frame condition $\mathbb{C}$ (and vice versa) iff for all frames $\mathfrak{F}$ it holds that $\mathfrak{F}$ satisfies $\mathbb{C}$ iff (for every model $\mathfrak{M}$ based on $\mathfrak{F}$ and every state w of $\mathfrak{M}$, if $\mathfrak{M}, w \models^+ B_1, \ldots, \mathfrak{M}, w \models^+ B_n$, then $\mathfrak{M}, w \models^+ A$). A statement $\{B_1, \ldots, B_n\} \vdash A$ S-corresponds to a frame condition $\mathbb{C}$ (and vice versa) iff for all general frames $\mathfrak{F}$ it holds that $\mathfrak{F}$ satisfies $\mathbb{C}$ iff (for every general model $\mathfrak{M}$ based on $\mathfrak{F}$ and every state w of $\mathfrak{M}$, if $\mathfrak{M}, w \models^+ B_1, \ldots, \mathfrak{M}, w \models^+ B_n$, then $\mathfrak{M}, w \models^+ A$).

The condition $\mathbb{C}_{A\Box\!\!\to A}$ is not only a very natural condition in view of the familiar understanding of the set $\{w' \in W \mid wR_{[\![A]\!]}w'\}$. It can easily be seen that $\mathbb{C}_{A\Box\!\!\to A}$ C-corresponds to the schematic formula $A \Box\!\!\to A$ (in $\mathbf{CK}_{\mathrm{FDE}}$, $\mathbf{cCL}$, $\mathbf{CKR}_{\mathrm{FDE}}$, and $\mathbf{CCL}$).

Let us now focus on the parameter P. In [Unterhuber, 2013, p. 199], essential use is made of Segerberg frames in the Henkin-style completeness proof for a number of conditional logics from a lattice of extensions of $\mathbf{CK}$. The impact of the parameter P is in a way minimized due to the fact that a conditional logic from that lattice is complete with respect to a class of Segerberg frames just in case it complete with respect to some class of Chellas models. Moreover, it can be shown that *any* conditional logic whatsoever from the mentioned lattice of systems is complete with respect to some class of Chellas models. To avoid such a triviality result for Segerberg frames, trivial and non-trivial frame conditions are distinguished, where only the former make use of use of inessential expressions as defined in [Unterhuber and Schurz, 2014] and can be characterized in terms of a standardized translation from logical principles into frame conditions. For example, the principle $(A \boxright B) \supset B$ corresponds to the trivial frame condition $(\forall X \subseteq W)(\forall w \in W)(\forall w'(wR_X w' \Rightarrow w' \in Y) \Rightarrow w \in Y)$ as well as the non-trivial frame condition $(\forall X \subseteq W)(\forall w \in W)(wR_X w)$, where the trivial frame condition uses inessential expressions, such as Y and w', and results from a standard translation of $(A \boxright B) \supset B$ (cf. [Unterhuber and Schurz, 2014]).[9] Note that none of the frame conditions employ the parameter P, although the parameter P plays an integral role in the completeness proof. The resulting completeness proofs are then non-trivial by requiring that extensions of Segerberg frames are restricted to non-trivial frame conditions.

However, with **FDE** as our base logic and general frames as introduced in Definitions 3 and 6, the situation is different.We have to explicitly use the parameter P for an adequate formulation of frame conditions. In extensions of **CK**, with classical logic as its base logic, the formula $({\sim}A \wedge A) \boxright {\sim}A$, for example, is valid on a frame iff the frame satisfies the following condition, where $\overline{X}$ is the complement of X: $(\forall X \subseteq W)(\forall w, w' \in W)(xR_{\overline{X} \cap X}\ w' \Rightarrow w' \in \overline{X})$. In the semantics for $\mathbf{CK}_{\mathrm{FDE}}$ and **cCL**, the evaluation of a strongly negated propositional variable ${\sim}p$ in a model is completely independent of the evaluation of p, and it is not clear how to capture the formula $({\sim}A \wedge A) \boxright {\sim}A$ by an $\mathfrak{L}_{\mathbf{FC}}$-formula that does not exhibit the binary predicate P. The following general frame condition, however, corresponds to $({\sim}A \wedge A) \boxright {\sim}A$:[10]

$$(\forall X, Y \subseteq W)(\forall w, w' \in W)(\langle X, Y\rangle \in P \Rightarrow (wR_{Y \cap X} w' \Rightarrow w' \in Y)).$$

If $\mathfrak{M} = \langle W, R, P, v^+, v^-\rangle$ is a general model, then $\langle [\![A]\!], [\![{\sim}A]\!]\rangle \in P$. Moreover, $[\![{\sim}A]\!] \cap [\![A]\!] = [\![{\sim}A \wedge A]\!]$. The condition thus ensures the

[9] We chose this example due to its simplicity and not because it is plausible as a conditional logic principle.

[10] Observe that a genuine frame correspondence result can be established for $({\sim}A \wedge A) \boxright {\sim}A$ only for systems $\mathbf{CK}_{\mathrm{FDE}}$ and **cCL**. This is not possible for systems $\mathbf{CKR}_{\mathrm{FDE}}$ and **CCL**, where $({\sim}A \wedge A) \boxright {\sim}A$ is validated already due to assuming condition $\mathbb{C}_{A \boxright A}$.

validity of $(\sim A \wedge A) \boxright \sim A$ in $\mathfrak{M}$. If a general frame $\mathfrak{F}$ does not satisfy the condition, then

$$(\exists X, Y \subseteq W)(\exists w, w' \in W)(\langle X, Y\rangle \in P \curlywedge (wR_{Y\cap X}w' \curlywedge w' \notin Y)),$$

and there is a general model based on $\mathfrak{F}$ and an instance of $(\sim A \wedge A) \boxright \sim A$ the truth of which is not supported at w. Consider $(\sim p \wedge p) \boxright \sim p$ and set $Y := [\![\sim p]\!]$, $X := [\![p]\!]$.

Similarly, $(\sim A \wedge B) \boxright \sim A$ corresponds to

(*) $(\forall X, Y, X', Y' \subseteq W)(\forall w, w' \in W)(\langle X, Y\rangle \in P \Rightarrow (\langle X', Y'\rangle \in P \Rightarrow (wR_{Y\cap X'}w' \Rightarrow w' \in Y)))$.

If $\mathfrak{M} = \langle W, R, P, v^+, v^-\rangle$ is a general model, then $\langle [\![A]\!], [\![\sim A]\!]\rangle, \langle [\![B]\!], [\![\sim B]\!]\rangle \in P$. Moreover, $[\![\sim A]\!] \cap [\![B]\!] = [\![\sim A \wedge B]\!]$. The condition thus ensures the validity of $(\sim A \wedge B) \boxright \sim A$ in $\mathfrak{M}$. If a general frame $\mathfrak{F}$ does not satisfy the condition, then

$$(\exists X, Y, X', Y' \subseteq W)(\exists w, w' \in W)(\langle X, Y\rangle \in P \curlywedge (\langle X', Y'\rangle \in P \curlywedge (wR_{Y\cap X'}w' \curlywedge w' \notin Y)))$$

and there is a general model based on $\mathfrak{F}$ and an instance of $(\sim A \wedge B) \boxright \sim A$ the truth of which is not supported at w. Consider $(\sim p \wedge q) \boxright \sim p$ and set $Y := [\![\sim p]\!]$, $X' := [\![q]\!]$. The condition (*) above is trivial in the sense that it involves quantification over the set variables X and Y' that do not occur at argument places of 'R', but at argument places of 'P'.

Remark 3. The four-place accessibility relation $R \subseteq W \times W \times \mathrm{Pow}(W) \times \mathrm{Pow}(W)$ introduced in Remark 1 would bring us back to pure frame correspondence instead of general frame correspondence. The formulas $(\sim A \wedge A) \boxright \sim A$ and $(\sim A \wedge B) \boxright \sim A$, for example, correspond to

$$(\forall X, Y \subseteq W)(\forall w, w' \in W)(wR_{\langle Y\cap X, X\cup Y\rangle}w' \Rightarrow w' \in Y)$$

and

(**) $(\forall X, Y, X', Y' \subseteq W)(\forall w, w' \in W)(wR_{\langle Y\cap X', X\cup Y'\rangle}w' \Rightarrow w' \in Y))$,

respectively. The latter condition is non-trivial in the sense that all set variables involved do occur at argument places of 'R' [cf. Unterhuber and Schurz, 2014, Definition 5.3]. For $(\sim A \wedge B) \boxright \sim A$, let $\mathfrak{M} = \langle W, R, v^+, v^-\rangle$ be a model. Since , $[\![\sim A]\!] \cap [\![B]\!] = [\![\sim A \wedge B]\!]$ and $[\![A]\!] \cup [\![\sim B]\!] = [\![\sim(\sim A \wedge B)]\!]$, the condition guarantees the validity of $(\sim A \wedge B) \boxright \sim A$ in $\mathfrak{M}$. If a frame $\mathfrak{F}$ does not satisfy the condition, then $(\exists X, Y \subseteq$

$W)(\exists w, w' \in W)(wR_{\langle Y \cap X, X \cup Y \rangle} w' \curlywedge w' \notin Y)$, and there is a model based on $\mathfrak{F}$ and an instance of $(\sim A \wedge B) \;\Box\!\!\rightarrow \sim A$ the truth of which is not supported at w. Consider again $(\sim p \wedge q) \;\Box\!\!\rightarrow \sim p$ and set $Y := [\![\sim p]\!]$, $X' := [\![q]\!]$.

3. Tableaux

We shall extend the tableau calculus for **FDE** presented in [Priest, 2008, Ch. 8] by tableau rules for $\Box\!\!\rightarrow$ that suitably modify the rules for $\Box\!\!\rightarrow$ from [Priest, 2008, Ch. 5] and by tableau rules for $\Diamond\!\!\rightarrow$, in order to obtain a tableau calculus for $\mathbf{CK}_{\mathrm{FDE}}$ in the language $\mathfrak{L}$. We will assume some familiarity with the tableau method as applied by Priest.

In tableaux for $\mathbf{CK}_{\mathrm{FDE}}$, tableau nodes consist of expressions of the form $A, +i$, or $A, -i$, or $ir_A j$, where A is an $\mathfrak{L}$-formula, i and j are natural numbers representing information states, $+$ indicates support of truth ($\models^+$), $-$ indicates failure of support of truth ($\not\models^+$), and r_A represents the accessibility relation $R_{[\![A]\!]}$ in the countermodel one tries to construct by unfolding a tableau. Tableau for a single conclusion derivability statement $\Delta \vdash B$ start with nodes of the form $A, +0$ for every premise A from the finite premise set Δ and a node of the form $B, -0$. Then tableau rules are applied (if that is possible) to tableau nodes leading to a more complex tableau. A branch of the tableau closes iff it contains a pair of nodes $A, +i$ and $A, -i$. The tableau closes iff all of its branches close. If a tableau (tableau branch) is not closed, it is called open. A tableau branch is said to be complete iff no more rules can be applied to expand it. A tableau is said to be complete iff each of its branches is complete. Closed branches will be marked by '×'.

The tableau rules for the connectives of **FDE** can be stated as follows:

$$\begin{array}{c} A \wedge B, +i \\ \downarrow \\ A, +i \\ B, +i \end{array} \qquad \begin{array}{c} A \wedge B, -i \\ \swarrow \quad \searrow \\ A, -i \quad B, -i \end{array} \qquad \begin{array}{c} A \vee B, +i \\ \swarrow \quad \searrow \\ A, +i \quad B, +i \end{array} \qquad \begin{array}{c} A \vee B, -i \\ \downarrow \\ A, -i \\ B, -i \end{array}$$

$$\begin{array}{c} \sim\sim A, \underline{+}i \\ \downarrow \\ A, \underline{+}i \end{array} \qquad \begin{array}{c} \sim(A \wedge B), \underline{+}i \\ \downarrow \\ \sim A \vee \sim B, \underline{+}i \end{array} \qquad \begin{array}{c} \sim(A \vee B), \underline{+}i \\ \downarrow \\ \sim A \wedge \sim B, \underline{+}i \end{array}$$

where the symbol $\underline{+}$ is to be read uniformly either as $+$ or as $-$. The tableau rules for the conditional $\Box\!\!\rightarrow$ and $\Diamond\!\!\rightarrow$ in $\mathbf{CK}_{\mathrm{FDE}}$ then are as follows ("CP" for "Conditional Logic Principle"):

$$
\begin{array}{cccc}
\text{CP1} & \text{CP2} & \text{CP3} & \text{CP4} \\
A \mathrel{\Box\!\!\rightarrow} B, +i & A \mathrel{\Diamond\!\!\rightarrow} B, -i & & \\
ir_A j & ir_A j & \sim(A \mathrel{\Box\!\!\rightarrow} B), \underline{+}i & \sim(A \mathrel{\Diamond\!\!\rightarrow} B), \underline{+}i \\
\downarrow & \downarrow & \downarrow & \downarrow \\
B, +j & B, -j & A \mathrel{\Diamond\!\!\rightarrow} \sim B, \underline{+}i & A \mathrel{\Box\!\!\rightarrow} \sim B, \underline{+}i
\end{array}
$$

$$
\begin{array}{cc}
\text{CP5} & \text{CP6} \\
A \mathrel{\Box\!\!\rightarrow} B, -i & A \mathrel{\Diamond\!\!\rightarrow} B, +i \\
\downarrow & \downarrow \\
ir_A j & ir_A j \\
B, -j & B, +j
\end{array}
$$

The leftmost two rules are applied whenever a node $ir_A j$ occurs on the branch; the rules CP5 and CP6 require the introduction of a new natural number j not already occurring in the tableau. We need, however, two more rules to account for what is known as "Left Logical Equivalence", LLE, namely the validity of the rules

$$
\frac{A \mathrel{\Box\!\!\rightarrow} C}{B \mathrel{\Box\!\!\rightarrow} C} \qquad\qquad \frac{A \mathrel{\Diamond\!\!\rightarrow} C}{B \mathrel{\Diamond\!\!\rightarrow} C}
$$

for logically equivalent formulas A and B.[11] To account for LLE, the following two rules ($reg \mathrel{\Box\!\!\rightarrow}$) and ($reg \mathrel{\Diamond\!\!\rightarrow}$), respectively, are added, where k and l have to be new numerals:

$$
\begin{array}{ccc}
 & B \mathrel{\Box\!\!\rightarrow} C, \pm i & \\
 & ir_A j & \\
\swarrow & \downarrow & \searrow \\
A, +k & B, +l & ir_B j \\
B, -k & A, -l &
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & B \mathrel{\Diamond\!\!\rightarrow} C, \pm i & \\
 & ir_A j & \\
\swarrow & \downarrow & \searrow \\
A, +k & B, +l & ir_B j \\
B, -k & A, -l &
\end{array}
$$

The rules can be read as saying that if state j is accessible from state i via $R_{[\![A]\!]}$ and if A and B are logically equivalent (if there is no state that supports the truth of one formula but fails to support the truth of the other), then state j is accessible from state i via $R_{[\![B]\!]}$. The requirement that $B \mathrel{\Box\!\!\rightarrow} C$, respectively $B \mathrel{\Diamond\!\!\rightarrow} C$, occurs on the branch is imposed for ($reg \mathrel{\Box\!\!\rightarrow}$) and ($reg \mathrel{\Diamond\!\!\rightarrow}$), respectively, because otherwise, for any expression $ir_A j$ on a branch, an infinite number of new expressions $ir_B j$ could be introduced — each due to a new formula B. A complete tableau could

[11] They are not valid in the basic conditional logic in [Priest, 2008, Ch. 5], which lacks corresponding rules. Note that Priest uses only formula-annotated accessibility relations in his semantics (cf. Section 5).

thus contain infinite branches, rendering the notion of a complete branch effectively inapplicable if no blocking techniques are used. The above set of tableau rules for the connectives from $\mathfrak{L}$ constitutes a tableau calculus T$\mathbf{CK}_{\mathrm{FDE}}$ for $\mathbf{CK}_{\mathrm{FDE}}$.

To obtain a tableau calculus for $\mathbf{cCL}$, the weakly connexive variant of $\mathbf{CK}_{\mathrm{FDE}}$, we replace CP3 and CP4 by the following rules, respectively:

$$\begin{array}{cc} \text{CP3}^* & \text{CP4}^* \\ {\sim}(A \mathrel{\Box\!\!\rightarrow} B), \underline{+}i & {\sim}(A \mathrel{\Diamond\!\!\rightarrow} B), \underline{+}i \\ \downarrow & \downarrow \\ A \mathrel{\Box\!\!\rightarrow} {\sim}B, \underline{+}i & A \mathrel{\Diamond\!\!\rightarrow} {\sim}B, \underline{+}i \end{array}$$

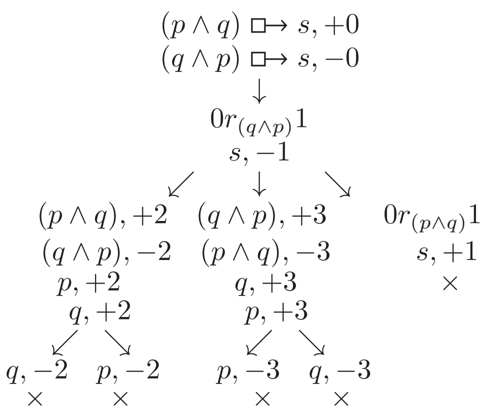
We shall refer to this modification of T$\mathbf{CK}_{\mathrm{FDE}}$ as T$\mathbf{cCL}$.

To obtain the tableau calculi T$\mathbf{CKR}_{\mathrm{FDE}}$ and T$\mathbf{CCL}$ for $\mathbf{CKR}_{\mathrm{FDE}}$ and $\mathbf{CCL}$, we add the following tableau rule to T$\mathbf{CK}_{\mathrm{FDE}}$ and T$\mathbf{cCL}$:

$$R_{A \mathrel{\Box\!\!\rightarrow} A} \qquad \begin{array}{c} ir_A j \\ \downarrow \\ A, +j \end{array}$$

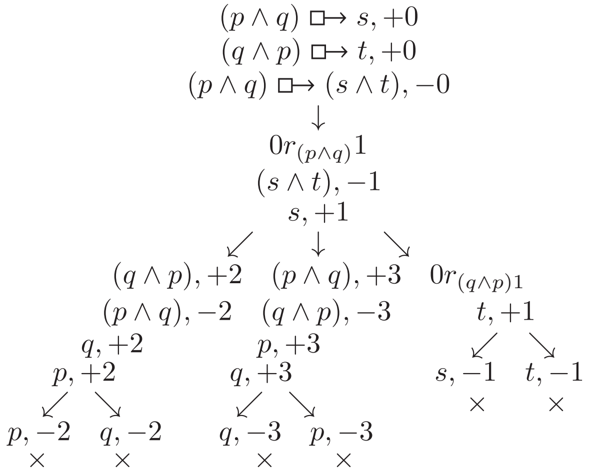
DEFINITION 13. Let $\mathbf{L}$ be a logic and let T$\mathbf{L}$ be a tableau calculus for $\mathbf{L}$. If $\Delta = \{B_1, \ldots, B_n, A\}$ is a finite set of $\mathfrak{L}$-formulas, then A is derivable from Δ in T$\mathbf{L}$ (in symbols: $\Delta \vdash_{\mathrm{T}\mathbf{L}} A$) iff there exists a closed and complete tableau for $B_1, +0, \ldots, B_n, +0, A, -0$ in T$\mathbf{L}$.

As a first example of a tableau proof, we present a proof of a very simple instance of LLE. We show that $\{(p \wedge q) \mathrel{\Box\!\!\rightarrow} r\} \vdash (q \wedge p) \mathrel{\Box\!\!\rightarrow} r$ in T$\mathbf{CK}_{\mathrm{FDE}}$:[12]



[12] To save space, we sometimes omit arrows.

Also the following example makes use of the rule (*reg* $\Box\!\!\rightarrow$). We show that $\{(p \wedge q) \Box\!\!\rightarrow s, (q \wedge p) \Box\!\!\rightarrow t\} \vdash (p \wedge q) \Box\!\!\rightarrow (s \wedge t)$ in $\mathbf{T}\mathbf{CK}_{\mathrm{FDE}}$:



Rule CP3* seems to suggest that System T**cCL** validates the principle of conditional excluded middle (CEM), a principle which is specific to Stalnaker's conditional logic [Stalnaker, 1968, 1980], but is not valid in a number of standard conditional logics, including Lewis' [1973] preferred system for counterfactuals. This becomes more evident if we use CEM*, $\sim(A \Box\!\!\rightarrow B) \supset (A \Box\!\!\rightarrow \sim B)$, which is — in the referenced systems — logically equivalent to CEM, i.e., $(A \Box\!\!\rightarrow B) \vee (A \Box\!\!\rightarrow \sim B)$. However, the effect of using rule CP3* differs from adopting CEM*: In the referenced systems we can infer $A \Box\!\!\rightarrow B$ from $A \Diamond\!\!\rightarrow B$ (by either CEM or CEM*), whereas this inference is not valid in system T**cCL**.

Definition 14. Let $\mathfrak{M} = \langle W, R, v^+, v^- \rangle$ ($\mathfrak{M} = \langle W, R, P, v^+, v^- \rangle$) be any model (general model) for $\mathbf{CK}_{\mathrm{FDE}}$ or **cCL** and let *br* be a tableau branch. The model $\mathfrak{M}$ is said to be faithful to *br* iff there is a function f from the set of all natural numbers to W such that:

1. for every node $A, +i$ on *br*, $\mathfrak{M}, f(i) \models^+ A$;
2. for every node $A, -i$ on *br*, $\mathfrak{M}, f(i) \not\models^+ A$;
3. for every node $jr_A k$ on *br*, $f(j) R_{[\![A]\!]} f(k)$.

The function f is said to show that $\mathfrak{M}$ is faithful to branch *br*.

Lemma 3 (Soundness lemma). *Let $\mathfrak{M}$ be any model (general model) for* $\mathbf{CK}_{\mathrm{FDE}}$ *or* **cCL** *and br be any tableau branch of a tableau in* $\mathbf{T}\mathbf{CK}_{\mathrm{FDE}}$ *or*

T**cCL**, *respectively. If* $\mathfrak{M}$ *is faithful to* br *and a tableau rule is applied to* br*, then the application produces at least one extension* br' *of* br*, such that* $\mathfrak{M}$ *is faithful to* br'*.*

PROOF. By induction on the construction of tableaux. The rules for **FDE** are treated in [Priest, 2008, Ch. 8]. Here we consider the remaining rules and start with the rules shared by $\mathbf{CK}_{\mathrm{FDE}}$ and **cCL**. (1) Consider the rule for $(A \mathrel{\Box\!\!\rightarrow} B), +i$. Suppose that the function f shows $\mathfrak{M}$ to be faithful to a branch br containing $(A \mathrel{\Box\!\!\rightarrow} B), +i$ and $ir_A j$ occurs on br. Then $f(i)R_{[\![A]\!]}f(j)$ and $\mathfrak{M}, f(i) \models^+ (A \mathrel{\Box\!\!\rightarrow} B)$. Therefore $\mathfrak{M}, f(j) \models^+ B$ and the function f shows $\mathfrak{M}$ to be faithful to the extended branch. (2) The case of the rule for $(A \mathrel{\Diamond\!\!\rightarrow} B), -i$ is similar. (3) Consider the rule for $(A \mathrel{\Box\!\!\rightarrow} B), -i$ and assume that $\mathfrak{M}, f(i) \not\models^+ (A \mathrel{\Box\!\!\rightarrow} B)$. Then in $\mathfrak{M}$'s set of states W, there is a state w' with $f(i)R_{[\![A]\!]^{\mathfrak{M}}}w'$ and $\mathfrak{M}, w' \not\models^+ B$. The function f' that is exactly like f except that $f'(j) = w'$ shows $\mathfrak{M}$ to be faithful to the extended branch. (4) The case of the rule for $(A \mathrel{\Diamond\!\!\rightarrow} B), +i$ is similar. The reasoning for the tableau rules with respect to which $\mathbf{CK}_{\mathrm{FDE}}$ and **cCL** differ is straightforward. (5) Consider the rule for $\sim(A \mathrel{\Box\!\!\rightarrow} B), +i$ in $\mathbf{CK}_{\mathrm{FDE}}$. Suppose that the function f shows $\mathfrak{M}$ to be faithful to a branch br containing $\sim(A \mathrel{\Box\!\!\rightarrow} B), +i$. Then $\mathfrak{M}, f(i) \models^+ \sim(A \mathrel{\Box\!\!\rightarrow} B)$ iff $\mathfrak{M}, f(i) \models^- (A \mathrel{\Box\!\!\rightarrow} B)$ iff [there is a $w' \in W$ such that $f(i)R_{[\![A]\!]}w'$ and $\mathfrak{M}, w' \models^- B$] iff [there is a $w' \in W$ such that $f(i)R_{[\![A]\!]}w'$ and $\mathfrak{M}, w' \models^+ \sim B$]. The function f thus shows $\mathfrak{M}$ to be faithful to the extended branch. The rules for $\sim(A \mathrel{\Box\!\!\rightarrow} B), -i$ and $\sim(A \mathrel{\Diamond\!\!\rightarrow} B), \underline{+}i$ are also not difficult. Suppose, e.g., $\sim(A \mathrel{\Diamond\!\!\rightarrow} B), -i$ occurs on br and f shows $\mathfrak{M}$ to be faithful to br so that $\mathfrak{M}, f(i) \not\models^+ \sim(A \mathrel{\Diamond\!\!\rightarrow} B)$. Then $\mathfrak{M}, f(i) \not\models^- (A \mathrel{\Diamond\!\!\rightarrow} B)$ iff [there is a state $w' \in W$ with $f(i)R_{[\![A]\!]^{\mathfrak{M}}}w'$ and $\mathfrak{M}, w' \not\models^- B$] iff [there is a state $w' \in W$ with $f(i)R_{[\![A]\!]^{\mathfrak{M}}}w'$ and $\mathfrak{M}, w' \not\models^+ \sim B$] iff $\mathfrak{M}, f(i) \not\models^+ (A \mathrel{\Box\!\!\rightarrow} \sim B)$. (6) The corresponding rules for **cCL** are dealt with in a similar way. We consider just the rule for $\sim(A \mathrel{\Diamond\!\!\rightarrow} B), -i$. Assume then that $\sim(A \mathrel{\Diamond\!\!\rightarrow} B), -i$ occurs on br and that $\mathfrak{M}$ is shown to be faithful to br by function f. Then $\mathfrak{M}, f(i) \not\models^+ \sim(A \mathrel{\Diamond\!\!\rightarrow} B)$ iff $\mathfrak{M}, f(i) \not\models^- (A \mathrel{\Diamond\!\!\rightarrow} B)$ iff [for all $w' \in W$, $f(i)R_{[\![A]\!]}w'$ implies $\mathfrak{M}, w' \not\models^- B$)] iff [for all $w' \in W$, $f(i)R_{[\![A]\!]}w'$ implies $\mathfrak{M}, w' \not\models^+ \sim B$)] iff $\mathfrak{M}, f(i) \not\models^+ (A \mathrel{\Diamond\!\!\rightarrow} \sim B)$. (7) The claim remains to be shown for ($reg \mathrel{\Box\!\!\rightarrow}$) and ($reg \mathrel{\Diamond\!\!\rightarrow}$). Here we consider ($reg \mathrel{\Diamond\!\!\rightarrow}$); the case of ($reg \mathrel{\Box\!\!\rightarrow}$) is analogous. Assume that $(B \mathrel{\Diamond\!\!\rightarrow} C), +i$ and $ir_A j$ occur on br and that f shows $\mathfrak{M}$ to be faithful to br. Then $f(i)R_{[\![A]\!]}f(j)$. Suppose that neither of the following cases holds: (a) $\mathfrak{M}, w_1 \models^+ A$ and

$\mathfrak{M}, w_1 \not\models^+ B$, for some $w_1 \in W$, (b) $\mathfrak{M}, w_2 \models^+ B$ and $\mathfrak{M}, w_2 \not\models^+ A$, for some $w_2 \in W$, and (c) $f(i)R_{[\![B]\!]}f(j)$. Thus, $[\![A]\!] = [\![B]\!]$ and not $f(i)R_{[\![B]\!]}f(j)$. Then $f(i)R_{[\![A]\!]}f(j)$ and not $f(i)R_{[\![B]\!]}f(j)$, a contradiction, by the definition of the models. So either f already shows $\mathfrak{M}$ to be faithful to br extended by $ir_B j$, or one of the following two cases holds: (i) there is a $w_1 \in W$ with $\mathfrak{M}, w_1 \models^+ A$ and $\mathfrak{M}, w_1 \not\models^+ B$ or (ii) there is a $w_2 \in W$ with $\mathfrak{M}, w_2 \models^+ B$ and $\mathfrak{M}, w_1 \not\models^+ A$. Then the function f' that is exactly like f except that $f'(k) = w_1$ or the function f'' that is exactly like f except that $f''(l) = w_2$ shows $\mathfrak{M}$ to be faithful to the extended branch. The case that $(B \mathbin{\Diamond\!\!\rightarrow} C), -i$ and $ir_A j$ occur on br is dealt with in the same way. □

DEFINITION 15. Let br be a complete open tableau branch. Then the structure $\mathfrak{M}_{br} = \langle W_{br}, R_{br}, v^+_{br}, v^-_{br} \rangle$ is defined as follows:

- $W_{br} := \{w_j \mid j \text{ occurs on } br\}$,
- $w_j R_{brX} w_k$ iff there is an A with $X = [\![A]\!]$ and $jr_A k$ occurs on br (for $X \subseteq W_{br}$, $w_j, w_k \in W_{br}$),
- $w_j \in v^+_{br}(p)$ iff $p, +j$ occurs on br (for any propositional variable p),
- $w_j \in v^-_{br}(p)$ if $\sim p, +j$ occurs on br (for any propositional variable p).

We call $\mathfrak{M}_{br}$ the model for $\mathbf{CK}_{\text{FDE}}$ ($\mathbf{cCL}$) induced by br and assume that $[\![A]\!]$ is defined as in models for $\mathbf{CK}_{\text{FDE}}$ ($\mathbf{cCL}$).

The model $\mathfrak{M}_{br}$ for $\mathbf{CK}_{\text{FDE}}$ ($\mathbf{cCL}$) induced by br is well-defined, i.e., is indeed a model for $\mathbf{CK}_{\text{FDE}}$ ($\mathbf{cCL}$). For formulas A that contain $\Box\!\!\rightarrow$ or $\Diamond\!\!\rightarrow$ it is maybe not obvious that the relation R_{brX} is well-defined. To see this, we can define the depth of modal nesting of a formula A, $dm(A)$. If A contains no operator $\Box\!\!\rightarrow$ or $\Diamond\!\!\rightarrow$, then $dm(A) = 0$. If A has the form $\sim B$, then $dm(A) = dm(B)$. If A is a conjunction $(B \wedge C)$ or a disjunction $(B \vee C)$, then $dm(A)$ is $max(dm(B), dm(C))$. If A has the form $B \mathbin{\Box\!\!\rightarrow} C$ or $B \mathbin{\Diamond\!\!\rightarrow} C$, then $dm(A) = max(dm(B), dm(C)) + 1$. We can show that R_{brX} is well-defined for any A such that $X = [\![A]\!]$ by a double induction first on the depth of modal nesting and then on the construction of A. If $dm(A) = 0$, then the truth set $[\![A]\!]$ is well-defined because it is defined independently of the ternary relation, and thus R_{brX} is well-defined. Suppose that $dm(A) = n + 1$, and R_{brX} is well-defined for formulas B with $dm(B) \leq n$, i.e., $[\![B]\!]$ is well-defined. Then (i) A has the shape $B \mathbin{\Box\!\!\rightarrow} C$ or $B \mathbin{\Diamond\!\!\rightarrow} C$ or (ii) A has the form $(B \wedge C)$ or $(B \vee C)$ with $dm(B) \leq n+1$ and $dm(C) \leq n+1$ or (iii) A has the form $\sim B$. In case (i), we may use the first induction hypothesis to conclude that $[\![B]\!]$ is well-defined and thus also $[\![A]\!]$ and $R_{br[\![A]\!]}$. In case (ii), we may note, for example, that $[\![(B \vee C)]\!] = [\![B]\!] \cup [\![C]\!]$. By induction on the construction of A, $[\![B]\!]$ and $[\![C]\!]$ are well-defined, and hence their union is well-defined. In case (iii), $dm(A) = dm(B)$. We show by induction on B that if $R_{[\![B]\!]}$ is well-defined, then so is $R_{[\![\sim B]\!]}$. If B is a propositional variable, then $[\![B]\!]$ and $[\![\sim B]\!]$ and hence$R_{[\![B]\!]}$ and $R_{[\![\sim B]\!]}$ are well-defined. If B has the form $\sim C$, then, by the induction hypothesis, if $R_{[\![C]\!]}$ is well-defined, then so is $R_{[\![\sim C]\!]}$, and since $[\![C]\!] = [\![\sim\sim C]\!]$, if $R_{[\![\sim C]\!]}$ is well-defined, then so is $R_{[\![\sim B]\!]}$. If B is a disjunction $(D \vee E)$, $[\![D]\!] \cup [\![E]\!]$ and thus also $[\![D]\!]$ and $[\![E]\!]$ are well-defined. Therefore, by the induction hypothesis, $[\![\sim D]\!]$ and $[\![\sim E]\!]$ are well-defined. Hence $[\![\sim D]\!] \cup [\![\sim E]\!] = [\![\sim(D \wedge E)]\!]$ and thus also $R_{[\![\sim(D \wedge E)]\!]}$ are well-defined. The case in which B is a conjunction is similar. If B has the form $(C \,\Box\!\!\rightarrow D)$, then, if $R_{[\![(C \Box\!\rightarrow D)]\!]}$ is well defined, then so is $R_{[\![C]\!]}$; but then $R_{[\![B]\!]}$ is also well-defined. The case that B has the form $(C \,\Diamond\!\!\rightarrow D)$ is analogous. Thus, the relations R_{brX} are well-defined for every formula A. That $\mathfrak{M}_{br}$ satisfies

$$[w_j R_{br[\![A]\!]} w_k \text{ iff } w_j R_{br[\![B]\!]} w_k] \text{ if } [\![A]\!] = [\![B]\!]$$

follows from the definition of R_{br} due to fact the rules (*reg* $\Box\!\!\rightarrow$) and (*reg* $\Diamond\!\!\rightarrow$) have been applied in the *complete* open branch *br*.

Definition 16. Let $\mathfrak{M}_{br} = \langle W_{br}, R_{br}, v^{+}_{br}, v^{-}_{br}\rangle$ be defined as in Definition 15. Then the structure $\mathfrak{M}_{br} = \langle W_{br}, R_{br}, P_{br}, v^{+}_{br}, v^{-}_{br}\rangle$ is defined by the requirement that P_{br} is the *smallest* subset of $(\text{Pow}(W_{br}) \times \text{Pow}(W_{br}))$ such that $\langle [\![q]\!], [\![\sim q]\!]\rangle \in P_{br}$ for every $q \in \text{PV}$ and such that P_{br} satisfies the following conditions:

1. if $\langle X, Y\rangle \in P_{br}$, then $\langle Y, X\rangle \in P_{br}$,
2. if $\langle X, Y\rangle, \langle X', Y'\rangle \in P_{br}$, then $\langle X \cap X', Y \cup Y'\rangle \in P_{br}, \langle X \cup X', Y \cap Y'\rangle \in P_{br}$,
3. if $\langle X, Y\rangle, \langle X', Y'\rangle \in P_{br}$, then $\langle \{w \in W_{br} \mid \forall w' \in W_{br}(wR_X w' \Rightarrow w' \in X')\}, \{w \in W_{br} \mid \exists w' \in W_{br}(wR_X w' \curlywedge w' \in Y')\} \in P_{br}$,
4. if $\langle X, Y\rangle, \langle X', Y'\rangle \in P_{br}$, then $\langle \{w \in W_{br} \mid \exists w' \in W_{br}(wR_X w' \curlywedge w' \in X')\}, \{w \in W_{br} \mid \forall w' \in W_{br}(wR_X w' \Rightarrow w' \in Y')\} \in P_{br}$.

(1 and 2 above as well as

3′. if $\langle X, Y\rangle, \langle X', Y'\rangle \in P_{br}$, then $\langle \{w \in W_{br} \mid \forall w' \in W_{br}(wR_X w' \Rightarrow w' \in X')\}, \{w \in W_{br} \mid \forall w' \in W_{br}(wR_X w' \Rightarrow w' \in Y')\} \in P_{br}$,

4′. if $\langle X, Y\rangle, \langle X', Y'\rangle \in P_{br}$, then $\langle \{w \in W_{br} \mid \exists w' \in W_{br}(wR_X w' \curlywedge w' \in X')\}, \{w \in W_{br} \mid \exists w' \in W_{br}(wR_X w' \curlywedge w' \in Y')\} \in P_{br}$).

Clearly, $\mathfrak{M}_{br}$ is a general model for $\mathbf{CK}_{\mathrm{FDE}}$ ($\mathbf{cCL}$); we call it the general model $\mathbf{CK}_{\mathrm{FDE}}$ ($\mathbf{cCL}$) induced by br.

Lemma 4 (Completeness lemma). *Suppose that br is a complete open tableau branch of a tableau in* T$\mathbf{CK}_{\mathrm{FDE}}$ (T$\mathbf{cCL}$), *and let* $\langle W_{br}, R_{br}, P_{br}, v^+_{br}, v^-_{br}\rangle = \mathfrak{M}_{br}$ *be the general model induced by br. Then*

- *if $A, +i$ occurs on br, then $\mathfrak{M}_{br}, w_i \models^+ A$,*
- *if $A, -i$ occurs on br, then $\mathfrak{M}_{br}, w_i \not\models^+ A$,*
- *if $\sim A, +i$ occurs on br, then $\mathfrak{M}_{br}, w_i \models^- A$,*
- *if $\sim A, -i$ occurs on br, then $\mathfrak{M}_{br}, w_i \not\models^- A$.*

Proof. By induction on the complexity of A. Let A be a propositional variable, p. If $p, +i$ occurs on br, then $\mathfrak{M}_{br}, w_i \models^+ p$, by definition. If $p, -i$ occurs on br, then $p, +i$ does not occur on br because br is an open branch. By definition, $\mathfrak{M}_{br}, w_i \not\models^+ p$. Similarly, if $\sim p, +i$ occurs on br, then $\mathfrak{M}_{br}, w_i \models^- p$, by definition. If $\sim p, -i$ occurs on br, then $\sim p, +i$ does not occur on br because br is an open branch. By definition, $\mathfrak{M}_{br}, w_i \not\models^- p$. The remaining cases are straightforward and make use of the fact that br is complete. Here we present only the case that A has the form $B \diamond\!\!\rightarrow C$ and that we are working in T$\mathbf{cCL}$. Thus, suppose $B \diamond\!\!\rightarrow C, +i$ occurs on branch br. By completeness of br, $iR_B j$ and $C, +j$ are on br for some j. By the induction hypothesis, $\mathfrak{M}_{br}, w_j \models^+ C$ and $w_i R_{br[\![B]\!]} w_j$. But then $\mathfrak{M}_{br}, w_i \models^+ B \diamond\!\!\rightarrow C$. If $B \diamond\!\!\rightarrow C, -i$ occurs on br, then for all j with $ir_B j$ on br, the node $C, -j$ is on br as well. By the construction of $\mathfrak{M}_{br}$ and the induction hypothesis, for all w_j such that $w_i R_{br[\![B]\!]} w_j$, $\mathfrak{M}_{br}, w_j \not\models^+ C$. Thus, $\mathfrak{M}_{br}, w_i \not\models^+ B \diamond\!\!\rightarrow C$. Next, assume that $\sim(B \diamond\!\!\rightarrow C), +i$ occurs on br. By completeness of br, $(B \diamond\!\!\rightarrow \sim C), +i$ is on br and for some j, also $ir_B j$ and $\sim C, +j$ are on br. By the induction hypothesis, $w_i R_{br[\![B]\!]} w_j$ and $\mathfrak{M}_{br}, w_j \models^+ \sim C$, i.e., $\mathfrak{M}_{br}, w_j \models^- C$. Thus, $\mathfrak{M}_{br}, w_i \models^- (B \diamond\!\!\rightarrow C)$, i.e., $\mathfrak{M}_{br}, w_i \models^+ \sim(B \diamond\!\!\rightarrow C)$. If $\sim(B \diamond\!\!\rightarrow C), -i$ occurs on br, then, by completeness of br, $(B \diamond\!\!\rightarrow \sim C), -i$ is on br and for all w_j such that $ir_B j$ is on br, also $\sim C, -j$ is on br. By the construction of $\mathfrak{M}_{br}$ and the induction hypothesis, for all w_j such that $w_i R_{br[\![B]\!]} w_j$, $\mathfrak{M}_{br}, w_j \not\models^+ \sim C$, i.e., $\mathfrak{M}_{br}, w_j \not\models^- C$. Thus, $\mathfrak{M}_{br}, w_i \not\models^- (B \diamond\!\!\rightarrow C)$, i.e., $\mathfrak{M}_{br}, w_i \not\models^+ \sim(B \diamond\!\!\rightarrow C)$. □

Remark 4. Note that if $\mathfrak{M} = \langle W, R, P, v^+, v^-\rangle$ is a general model for $\mathbf{CK}_{\mathrm{FDE}}$ ($\mathbf{cCL}$), then $\mathfrak{M}' = \langle W, R, v^+, v^-\rangle$ is a model for $\mathbf{CK}_{\mathrm{FDE}}$ ($\mathbf{cCL}$), and for every $\mathfrak{L}$-formula A and every $w \in W$, $\mathfrak{M}, w \models^+ A$ iff $\mathfrak{M}', w \models^+ A$. The first claim is immediate, and the second follows from the fact that the support of truth and support of falsity conditions for $\mathfrak{M}$ and $\mathfrak{M}'$ coincide.

From the previous two lemmas, it follows by familiar reasoning that for finite premise sets, T**CK**$_{\mathrm{FDE}}$ (T**cCL**) is sound and complete with respect to **CK**$_{\mathrm{FDE}}$ (**cCL**).

Theorem 1. *Let* $\Delta \cup \{A\}$ *be a finite set of* $\mathfrak{L}$*-formulas. Then* $\Delta \models_{\mathbf{CK}_{\mathrm{FDE}}} A$ *iff* $\Delta \vdash_{\mathrm{T}\mathbf{CK}_{\mathrm{FDE}}} A$ *and* $\Delta \models_{\mathbf{cCL}} A$ *iff* $\Delta \vdash_{\mathrm{T}\mathbf{cCL}} A$.

Proof. Let $\Delta = \{B_1, \ldots, B_n\}$. If $\Delta \not\models_{\mathbf{CK}_{\mathrm{FDE}}} A$ ($\Delta \not\models_{\mathbf{cCL}} A$), then there is a model for **CK**$_{\mathrm{FDE}}$ (**cCL**) $\mathfrak{M} = \langle W, R, v^+, v^- \rangle$ and $w \in W$ such that $\mathfrak{M}, w \models^+ B$ for every $B \in \Delta$ and $\mathfrak{M}, w \not\models^+ A$. Any map f from the set of all natural numbers to W with $f(w) = 0$ shows $\mathfrak{M}$ to be faithful to the list $B_1, +0, \ldots, B_n, +0, A, -0$. By the soundness lemma, there exists at least one branch, *br*, such that $\mathfrak{M}$ is faithful to every initial segment of it. The branch *br* cannot be closed because otherwise there is an initial segment of *br* containing a pair $C, +j$ and $C, -j$, which contradicts the faithfulness of $\mathfrak{M}$. Hence *br* is open, and thus $\Delta \nvdash_{\mathrm{T}\mathbf{CK}_{\mathrm{FDE}}} A$ ($\Delta \nvdash_{\mathrm{T}\mathbf{cCL}} A$). Conversely, if $\Delta \nvdash_{\mathrm{T}\mathbf{CK}_{\mathrm{FDE}}} A$ ($\Delta \nvdash_{\mathrm{T}\mathbf{cCL}} A$), then there is an open branch, *br*, of a tableau for $B_1, +0, \ldots, B_n, +0, A, -0$. By the completeness lemma, the induced general model, $\mathfrak{M}_{br}$, has it that $\mathfrak{M}_{br}, w_0 \models^+ B$ for every $B \in \Delta$ and $\mathfrak{M}_{br}, w_0 \not\models^+ A$. By Remark 4, there is a countermodel to $\Delta \vdash_{\mathrm{T}\mathbf{CK}_{\mathrm{FDE}}} A$ ($\Delta \vdash_{\mathrm{T}\mathbf{cCL}} A$). □

The proof of the completeness lemma, Lemma 4, did not make use of the induced set P_{br} of admissible extension/anti-extension pairs. To verify that for finite premise sets, T**CKR**$_{\mathrm{FDE}}$ (T**CCL**) is not only sound but also complete with respect to **CKR**$_{\mathrm{FDE}}$ (**CCL**), we have to make use of P_{br}, which means that we must consider a model based on a *general* frame for **CKR**$_{\mathrm{FDE}}$ (**CCL**).

Theorem 2. $\Delta \models_{\mathbf{CKR}_{\mathrm{FDE}}} A$ *iff* $\Delta \vdash_{\mathrm{T}\mathbf{CKR}_{\mathrm{FDE}}} A$ *and* $\Delta \models_{\mathbf{CCL}} A$ *iff* $\Delta \vdash_{\mathrm{T}\mathbf{CCL}} A$.

Proof. For soundness it suffices to show that Lemma 3 can be extended to the logics **CKR**$_{\mathrm{FDE}}$ and **CCL**, so consider rule $R_{A \Box\!\!\rightarrow A}$. Suppose that $ir_A j$ occurs on a tableau branch *br* and that the function f shows the model $\mathfrak{M}$ to be faithful to *br*. Then $f(i) R_{[\![A]\!]} f(j)$ and by condition $\mathbb{C}_{A \Box\!\!\rightarrow A}$, $\mathfrak{M}, f(j) \models^+ A$, so that f shows $\mathfrak{M}$ to be faithful to the extended branch. To prove completeness, it is enough to show that the frame of the induced general model, $\mathfrak{M}_{br}$, satisfies $\mathbb{C}_{A \Box\!\!\rightarrow A}$:

$(\forall X \subseteq W_{br})(\forall w_i, w_j \in W_{br})(w_i R_{brX} w_j \Rightarrow w_j \in X)$. By Lemmas 1 and 2 and the definition of P_{br} in the general model $\mathfrak{M}_{br}$, it follows that $Pow(W_{br}) = \{[\![A]\!]^{\mathfrak{M}_{br}} \mid A$ is an $\mathfrak{L}$-formula$\}$. Thus, it is enough to show that for every $\mathfrak{L}$-formula A and every w_i, $w_j \in W_{br}$, $w_i R_{br[\![A]\!]^{\mathfrak{M}_{br}}} w_j$ implies $w_j \in [\![A]\!]^{\mathfrak{M}_{br}}$. Suppose, w_i, $w_j \in W_{br}$ and $w_i R_{br[\![A]\!]^{\mathfrak{M}_{br}}} w_j$. Then $ir_A j$ occurs on br and by completeness of br and rule $R_{A \Box\!\!\rightarrow A}$, $A, +j$ occurs on br. Therefore, by Lemma 4, $\mathfrak{M}_{br}, w_j \models^+ A$, i.e., $w_j \in [\![A]\!]^{\mathfrak{M}_{br}}$. □

We have made use of the fact that every general model gives rise to an equivalent model (Remark 4). There is another tight relationship between models and general models.

Lemma 5. *Let $\mathfrak{M} = \langle W, R, v^+, v^- \rangle$ by a model for* $\mathbf{CK}_{\mathrm{FDE}}$ *(*$\mathbf{cCL}$*). Then* (1) *$\mathfrak{M}' = \langle W, R, P, v^+, v^- \rangle$ with $P = \{\langle [\![A]\!]^{\mathfrak{M}}, [\![{\sim}A]\!]^{\mathfrak{M}} \rangle \mid A$ is an $\mathfrak{L}$-formula$\}$ is a general model for* $\mathbf{CK}_{\mathrm{FDE}}$ *(*$\mathbf{cCL}$*), and* (2) *for every $\mathfrak{L}$-formula A and every $w \in W$, $\mathfrak{M}, w \models^+ A$ iff $\mathfrak{M}', w \models^+ A$.*

Proof. To establish (1), it must be shown that P satisfies the conditions 1–4 (1, 2, 3′, 4′) from Definition 3 (Definition 6). 1: Suppose $\langle [\![A]\!], [\![{\sim}A]\!] \rangle \in P$. By Lemmas 1 and 2, $\langle [\![{\sim}A]\!], [\![A]\!] \rangle \in P$. 2: Suppose $\langle [\![A]\!], [\![{\sim}A]\!] \rangle$, $\langle [\![B]\!], [\![{\sim}B]\!] \rangle \in P$. Then $\langle [\![A]\!] \cap [\![B]\!], [\![{\sim}A]\!] \cup [\![{\sim}B]\!] \rangle = \langle [\![A \wedge B]\!], [\![{\sim}A \vee {\sim}B]\!] \rangle = \langle [\![A \wedge B]\!], [\![{\sim}(A \wedge B)]\!] \rangle \in P$. The proof of the second part of 2 is dual. 3 and 4: We consider the case of $\mathbf{cCL}$; the case of $\mathbf{CK}_{\mathrm{FDE}}$ is analogous. Assume that $\langle [\![A]\!], [\![{\sim}A]\!] \rangle$, $\langle [\![B]\!], [\![{\sim}B]\!] \rangle \in P$. Then $\langle \{w \in W \mid \forall w' \in W(wR_{[\![A]\!]}w' \Rightarrow w' \in [\![B]\!])\}, \{w \in W \mid \forall w' \in W(wR_{[\![A]\!]}w' \Rightarrow w' \in [\![{\sim}B]\!])\} \rangle = \langle [\![A \Box\!\!\rightarrow B]\!], [\![{\sim}(A \Box\!\!\rightarrow B)]\!] \rangle \in P$.

Moreover, $\langle \{w \in W \mid \exists w' \in W(wR_{[\![A]\!]}w' \curlywedge w' \in [\![B]\!])\}, \{w \in W \mid \exists w' \in W(wR_{[\![A]\!]}w' \curlywedge w' \in [\![{\sim}B]\!])\} \rangle = \langle [\![A \Diamond\!\!\rightarrow B]\!], [\![{\sim}(A \Diamond\!\!\rightarrow B)]\!] \rangle \in P$.

Claim (2) follows from the fact that the support of truth and support of falsity conditions for $\mathfrak{M}$ and $\mathfrak{M}'$ coincide. □

Note that the properties stated in Lemma 5 generalize facts from classical Chellas-Segerberg semantics [Unterhuber, 2013, Section 4.3].

The preceding lemma establishes the following observation.

Corollary 1. *Let $\mathfrak{C}$ be a class of frames for* $\mathbf{CK}_{\mathrm{FDE}}$ *(*$\mathbf{cCL}$*) and $\mathfrak{C}'$ be the class of all general frames $\langle W, R, P \rangle$ such that $\langle W, R \rangle$ belongs to $\mathfrak{C}$. An $\mathfrak{L}$-formula A is valid with respect to $\mathfrak{C}$ iff A is valid with respect to $\mathfrak{C}'$.*

Proof. Suppose that A is not valid with respect to $\mathfrak{C}$. Then there is a frame $\mathfrak{F} \in \mathfrak{C}$, a model $\mathfrak{M} = \langle W, R, v^+, v^- \rangle$ based on $\mathfrak{F}$, and a $w \in W$ with $\mathfrak{M}, w \not\models^+ A$. By Lemma 5, for $\mathfrak{M}' = \langle W, R, P, v^+, v^- \rangle$, $\mathfrak{M}', w \not\models^+ A$.

Conversely, suppose that A is not valid with respect to $\mathfrak{C}'$. Then there is a frame $\mathfrak{F} \in \mathfrak{C}'$, a general model $\mathfrak{M} = \langle W, R, P, v^+, v^- \rangle$ based on $\mathfrak{F}$, and a $w \in W$ with $\mathfrak{M}, w \not\models^+ A$. By Remark 4, for $\mathfrak{M}' = \langle W, R, v^+, v^- \rangle$, $\mathfrak{M}', w \not\models^+ A$. □

Corollary 2. *The logic* $\mathbf{CK}_{\mathrm{FDE}}$ ($\mathbf{cCL}$) *is the set of all* $\mathfrak{L}$*-formulas valid with respect to the class of all general models for* $\mathbf{CK}_{\mathrm{FDE}}$ ($\mathbf{cCL}$). *The logic* $\mathbf{CKR}_{\mathrm{FDE}}$ ($\mathbf{CCL}$) *is the set of all* $\mathfrak{L}$*-formulas valid with respect to the class of all general models for* $\mathbf{CK}_{\mathrm{FDE}}$ ($\mathbf{cCL}$) *satisfying* $\mathbb{C}_{A \Box\!\!\rightarrow A}$.

4. Some properties of the systems cCL and CCL

The system **cCL** validates Boethius' theses in rule form:

$$(A \Box\!\!\rightarrow B) \vdash {\sim}(A \Box\!\!\rightarrow {\sim}B), \qquad (A \Box\!\!\rightarrow {\sim}B) \vdash {\sim}(A \Box\!\!\rightarrow B).$$

Observe that the first principle is not equivalent to $(A \Box\!\!\rightarrow B) \vdash (A \Diamond\!\!\rightarrow B)$, where the latter might be considered a generalization of principle D from classical modal logic in the following sense: conditionals $A \Box\!\!\rightarrow B$ and $A \Diamond\!\!\rightarrow B$ can be interpreted as necessity and possibility statements of the form $[A]B$ and $\langle A \rangle B$, where for each antecedent A a different type of necessity and possibility is described, respectively.

The soundness of both principles can be shown by the following tableau proofs in T**cCL**:

$$\begin{array}{c} A \Box\!\!\rightarrow B, +0 \\ {\sim}(A \Box\!\!\rightarrow {\sim}B), -0 \\ A \Box\!\!\rightarrow {\sim}{\sim}B, -0 \\ 0r_A 1 \\ {\sim}{\sim}B, -1 \\ B, -1 \\ B, +1 \\ \times \end{array} \qquad \begin{array}{c} A \Box\!\!\rightarrow {\sim}B, +0 \\ {\sim}(A \Box\!\!\rightarrow B), -0 \\ A \Box\!\!\rightarrow {\sim}B, -0 \\ \times \end{array}$$

Figure 1 shows that in T**CCL** Aristotle's and Boethius' theses are provable by making use of rule $R_{A \Box\!\!\rightarrow A}$. Note that Boethius' theses are nested conditionals. It is valid despite the fact that system T**cCL** does not include principles that specifically allow for inferences between formulas with different levels of nestings. An example of the latter type of principle is the following: conclude $A \Box\!\!\rightarrow (A \Box\!\!\rightarrow B)$ from $A \Box\!\!\rightarrow B$. The

$$\begin{array}{c} \sim(A \,\Box\!\!\rightarrow \sim A), -0 \\ (A \,\Box\!\!\rightarrow \sim\sim A), -0 \\ 0r_A 1 \\ \sim\sim A, -1 \\ A, -1 \\ A, +1 \\ \times \end{array} \qquad \begin{array}{c} \sim(\sim A \,\Box\!\!\rightarrow A), -0 \\ (\sim A \,\Box\!\!\rightarrow \sim A), -0 \\ 0r_{\sim A} 1 \\ \sim A, -1 \\ \sim A, +1 \\ \times \end{array}$$

$$\begin{array}{c} (A \,\Box\!\!\rightarrow B) \,\Box\!\!\rightarrow \sim(A \,\Box\!\!\rightarrow \sim B), -0 \\ 0r_{A\Box\!\!\rightarrow B} 1 \\ \sim(A \,\Box\!\!\rightarrow \sim B), -1 \\ (A \,\Box\!\!\rightarrow \sim\sim B), -1 \\ 1r_A 2 \\ \sim\sim B, -2 \\ B, -2 \\ A \,\Box\!\!\rightarrow B, +1 \\ B, +2 \\ \times \end{array} \qquad \begin{array}{c} (A \,\Box\!\!\rightarrow \sim B) \,\Box\!\!\rightarrow \sim(A \,\Box\!\!\rightarrow B), -0 \\ 0r_{A\Box\!\!\rightarrow \sim B} 1 \\ \sim(A \,\Box\!\!\rightarrow B), -1 \\ (A \,\Box\!\!\rightarrow \sim B), -1 \\ 1r_A 2 \\ \sim B, -2 \\ A \,\Box\!\!\rightarrow \sim B, +1 \\ \sim B, +2 \\ \times \end{array}$$

Figure 1. Proofs of Aristotle's and Boethius' theses in T**CCL**

latter principle can be rendered valid in T**cCL** if we require R_A to be transitive, i.e., if iR_Aj and jR_Ak then iR_Ak.

Rather, Boethius' thesis is valid due to a principle called "supraclassicality" [see, e.g., Schurz, 1998, p. 84; Unterhuber, 2013, p. 278]: infer $A \,\Box\!\!\rightarrow B$ if A logically implies B. It is easy to see why Boethius' theses can be rendered valid in T**cCL** by this principle, given that $A \,\Box\!\!\rightarrow B$ logically implies $\sim(A \,\Box\!\!\rightarrow \sim B)$.

Storrs McCall [2012] classifies the principles which he calls "Abelard's first principle" and "Aristotle's second thesis" as connexive principles:

- Abelard's First Principle: $\sim((A \rightarrow B) \wedge (A \rightarrow \sim B))$,
- Aristotle's Second Thesis: $\sim((A \rightarrow B) \wedge (\sim A \rightarrow B))$.

His classification of both principles as connexive hinges on the idea of negation as cancellation. In [Wansing and Skurt, 2018] it is argued that one should not consider these schematic formulas as connexive principles because the idea of negation as cancelation is conceptually unclear and should therefore not be used as a basis for any validity claims. In the connexive logic **C**, Abelard's first principle and Aristotle's second thesis fail to be valid, and similarly in **CCL**, Abelard's first principle and

Aristotle's second thesis fail to be valid for the conditional $\boxminus\!\!\rightarrow$. Figure 2 specifies open tableaux for substitution instances of these principles and thus provides countermodels.

$$\begin{array}{c}
{\sim}((p \boxminus\!\!\rightarrow q) \wedge (p \boxminus\!\!\rightarrow {\sim}q)), -0\\
{\sim}(p \boxminus\!\!\rightarrow q) \vee {\sim}(p \boxminus\!\!\rightarrow {\sim}q), -0\\
{\sim}(p \boxminus\!\!\rightarrow q), -0\\
{\sim}(p \boxminus\!\!\rightarrow {\sim}q), -0\\
(p \boxminus\!\!\rightarrow {\sim}q), -0\\
(p \boxminus\!\!\rightarrow {\sim}{\sim}q), -0\\
0r_p1\\
{\sim}q, -1\\
p, +1\\
0r_p2\\
{\sim}{\sim}q, -2\\
q, -2\\
p, +2\\
{\sim}q, -2\\
{\sim}{\sim}q, -1\\
q, -1
\end{array}
\qquad
\begin{array}{c}
{\sim}((p \boxminus\!\!\rightarrow q) \wedge ({\sim}p \boxminus\!\!\rightarrow q)), -0\\
{\sim}(p \boxminus\!\!\rightarrow q) \vee {\sim}({\sim}p \boxminus\!\!\rightarrow q), -0\\
{\sim}(p \boxminus\!\!\rightarrow q), -0\\
{\sim}({\sim}p \boxminus\!\!\rightarrow q), -0\\
(p \boxminus\!\!\rightarrow {\sim}q), -0\\
({\sim}p \boxminus\!\!\rightarrow {\sim}q), -0\\
0r_p1\\
{\sim}q, -1\\
p, +1\\
0r_{\sim p}2\\
{\sim}q, -2\\
{\sim}p, +2
\end{array}$$

Figure 2. Examples of open tableaux

As to Figure 2, a model $\langle\{0,1,2\}, R, v^+, v^-\rangle$ for **CCL** is a countermodel for ${\sim}((p \boxminus\!\!\rightarrow q) \wedge (p \boxminus\!\!\rightarrow {\sim}q))$ if $0R_{[\![p]\!]}1$, $0R_{[\![p]\!]}2$, $v^+(p) = \{1,2\}$, $1 \notin v^+(q)$, $2 \notin v^+(q)$, $1 \notin v^-(q)$, and $2 \notin v^-(q)$. A model $\langle\{0,1,2\}, R, v^+, v^-\rangle$ for **CCL** is a countermodel for ${\sim}((p \boxminus\!\!\rightarrow q) \wedge ({\sim}p \boxminus\!\!\rightarrow q))$ if $0R_{[\![p]\!]}1$, $0R_{[\![{\sim}p]\!]}2$, $v^+(p) = \{1\}$, $v^-(p) = \{2\}$, $1 \notin v^-(q)$, $2 \notin v^-(q)$.

Although the conditional $\boxminus\!\!\rightarrow$ in **CCL** is reflexive, it is still a very weak conditional. It does not, for example, validate Modus Ponens if we add to **cCL** the following tableau rule:

$$R_{MP} \qquad \begin{array}{c} A, +i\\ (A \boxminus\!\!\rightarrow B), +i\\ \downarrow\\ ir_Ai \end{array}$$

we can prove the derivability statement $\{A, A \boxright B\} \vdash B$:

$$\begin{array}{c} A, +0 \\ A \boxright B, +0 \\ B, -0 \\ 0r_A0 \\ B, +0 \\ \times \end{array}$$

The statement $\{A, A \boxright B\} \vdash B$ C-corresponds to

$$\mathbb{C}_{MP} \quad (\forall X \subseteq W)(\forall w \in W)\, w \in X \Rightarrow wR_X w.$$ [13]

Suppose that $\mathfrak{M} = \langle W, R, v^+, v^- \rangle$ is a model and that $w \in W$ with $\mathfrak{M}, w \models^+ A$ and $\mathfrak{M}, w \models^+ A \boxright B$. If $\mathbb{C}_{MP}$ holds, then $wR_{[\![A]\!]}w$ and thus $\mathfrak{M}, w \models^+ B$. If $\mathbb{C}_{MP}$ is not satisfied, then there is a frame $\mathfrak{F} = \langle W, R \rangle$, $X \subseteq W$, and $w \in W$ such that it is not the case that $wR_X w$. But there is a model $\mathfrak{M} = \langle W, R, v^+, v^- \rangle$ with $\mathfrak{M}, w \models^+ p$, $\mathfrak{M}, w \models^+ p \boxright q$, and $\mathfrak{M}, w \not\models^+ q$ for $[\![p]\!]^{\mathfrak{M}} = \{w\}$, and $[\![q]\!]^{\mathfrak{M}} = \{w' \mid wR_{[\![p]\!]^{\mathfrak{M}}} w'\} = \varnothing$.

The logic defined as the set of all $\mathfrak{L}$-formulas that are valid in the class of all (Chellas) models satisfying $\mathbb{C}_{MP}$ validates Modus Ponens ([cf. Unterhuber, 2013, Ch. 5] and [Unterhuber and Schurz, 2014]). But even if we assume both $\mathbb{C}_{A \boxright A}$ and $\mathbb{C}_{MP}$ and add the rules $R_{A \boxright A}$ and R_{MP} to T**cCL**, the conditional $\boxright$ still is much weaker than intuitionistic implication. We have the following open tableaux in Figure 3.

$$\begin{array}{c} A \boxright (B \boxright A), -0 \\ 0r_A1 \\ (B \boxright A), -1 \\ 1r_B2 \\ A, -2 \\ A, +1 \\ B, +2 \\ 1r_A1 \\ 2r_B2 \end{array}$$

Figure 3. Another open tableau

[13] We use here the fact that for the support of truth and support of falsity conditions of conditionals in models for **cCL** and **CCL** (Definition 2), the accessibility relation R is only relativized to extensions and not to anti-extensions (cf. Remark 1).

Like the connexive logic **C**, the system **CCL** is a non-trivial but inconsistent logic. Both $(A \wedge {\sim}A) \boxright A$ and ${\sim}((A \wedge {\sim}A) \boxright A)$, for example, are provable in T**CCL**.

5. Extension by a constructive conditional — formula-annotated Lewis-Nelson models — and neutralization-extension logics

In this section, we aim to contrast our approach with two recent alternatives. This comparison serves then as a motivation for constructive versions **cCCL** and **CCCL** (Section 6) of the logics **cCL** and **CCL**, respectively (Sections 2–4). We will first focus on the constructive conditional logic by Kapsner and Omori [2017] in terms of what we shall call "formula-annotated Lewis-Nelson models" and then turn to a recent alternative approach concerning restrictedly connexive conditional logics by Vidal [2017b], who presents a semantics of conditionals in terms of so-called neutralization and extension functions.

The logic **FDE** lacks a conditional that satisfies Modus Ponens. This may be seen as a defect, which is overcome in David Nelson's four-valued constructive logic with strong negation **N4**, which results from **FDE** by adding a constructive implication, $\rightarrow$. The system **N4** is both paracomplete (in the sense that $A \vee {\sim}A$ is not valid) and paraconsistent. If we add $\rightarrow$ to our language $\mathfrak{L}$, we obtain the language $\mathfrak{L}_{\rightarrow}$, which is given by the following grammar:[14]

$$A := p \mid {\sim}A \mid (A \wedge A) \mid (A \vee A) \mid (A \rightarrow A) \mid (A \boxright A) \mid (A \diamondright A).$$

[14] The language $\mathfrak{L}$ adds *two* binary connectives to the vocabulary of **FDE**: $\boxright$ and $\diamondright$. Insofar as constructive implication, $\rightarrow$, is a kind of strict Boolean implication with respect to support of truth, interpreted by restricted universal quantification over information states, one might think of adding another binary connective $\rightsquigarrow$ that is interpreted with respect to support of truth and support of falsity as follows:

$\mathfrak{M}, w \models^{+} A \rightsquigarrow B$ iff for some $w' \in W$ such that $w \leq w'$ it holds that $\mathfrak{M}, w \not\models^{+} A$ or $\mathfrak{M}, w \models^{+} B$

$\mathfrak{M}, w \models^{-} A \rightsquigarrow B$ iff $\mathfrak{M}, w \models^{+} A$ and $\mathfrak{M}, w \models^{-} B$.

But the support of truth and the support of falsity conditions for $\rightarrow$ in **N4** exhibit a certain asymmetry. The support of truth conditions are "dynamic" and refer to states different from the state of evaluation, whereas the support of falsity conditions are "static" and evaluate an implication "on the spot". So maybe it is not clear how to formulate the support of falsity conditions for formulas $A \rightarrow B$. A discussion of various such conditions can be found in [Wansing, 2008].

Recently, Kapsner and Omori [2017] suggested to add a restrictedly connexive conditional, $\Box\!\!\rightarrow$ (written as $\sqsupset$), to Nelson's *three-valued* logic **N3**, which is paracomplete but not paraconsistent.[15] It is well-known that if the modification of the support of falsity-conditions for the constructive implication that leads from **N4** to the connexive logic **C** is applied to **N3**, the result is the trivial system in the language of **N3**. In order to avoid triviality, Kapsner and Omori impose a consistency constraint on both the support of truth and the support of falsity conditions for $\Box\!\!\rightarrow$. The following definition presents their formula-annotated Lewis-Nelson models in a way that facilitates comparison with the models based on Chellas frames for **CCL**. With notational adjustment, Kapsner and Omori thus consider the language $\mathfrak{L}^{-}_{\rightarrow}$, given by the following grammar:

$$A := p \mid {\sim}A \mid (A \wedge A) \mid (A \vee A) \mid (A \rightarrow A) \mid (A \mathrel{\Box\!\!\rightarrow} A).$$

DEFINITION 17. A formula-annotated Lewis-Nelson model is a structure $\mathfrak{M} = \langle W, \leq, \{R_A \mid A \text{ is an } \mathfrak{L}^{-}_{\rightarrow}\text{-formula}\}, v^+, v^-\rangle$, where W is a non-empty set (of states), $\leq$ is a partial order on W, $\{R_A \mid A$ is an $\mathfrak{L}^{-}_{\rightarrow}$-formula$\}$ is a collection of binary relations on W, and v^+ and v^- are valuation functions $v^+\colon \mathrm{PV} \longrightarrow \mathrm{Pow}(W)$ and $v^-\colon \mathrm{PV} \longrightarrow \mathrm{Pow}(W)$. For all $p \in \mathrm{PV}$ and for all $w, w' \in W$, (i) $v^+(p) \cap v^-(p) = \varnothing$, and (ii) if $w \in v^*(p)$ and $w \leq w'$, then $w' \in v^*(p)$, for $* \in \{+, -\}$, and the relations R_A satisfy the following conditions, where $f_A(w) := \{w' \in W \mid wR_Aw'\}$:

1. $f_A(w) \subseteq [\![A]\!]$ (i.e., $(\forall w, w' \in W) wR_Aw' \Rightarrow w' \in [\![A]\!]$).
2. If $w \in [\![A]\!]$, then $w \in f_A(w)$ (i.e., $(\forall w \in W) w \in [\![A]\!] \Rightarrow wR_Aw$).
3. If $[\![A]\!] \neq \varnothing$, then $f_A(w) \neq \varnothing$.
4. If $f_A(w) \subseteq [\![B]\!]$ and $f_B(w) \subseteq [\![A]\!]$, then$f_A(w) = f_B(w)$.
5. If $f_A(w) \cap [\![B]\!] = \varnothing$, then $f_{A\wedge B}(w) \subseteq f_B(w)$.
6. If $w \in [\![A]\!]$ and $w' \in f_A(w)$, then $w = w'$.

The support of truth and support of falsity conditions in formula-annotated Lewis-Nelson models coincide with those from Definition 2, except that

$$\begin{array}{lll} \mathfrak{M}, w \models^+ A \rightarrow B & \text{iff} & \text{for all } w' \in W \text{ such that } w \leq w' \text{ it holds that} \\ & & \mathfrak{M}, w' \not\models^+ A \text{ or } \mathfrak{M}, w' \models^+ B \\ \mathfrak{M}, w \models^- A \rightarrow B & \text{iff} & \mathfrak{M}, w \models^+ A \text{ and } \mathfrak{M}, w \models^- B \end{array}$$

[15] For the Lewis part of their semantics, Kapsner and Omori refer to Priest [2008]. However, a definition of Lewis-type models based on a set selection function rather than a formula-annotated selection function (see below) can be found in [Delgrande, 1987, 1988]; for a critical discussion of Delgrande's approach see [Schurz, 1998].

$\mathfrak{M}, w \models^+ A \,\Box\!\!\rightarrow B$	iff	for some $w' \in W$, wR_Aw' and for all $w' \in W$ such that wR_Aw' it holds that $\mathfrak{M}, w' \models^+ B$
$\mathfrak{M}, w \models^- A \,\Box\!\!\rightarrow B$	iff	for some $w' \in W$, wR_Aw' and for all $w' \in W$ such that wR_Aw' it holds that $\mathfrak{M}, w' \models^- B$.

Moreover, if $\mathfrak{M} = \langle W, \leq, \{R_A \mid A \text{ is an } \mathfrak{L}^-_{\rightarrow}\text{-formula}\}, v^+, v^-\rangle$ is a formula-annotated Lewis-Nelson model, then the structure $\mathfrak{F} = \langle W, \leq, \{R_A| \; A \text{ is an } \mathfrak{L}^-_{\rightarrow}\text{-formula}\}\rangle$ is said to be a formula-annotated Lewis-Nelson frame, and $\mathfrak{M}$ is said to be based on $\mathfrak{F}$.

We refer to Kapsner and Omori's Lewis-Nelson models as "formula-annotated Lewis-Nelson models" for the following reason: The set selection function version of Lewis' [1973] semantics for counterfactuals is based on a set selection function that assigns sets of possible worlds to ordered pairs of worlds and *sets of worlds* (i.e., propositions, cf. Section 2.3) rather than pairs of worlds and *formulas* (formula-annotated), where Kapsner and Omori use the latter approach. In addition, Lewis' preferred semantics is not based on a set selection function. Rather, in his semantics different semantic structures are used that allows for infinite descending chains of possible worlds, thereby rejecting Stalnaker's limit assumption which is implicit in set selection functions (Section 1.4; cf. [Unterhuber, 2013, p. 75]). The fact that Lewis' semantics admits such infinite descending chains makes it also misleading to characterize Lewis' semantics in terms of closest possible worlds, insofar it is not guaranteed that such a set of possible worlds always exists. Lewis specifies a set selection variant of his semantics just to contrast his preferred semantics with alternatives. We only use the modifier "formula-annotated" though, since the use of this modifier should suffice to indicate that the semantics above differs in essential ways from Lewis' semantics.

Note that the conditions 1–6 are not purely structural insofar as they exhibit $\mathfrak{L}^-_{\rightarrow}$-formulas. Due to condition (i), the semantics gives rise to a system that fails to be paraconsistent. Moreover, for no formula A is it the case that a state supports both the truth and the falsity of A. Since condition (1) is assumed, which is similar to $\mathbb{C}_{A \Box\!\!\rightarrow A}$, the constraint in the support of truth and the support of falsity conditions for formulas $A \,\Box\!\!\rightarrow B$ that for some $w' \in W$, wR_Aw' restricts the support of truth and the support of falsity conditions of conditionals $A \,\Box\!\!\rightarrow B$ to consistent antecedents A for which $[\![A]\!] \neq \varnothing$. Validity is defined as support of truth at any state of any formula-annotated Lewis-Nelson model, and the entailment relation, $\models$, between sets of formulas and single formulas is defined in terms of preservation of support of truth: $\Delta \models A$ iff for all models $\mathfrak{M} = \langle W, \leq, \{R_A \mid A \text{ is a formula}\}, v^+, v^- \rangle$ and all $w \in W$, it holds that $\mathfrak{M}, w \models^+ A$ if $\mathfrak{M}, w \models^+ B$ for all $B \in \Delta$.

Since formula-annotated Lewis-Nelson frames $\langle W, \leq, \{R_A \mid A$ is an $\mathfrak{L}^-_{\rightarrow}$-formula$\}\rangle$ as defined by Kapsner and Omori use binary relations R_A on W, the semantics does not, however, allow for a purely structural correspondence theory based on *frames* (or general frames), making use of conditions that do not refer to formulas A.

Kapsner and Omori's semantically defined system has a number of noteworthy properties:

- The conditional $\Box\!\!\rightarrow$ is not reflexive: $(A \wedge {\sim}A) \Box\!\!\rightarrow (A \wedge {\sim}A)$ is not valid.
- Simplification fails for $\Box\!\!\rightarrow$ as neither $(A \wedge {\sim}A) \Box\!\!\rightarrow A$ nor $(A \wedge {\sim}A) \Box\!\!\rightarrow {\sim}A$ is valid. Actually, for no formula B, $(A \wedge {\sim}A) \Box\!\!\rightarrow B$ is valid; *contradictio nihil implicat.*
- Moreover, for no formula B, $((A \wedge {\sim}A) \Box\!\!\rightarrow (A \wedge {\sim}A)) \Box\!\!\rightarrow B$ is valid.
- Weakening fails for $\Box\!\!\rightarrow$ as $p \Box\!\!\rightarrow p$ is valid, but $(p \wedge {\sim}p) \Box\!\!\rightarrow p$ is not, and the logic is not closed under substitution because $p \Box\!\!\rightarrow p$ is valid, but $(p \wedge {\sim}p) \Box\!\!\rightarrow (p \wedge {\sim}p)$ is not.
- The constant U, the truth of which is never supported and the falsity of which is never supported, can be defined, for example, by $(p \wedge {\sim}p) \Box\!\!\rightarrow (p \wedge {\sim}p)$ for some $p \in \mathrm{PV}$. Aristotle's theses and Boethius' theses ${\sim}(A \Box\!\!\rightarrow {\sim}A)$, ${\sim}({\sim}A \Box\!\!\rightarrow A)$, $(A \Box\!\!\rightarrow B) \Box\!\!\rightarrow {\sim}(A \Box\!\!\rightarrow {\sim}B)$, and $(A \Box\!\!\rightarrow {\sim}B) \Box\!\!\rightarrow {\sim}(A \Box\!\!\rightarrow B)$ fail to be valid in Kapsner and Omori's system if A is instantiated by U or by a formula of the shape $A \wedge {\sim}A$.[16]

In addition to the lack of a purely structural correspondence theory, the semantics in terms of formula-annotated Lewis-Nelson models may be seen to have at least two other drawbacks. Kapsner and Omori [2017, p. 504] motivate adding their restrictedly connexive conditional to **N3** instead of **N4** by remarking that "the move to the **N4**-based logic works well technically, but philosophically is a doubtful one". This may be clearly criticized. There is no convincing reason to prefer truth value gaps over truth value gluts, and to prefer paracompleteness over paraconsistency in principle when it comes to information processing, and it is not without reason that **FDE**, Belnap and Dunn's useful four-valued logic, and

[16] Weak versions of Boethius' theses, however, are valid: $(A \Box\!\!\rightarrow B) \rightarrow {\sim}(A \Box\!\!\rightarrow {\sim}B)$, and $(A \Box\!\!\rightarrow {\sim}B) \rightarrow {\sim}(A \Box\!\!\rightarrow B)$.

Nelson's **N4** treat verification and falsification on a par. On the contrary, the four-valued semantics is well-motivated and natural not only from the point of view of information processing but also from the point of view of proof-theoretic semantics [cf. Wansing, 2016, 2017]. Moreover, the property *contradictio nihil implicat* echoes the idea of negation as cancellation. If the cancellation model of negation is meant to justify *contradictio nihil implicat*, this is a very problematic justificatory base [see Wansing and Skurt, 2018].

Let us now focus on Vidal's [2017b] alternative approach. Vidal defines a restrictedly connexive logic by building on [Vidal, 2017a] and uses a type of formula-annotated selection function.[17] Due to that, several points of criticism of Kapsner and Omori [2017] hold also for Vidal's logic, such as the lack of a purely structural correspondence theory. Vidal constructs his variant of a formula-annotated selection function by the use of two functions, where the image of the first function serves as one of two arguments for the second function. The first function maps worlds and sets of formulas to sets of worlds (neutralization function) and the second function assigns sets of worlds to pairs of sets of worlds and sets of formulas (expansion function). Although the general semantics allows for a third truth value, its application concerning conditionals assumes only the two classical truth values. Moreover, no nested conditionals are allowed, i.e., for $A \mathrel{\Box\!\!\rightarrow} B$ to be a formula, A and B are not allowed to contain the conditional connective $\Box\!\!\rightarrow$.

The core idea of the neutralization function is to abstract from the truth values of formulas at the given world. In that respect, Vidal follows Gärdenfors' [1988] treatment of counterfactuals in the framework of AGM belief revision. Gärdenfors argues that for determining the truth conditions of counterfactuals we have to abstract from the truth values as given in our world first. For the evaluation of conditionals, Vidal requires the image of the neutralization function only to be determined by the set of atomic proposition occurring in the antecedent of a conditional.[18] The resulting set is then mapped by the extension function to a set of possible worlds, relative to sets of formulas which only contain the antecedent.

[17] In contrast to formula-annotated selection functions as described above, this type of selection function takes worlds and sets of formulas as arguments rather than worlds and formulas.

[18] For the formal details we rely here on Vidal's [2017a] rather than his [2017b]. Vidal [2017a] also puts additional restrictions on the neutralization and expansion function, which we do not discuss here.

The conditional is true iff the image of the neutralization function is not empty and the set assigned by the extension function is a subset of the set of worlds in which the consequent is true. Vidal [2017b] describes then two ways in which a restrictedly connexive system can be achieved, based on this system. The first strategy requires for a conditional to be true that the set of worlds assigned to the antecedent and consequent by a model are neither allowed to be empty nor the whole set of possible worlds (restricted to the bivalent case). The second strategy is to introduce a contingency operator. The result of applying the contingency operator to a formula is true only if the respective formula is neither true at all worlds nor false at all worlds. Connexive principles hold then only if the antecedent and consequent are contingent as specified by the contingency operator. This strategy follows the treatment of connexive logics by Lewis [1973] and Priest [1999], who restrict connexive principles to cases in which the antecedent is possible.

Vidal's strategy is, however, unsatisfactory from a logical standpoint. In the latter two approaches, connexive principles are logically valid only if the antecedent is logically true, since for any other formula we can construct a model in which it is false, thus in a sense trivializing connexive principles from a logical point of view [see Unterhuber, 2016]. Note that Vidal fares actually worse since he requires the antecedent to be contingent rather than merely possible. Thus, no connexive principle seems to be logically valid in his semantics, although they can be valid in particular models.

In sum, both approaches — Kapsner and Omori's on the one hand and Vidal's on the other — do not yield a fully adequate connexive logic. In fact, in our terminology Vidal's system is not a connexive logic since it does not validate Aristotle's and Boethius' theses in full generality. Also, Kapsner and Omori's logic is only weakly connexive since it validates the rule form of Boethius' theses but fails to validate the unrestricted versions.

6. Extension by a constructive conditional: the constructive weakly connexive, respectively connexive logics cCCL and CCCL

Having introduced the weakly connexive, respectively, connexive logics **cCL** and **CCL** in Sections 2–4, we now aim to expand these logics by including a constructive conditional. In this regard, we follow Kapser and Omori insofar as they employ two conditionals, a (weakly) connexive one and a constructive one (see Section 5). The resulting constructive (weakly) connexive logics are named "**cCCL**" and "**CCCL**", respectively, and are formulated in the expanded language $\mathfrak{L}_{\rightarrow}$ (see Section 5). We do not strive to expand these systems, as Kaspner and Omori do, by making the respective proof theory for conditionals $\Box\!\!\rightarrow$ and $\Diamond\!\!\rightarrow$ stronger, but build for our system on [Priest, 2008] and [Wansing, 2005].

The use of a binary relation, $\leq$, for interpreting the constructive implication, and of relations R_X for every set of states X, requires a decision on how to extend the persistence (alias heredity, alias monotonicity) requirement from **N4**, i.e., condition (ii) from Definition 17: $(\forall w, w' \in W)$ if $w \in v^*(p)$ and $w \leq w'$, then $w' \in v^*(p)$, for $* \in \{+, -\}$, to all $\mathfrak{L}_{\rightarrow}$-formulas. Various options for guaranteeing persistence in intuitionistic modal logics are carefully discussed and compared with each other in [Simpson, 1994, Section 3.3]. A choice must be made between modifying the semantic evaluation clauses, imposing conditions on the interaction between the relations that are part of the models, or a combination of both approaches. Here we follow the first approach and use the conditions employed in [Bošić and Došen, 1984; Došen, 1985].

Definition 18. A constructive frame is a structure $\langle W, R, \leq\rangle$, where $\langle W, R\rangle$ is a Chellas frame (Definition 1) and

1. $\leq$ is a reflexive and transitive binary relation on W,
2. $(\forall X \subseteq W)\,(\leq \circ R_X) \subseteq (R_X \circ \leq)$,
3. $(\forall X \subseteq W)\,(\leq^{-1} \circ R_X) \subseteq (R_X \circ \leq^{-1})$,

where '$\circ$' stands for the composition of binary relations. If $\langle W, R, \leq\rangle$ is a constructive frame, then $\mathfrak{M} = \langle W, R, \leq, v^+, v^-\rangle$ is a constructive model iff v^+ and v^- are valuation functions $v^+\colon \mathrm{PV} \longrightarrow \mathrm{Pow}(W)$ and $v^-\colon \mathrm{PV} \longrightarrow \mathrm{Pow}(W)$ such that if $w \in v^*(p)$ and $w \leq w'$, then $w' \in v^*(p)$, for $* \in \{+, -\}$.

Definition 19. A constructive model $\mathfrak{M} = \langle W, R, \leq, v^+, v^-\rangle$ is a model for **cCCL** iff support of truth and support of falsity relations $\models^+$ and $\models^-$ between $\mathfrak{M}$, states $w \in W$, and formulas from $\mathfrak{L}_{\rightarrow}$ are inductively defined as in Definition 5, and using Nelson's clauses for the constructive implication, i.e.:

$$\begin{array}{lll}\mathfrak{M}, w \models^+ A \rightarrow B & \text{iff} & \text{for all } w' \in W \text{ such that } w \leq w' \text{ it holds that} \\ & & \mathfrak{M}, w' \not\models^+ A \text{ or } \mathfrak{M}, w' \models^+ B \\ \mathfrak{M}, w \models^- A \rightarrow B & \text{iff} & \mathfrak{M}, w \models^+ A \text{ and } \mathfrak{M}, w \models^- B.\end{array}$$

Lemma 6 (Persistence). *Let* $\mathfrak{M} = \langle W, R, \leq, v^+, v^- \rangle$ *be a model for* **cCCL**. *Then for every* w, $w' \in W$ *and every* $\mathfrak{L}_{\rightarrow}$*-formula* A, *it holds that if* $w \leq w'$ *and* $\mathfrak{M}, w \models^* A$, *then* $\mathfrak{M}, w' \models^* A$, *for* $* \in \{+, -\}$.

Proof. Since persistence is known to hold for **N4**, it suffices to consider formulas of the form $A \mathrel{\Box\!\!\rightarrow} B$ and of the form $A \mathrel{\Diamond\!\!\rightarrow} B$. Consider a formula of the form $A \mathrel{\Box\!\!\rightarrow} B$ and suppose that $w \leq w'$ as well as (*) $\mathfrak{M}, w \models^+ A \mathrel{\Box\!\!\rightarrow} B$. Let v be an arbitrary state from W such that $w' R_{[\![A]\!]} v$. Then $\langle w, v \rangle \in \leq \circ R_{[\![A]\!]}$. By condition 2, $\langle w, v \rangle \in R_{[\![A]\!]} \circ \leq$. There thus exists $u \in W$ such that $w R_{[\![A]\!]} u$ and $u \leq v$. By (*), $\mathfrak{M}, u \models^+ B$, and by the induction hypothesis applied to B, $\mathfrak{M}, v \models^+ B$. Therefore, $\mathfrak{M}, w' \models^+ A \mathrel{\Box\!\!\rightarrow} B$. Next, consider a formula of the form $A \mathrel{\Diamond\!\!\rightarrow} B$ and suppose that $w \leq w'$ as well as (**) $\mathfrak{M}, w \models^+ A \mathrel{\Diamond\!\!\rightarrow} B$. Then there exists $v \in W$ with $w R_{[\![A]\!]} v$ and $\mathfrak{M}, v \models^+ B$. Since $\langle w', v \rangle \in \leq^{-1} \circ R_{[\![A]\!]}$, by condition 3, $\langle w', v \rangle \in R_{[\![A]\!]} \circ \leq^{-1}$. There thus exists $u \in W$ with $w' R_{[\![A]\!]} u$ and $v \leq u$. By the induction hypothesis applied to B, $\mathfrak{M}, u \models^+ B$. Hence, there exists $u \in W$ with $w' R_{[\![A]\!]} u$ and $\mathfrak{M}, u \models^+ B$. In other words $\mathfrak{M}, w' \models^+ A \mathrel{\Diamond\!\!\rightarrow} B$. The reasoning for $\mathfrak{M}, w \models^- A \mathrel{\Box\!\!\rightarrow} B$ and $\mathfrak{M}, w \models^- A \mathrel{\Diamond\!\!\rightarrow} B$ is similar. □

We next define general frames and general models for **cCCL**.

Definition 20. A quadruple $\langle W, R, \leq, P \rangle$ is a general frame for **cCCL** iff $\langle W, R, \leq \rangle$ is a constructive frame, $\langle W, R, P \rangle$ is a general frame for **cCL** (Definition 6) and P in addition satisfies the following condition:

5. if $\langle X, Y \rangle, \langle X', Y' \rangle \in P$, then $\langle \{w \in W \mid \forall w' \in W(w \leq w' \Rightarrow (w' \notin X$ or $w' \in X'))\}, \{w \in W \mid w \in X \curlywedge w \in Y')\} \rangle \in P$.

Definition 21. Let $\langle W, R, \leq, P \rangle$ be a general frame for **cCCL**. The tuple $\langle W, R, \leq, P, v^+, v^- \rangle$ is a general model for **cCCL** iff $\langle W, R, \leq, v^+, v^- \rangle$ is a constructive model and $\langle [\![p]\!], [\![{\sim}p]\!] \rangle \in P$ for every $p \in \mathrm{PV}$. Support of truth and support of falsity relations $\models^+$ and $\models^-$ are defined as in the case of models for **cCCL**.

Lemma 7. *Let* $\langle W, R, \leq, P, v^+, v^- \rangle$ *be a general model for* **cCCL**. *Then for every* $\mathfrak{L}_{\rightarrow}$*-formula* A, $\langle [\![A]\!], [\![{\sim}A]\!] \rangle \in P$.

Proof. By induction on the complexity of A. We consider the case not already treated before. Let A be a formula $B \rightarrow C$ and assume that $\langle [\![B]\!], [\![{\sim}B]\!] \rangle \in P$ and $\langle [\![C]\!], [\![{\sim}C]\!] \rangle \in P$. Then by 5, $\langle \{w \in W \mid \forall w' \in W(w \leq w' \Rightarrow (w' \notin [\![B]\!]$ or $w' \in [\![C]\!]))\}, \{w \in W \mid w \in [\![B]\!] \curlywedge w \in [\![{\sim}C]\!])\} \rangle \in P$. By Definition 19, $\langle [\![B \rightarrow C]\!], [\![{\sim}(B{\rightarrow}C)]\!] \rangle \in P$. □

Eventually, we define our basic constructive conditional logics.

DEFINITION 22. The logic **cCCL** (**CCCL**) is the set of all $\mathfrak{L}_{\rightarrow}$-formulas valid with respect to the class of all models for **cCCL** (all models for **cCCL** that satisfy $\mathbb{C}_{A\Box\!\!\rightarrow A}$). If a model, general frame, or general model for **cCCL** satisfies $\mathbb{C}_{A\Box\!\!\rightarrow A}$ it will be called a model, general frame, or general model, respectively, for **CCCL**.

We define tableau calculi for **cCCL** and **CCCL** that generalize the tableau calculi for **cCL** and **CCL**. Tableau nodes may now also consist of expressions of the form iRj, to be understood as $i \leq j$ ("state j is a possible expansion of state i"). We introduce certain non-operational tableau rules and tableau rules for formulas $(A \rightarrow B)$. The latter rules (notation adjusted) can also be found in [Priest, 2008, p. 176].

DEFINITION 23. The tableau calculi T**cCCL** and T**CCCL** are obtained from the tableau systems T**cCL** and T**CCL**, respectively, by adding the following tableau rules:

$$(ref)\quad \begin{array}{c} \cdot \\ \downarrow \\ iRi \end{array} \qquad (tran)\quad \begin{array}{c} iRj \\ jRk \\ \downarrow \\ iRk \end{array} \qquad (per^{+})\quad \begin{array}{c} p, +i \\ iRj \\ \downarrow \\ p, +j \end{array} \qquad (per^{-})\quad \begin{array}{c} {\sim}p, +i \\ iRj \\ \downarrow \\ {\sim}p, +j \end{array}$$

$$(per\Box)\quad \begin{array}{c} iRj \\ jr_A k \\ \downarrow \\ ir_A l \\ lRk \end{array} \qquad (per\Diamond)\quad \begin{array}{c} jRi \\ jr_A k \\ \downarrow \\ ir_A l \\ kRl \end{array}$$

$$\begin{array}{c} (A \rightarrow B), +i \\ iRj \\ \swarrow\ \searrow \\ A, -j \quad B, +j \end{array} \qquad \begin{array}{c} (A \rightarrow B), -i \\ \downarrow \\ iRj \\ A, +j \\ B, -j \end{array} \qquad \begin{array}{c} {\sim}(A \rightarrow B), +i \\ \downarrow \\ A, +i \\ {\sim}B, +i \end{array} \qquad \begin{array}{c} {\sim}(A \rightarrow B), -i \\ \swarrow\ \searrow \\ A, -i \quad {\sim}B, -i \end{array}$$

The rule (*ref*) captures the reflexivity of $\leq$ and can be applied to any natural number on the tableau. The rule (*tran*) captures the transitivity of $\leq$, and it is well-known that the presence of this rule may lead to infinite tableau branches (in tableaux for modal logics, and in tableaux for T**cCL** and T**CCL** as well). The rules (per^+) and (per^-) secure persistence for propositional variables and their negations. The rules ($per\Box$) and ($per\Diamond$), where the number l must be new, reflect the conditions that secure persistence for arbitrary formulas (by ensuring persistence for formulas $A \Box\!\!\rightarrow B$ and $A \Diamond\!\!\rightarrow B$), i.e., $(\forall X \subseteq W)\,(\leq \circ R_X) \subseteq (R_X \circ \leq)$ and $(\forall X \subseteq W)\,(\leq^{-1} \circ R_X) \subseteq (R_X \circ \leq^{-1})$, respectively. Also the number j in the rule for $(A \rightarrow B), -i$ must be new.

Definition 24. Let $\mathfrak{M} = \langle W, R, \leq, v^+, v^- \rangle$ ($\mathfrak{M} = \langle W, R, \leq, P, v^+, v^- \rangle$) be any model (general model) for **cCCL** and let br be a tableau branch. The model $\mathfrak{M}$ is said to be faithful to br iff there is a function f from the set of all natural numbers to W such that the conditions 1–3 from Definition 14 are satisfied and, moreover:

4. for every node jRk on br, $f(j) \leq f(k)$.

The function f is said to show that $\mathfrak{M}$ is faithful to branch br.

Lemma 8 (Soundness lemma). *Let $\mathfrak{M} = \langle W, R, \leq, v^+, v^- \rangle$ (resp. $\mathfrak{M} = \langle W, R, \leq, P, v^+, v^- \rangle$) be any model (resp. general model) for* **cCCL** *and br be any tableau branch of a tableau in* T**cCCL**. *If $\mathfrak{M}$ is faithful to br and a tableau rule is applied to br, then the application produces at least one extension br' of br, such that $\mathfrak{M}$ is faithful to br'.*

Proof. By induction on the construction of tableaux. For applications of the rules (ref), ($tran$), (per^+), and (per^-) an appeal to the induction hypothesis, the reflexivity, respectively transitivity of $\leq$, and the persistence of propositional variables and their negations suffices to establish the claim. The case of the tableau rules for formulas $(A \rightarrow B)$ is dealt with in [Priest, 2008, p. 115]; the case of the tableau rules for formulas $\sim(A \rightarrow B)$ uses the induction hypothesis and the definition of support of truth and support of falsity for such formulas. Consider the rules that reflect the properties 2 and 3 from Definition 18. ($per\Box$): Suppose that iRj and jr_Ak occur on br and that f shows $\mathfrak{M}$ to be faithful to br. Then $f(i) \leq f(j)$ and $f(j)R_{[\![A]\!]}f(k)$. By condition 2, there exists $u \in W$ with $f(i)R_{[\![A]\!]}u$ and $u \leq f(k)$. The function f' that is exactly like f except that $f'(l) = u$ shows that $\mathfrak{M}$ is faithful to the extension of br by ir_Al and lRk. ($per\Diamond$): Let jRi and jr_Ak occur on br and suppose hat f shows $\mathfrak{M}$ to be faithful to br. Then $f(j) \leq f(i)$ and $f(j)R_{[\![A]\!]}f(k)$. By condition 3, there exists $u \in W$ with $f(i)R_{[\![A]\!]}u$ and $f(k) \leq u$. The function f' that is exactly like f except that $f'(l) = u$ shows that $\mathfrak{M}$ is faithful to the extension of br by ir_Al and kRl. $\Box$

Since conditions referring to arbitrary subsets of the set of all states are built into the definition of a constructive frame, we have to consider admissible extension/anti-extension pairs already in the completeness lemma for **cCCL**.

Definition 25. Let br be a complete open tableau branch, and let $\langle W_{br}, R_{br}, P_{br}, v^+_{br}, v^-_{br}\rangle$ be the general model for **cCL** induced by br. Then the structure $\mathfrak{M}_{br} = \langle W_{br}, R_{br}, \leq_{br}, P_{br}, v^+_{br}, v^-_{br}\rangle$ induced by br is defined by imposing the following conditions, in addition to conditions 1, 2, 3′, and 4′ from Definition 16:

- $w_j \leq_{br} w_k$ iff jRk occurs on br,

5. if $\langle X, Y\rangle, \langle X', Y'\rangle \in P_{br}$, then $\langle\{w \in W_{br} \mid \forall w' \in W_{br}(w \leq w' \Rightarrow (w' \notin X \text{ or } w' \in X'))\}, \{w \in W_{br} \mid w \in X \curlywedge w \in Y')\}\rangle \in P_{br}$.

That is, P_{br} is the *smallest* subset of (Pow(W) × Pow(W_{br})) such that the all of the above conditions are satisfied.

Lemma 9. *The structure* $\mathfrak{M}_{br} = \langle W_{br}, R_{br}, \leq_{br}, P_{br}, v^+_{br}, v^-_{br}\rangle$ *is a general model for* **cCCL**.

Proof. The rules (*ref*) and (*tran*) guarantee that $\leq_{br}$ is a preorder, and the rules (per^+) and (per^-) make sure that persistence holds for propositional variables and their negations. It remains to show that the conditions 2 and 3 from Definition 18 are satisfied. By Lemma 7 and the definition of $\mathfrak{M}_{br}$, it follows that $Pow(W_{br}) = \{[\![A]\!]^{\mathfrak{M}_{br}} \mid A$ is an $\mathfrak{L}_{\rightarrow}$-formula$\}$. 2: Suppose that $\langle w_i, w_k\rangle \in \leq_{br} \circ R_{br[\![A]\!]}$. Then there is a $j \in \mathbb{N}$ with iRj and $jr_A k$ on branch br. Therefore, for some $l \in \mathbb{N}$, $w_i R_{br[\![A]\!]} w_l$ and $w_l \leq_{br} w_k$, and thus $\langle w_i, w_k\rangle \in R_{br[\![A]\!]} \circ \leq_{br}$. The case of condition 3 is analogous. □

We call $\mathfrak{M}_{br} = \langle W_{br}, R_{br}, \leq_{br}, P_{br}, v^+_{br}, v^-_{br}\rangle$ the general model induced by br.

Lemma 10 (Completeness lemma). *Suppose that br is a complete open tableau branch of a tableau in* T**cCCL**, *and let* $\mathfrak{M}_{br} = \langle W_{br}, R_{br}, \leq_{br}, P_{br}, v^+_{br}, v^-_{br}\rangle$ *be the general model induced by br. Then*

- *If $A, +i$ occurs on br, then* $\mathfrak{M}_{br}, w_i \models^+ A$.
- *If $A, -i$ occurs on br, then* $\mathfrak{M}_{br}, w_i \not\models^+ A$.
- *If $\sim A, +i$ occurs on br, then* $\mathfrak{M}_{br}, w_i \models^- A$.
- *If $\sim A, -i$ occurs on br, then* $\mathfrak{M}_{br}, w_i \not\models^- A$.

PROOF. By induction on the complexity of A. In view of the proof of Lemma 4, it is enough to consider the case of formulas of the form $(B \to C)$. This case is treated in [Priest, 2008, Chapter 9]. □

As in the case of Theorem 1, from the soundness and completeness lemmas, it follows that for finite premise sets, T**cCCL** is sound and complete with respect to **cCCL**.

THEOREM 3. *Let $\Delta \cup \{A\}$ be a finite set of $\mathfrak{L}$-formulas. Then $\Delta \models_{\mathbf{cCCL}} A$ iff $\Delta \vdash_{\mathrm{T}\mathbf{cCCL}} A$.*

Remark 5. If $\mathfrak{M} = \langle W, R, \leq, P, v^+, v^- \rangle$ is a general model for **cCCL**, then (i) $\mathfrak{M}' = \langle W, R, \leq, v^+, v^- \rangle$ is a model for **cCCL**, and (ii) for every $\mathfrak{L}_{\to}$-formula A and every $w \in W$, $\mathfrak{M}, w \models^+ A$ iff $\mathfrak{M}', w \models^+ A$.

In analogy to the proof of Theorem 2, we obtain a characterization result for **CCCL**.

THEOREM 4. *$\Delta \models_{\mathbf{CCCL}} A$ iff $\Delta \vdash_{\mathrm{T}\mathbf{CCCL}} A$.*

The relationship between models and general models stated in Lemma 5 extends to the case of **CCCL**.

LEMMA 11. *Let $\mathfrak{M} = \langle W, R, \leq, v^+, v^- \rangle$ by a model for **CCCL**. Then (1) $\mathfrak{M}' = \langle W, R, \leq, P, v^+, v^- \rangle$ with $P = \{\langle [\![A]\!]^{\mathfrak{M}}, [\![{\sim}A]\!]^{\mathfrak{M}} \rangle \mid A$ is an $\mathfrak{L}_{\to}$-formula$\}$ is a general model for **CCCL**, and (2) for every $\mathfrak{L}_{\to}$-formula A and every $w \in W$, $\mathfrak{M}, w \models^+ A$ iff $\mathfrak{M}', w \models^+ A$.*

PROOF. To establish (1), it remains to be shown that P satisfies condition 5 from Definition 20. Assume that $\langle [\![B]\!], [\![{\sim}B]\!] \rangle$ and $\langle [\![C]\!], [\![{\sim}C]\!] \rangle$ belong to P. Since $\langle [\![B \to C]\!], [\![{\sim}(B \to C)]\!] \rangle$ belongs to P and $\langle [\![B \to C]\!], [\![{\sim}(B \to C)]\!] \rangle = \langle \{w \in W \mid \forall w' \in W (w \leq w' \Rightarrow (w' \notin [\![B]\!]$ or $w' \in [\![C]\!]))\}, \{w \in W \mid w \in [\![B]\!] \curlywedge w \in [\![{\sim}C]\!])\} \rangle \in P$, condition 5 is satisfied.

Claim (2) follows from the fact that the support of truth and support of falsity conditions for $\mathfrak{M}$ and $\mathfrak{M}'$ coincide. □

We obtain the following two corollaries.

COROLLARY 3. *Let $\mathfrak{C}$ be a class of frames for **cCCL** and $\mathfrak{C}'$ be the class of all general frames $\langle W, R, \leq, P \rangle$ such that $\langle W, R \leq \rangle$ belongs to $\mathfrak{C}$. An $\mathfrak{L}_{\to}$-formula A is valid with respect to $\mathfrak{C}$ iff A is valid with respect to $\mathfrak{C}'$.*

COROLLARY 4. *The logic **cCCL** (**CCCL**) is the set of all $\mathfrak{L}_{\to}$-formulas valid with respect to the class of all general models for **cCCL** (**CCCL**).*

Remark 6. Simpson [1994] criticizes that Bošić and Došen's [1984; 1985] intuitionistic modal logics violate certain desiderata which truly intuitionistic systems of modal logic ought to satisfy. These are his Requirement 4, saying that if $A \vee B$ is a theorem of an intuitionistic modal logic, then so is either A or B (that is, the disjunction property holds), and Requirement 5, according to which $\Box A$ ("it is necessary that A") and $\Diamond A$ ("it is possible that A") are independent and not mutually definable as in normal modal logics based on classical propositional logic. In Bošić and Došen's systems, however, $\Diamond A \vee \Box{\sim}A$ and $\Diamond A \leftrightarrow {\sim}\Box{\sim}A$ are theorems. Note that in **cCCL** and **CCCL**, neither $(A \mathrel{\Diamond\!\!\rightarrow} B) \vee (A \mathrel{\Box\!\!\rightarrow} {\sim}B)$ nor $(A \mathrel{\Diamond\!\!\rightarrow} B) \leftrightarrow {\sim}(A \mathrel{\Diamond\!\!\rightarrow} {\sim}B)$ are theorems.

7. Future work

The present first part of this paper focuses on the basic weakly connexive, respectively connexive conditional logics **cCL** and **CCL** and on tableau calculi that translate the relational semantics of **cCL** and **CCL** into tableau rules. We have seen that in extensions of **cCL** obtained by imposing frame conditions, we have to consider general frames to make the completeness lemma work. Moreover, it was noted that if we follow the tradition in conditional logic and use accessibility relations $R \subseteq W \times W \times \mathrm{Pow}(W)$, which makes the present approach more easily comparable to earlier work in conditional logic, there are axiomatic extensions of $\mathbf{CK}_{\mathrm{FDE}}$ and **cCL** that can be captured only by imposing general frame conditions.

The logics **cCL** and **CCL** are weak, and one immediate task for further investigations is to develop axiom systems for **cCL** and **CCL** as well as axiomatic extensions of these calculi and to investigate correspondences between additional axioms and frame conditions. An analogous development can be carried out for the systems **cCCL** and **CCCL** with two kinds of conditionals. These are topics that are left for a second part of the paper.

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