Logic and Logical Philosophy
Volume 28 (2019), 389–407
DOI: 10.12775/LLP.2018.011

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INCONSISTENT MODELS (AND INFINITE MODELS) FOR ARITHMETICS WITH CONSTRUCTIBLE FALSITY

Abstract. An earlier paper on formulating arithmetic in a connexive logic ended with a conjecture concerning $\mathsf{C}^\sharp$, the closure of the Peano axioms in Wansing's connexive logic C. Namely, the paper conjectured that $\mathsf{C}^\sharp$ is Post consistent relative to Heyting arithmetic, *i.e.*, is nontrivial if Heyting arithmetic is nontrivial. The present paper borrows techniques from relevant logic to demonstrate that $\mathsf{C}^\sharp$ is Post consistent *simpliciter*, rendering the earlier conjecture redundant. Given the close relationship between C and Nelson's paraconsistent $\mathsf{N4}$, this also supplements Nelson's own proof of the Post consistency of $\mathsf{N4}^\sharp$. Insofar as the present technique allows infinite models, this resolves Nelson's concern that $\mathsf{N4}^\sharp$ is of interest only to those accepting that there are finitely many natural numbers.

Keywords: strong negation; connexive logic; constructible falsity; first-order arithmetic; connexive arithmetic; Post consistency; paraconsistent logic

1. Introduction

Constructive mathematics (including, *e.g.*, ultrafinitist and intuitionistic mathematics) is monolithic as a demonstration of the the fruitful application of non-classical intuitions about reasoning to mathematical practice. However, its dominance also risks eclipsing analogous non-classical endeavors, such as relevant arithmetic and linear arithmetic,[1] that witness the breadth of the possibilities for non-classical mathematics.

[1] Relevant arithmetic is exhaustively examined in the unpublished monograph [6]. A discussion of linear arithmetic can be found in [15].

Special Issue: Advances in Connexive Logic. Guest Editors: Hitoshi Omori and Heinrich Wansing

Published online August 8, 2018

The paper [2] examined the prospects for carrying out mathematical practice in which reasoning obeyed the principles of a connexive logic. Despite some suggestive analogies between the connexive intuitions of Everett Nelson and the super-constructive framework of David Nelson, the results were ultimately discouraging. To implement even extraordinarily weak fragments of the Peano axioms against any of three of the most well-known connexive logics swiftly and decisively leads to severe pathologies. (*E.g.*, the theories of Peano arithmetic in first-order extensions of Priest's connexive logic of [14] are provably decidable, but only in virtue of the fact that the Peano axioms have *no* consequences in these systems.)

One of the final (and more promising) elements of that paper was a conjecture concerning the theory of the Peano axioms evaluated with Wansing's connexive system C of [17] as a background logic, namely, that the Post consistency of Heyting arithmetic entails the Post consistency of arithmetic in C.[2] As Wansing has shown, first-order C enjoys a faithful translation into first-order intuitionistic logic, a fact that lent considerable plausibility to the conjecture. (The presence of inconsistent theorems of C — a radical feature of the system — immediately rules out the negation consistency of arithmetic in C. Examples of these inconsistencies will be described in Section 2.)

The present paper demonstrates that this conjecture is *redundant* by providing a proof that arithmetic in C is indeed Post consistent *simpliciter* and, in particular, does not prove that $\mathbf{0} = \mathbf{1}$. The construction borrows heavily from the techniques developed by Meyer and Mortensen in [7] and Priest in [13] for proving the Post consistency of a number of paraconsistent arithmetics. This family of techniques demonstrates the nontriviality of relevant arithmetic ($\mathsf{R}^\sharp$) or arithmetic in LP ($\mathsf{LP}^\sharp$) by producing finite, inconsistent — yet non-trivial — models whose elements are equivalence classes of natural numbers. The importance of such models does not flow from any claim that they are adequate models of the natural numbers, but rather from their utility as witnesses that these arithmetics have non-theorems.

Despite their serviceability in proving metatheoretic properties, the artificiality of such finite models for mathematical practice is regrettable.

[2] Recall that a theory T is *Post consistent* if there is a sentence ψ that T does not prove while T is *negation consistent* if there are no sentences φ such that T proves both φ and $\sim\varphi$. In intuitionistic logic, the two coincide while in a paraconsistent logic like C, Post consistency is a strictly weaker property than negation consistency.

In the case of C, however, the intensional nature and constructive elements of C allow for more nuanced, interesting models. I would also like to discuss methods of equipping the models for arithmetic in C with more structure and subtlety than is available in, *e.g.*, Priest's collapsed models of arithmetic.

In tandem, this will also provide a demonstration that arithmetic in Nelson's constructive logic N4 from [9] is also Post consistent. Given the close relationship between N4 and C — the Kripke-style model theory of N4 is a core element for Wansing's semantics for C — this is rather natural, but it might be surprising that the same models demonstrating the non-triviality of arithmetic in C witness the non-triviality of arithmetic in N4. Of particular interest for the case of N4 is the description of an infinite and non-trivial model for $\mathsf{N4}^\sharp$ that may work to resolve Nelson's worry in [9] that arithmetic in N4 would be of interest only to those who are unsure that there exist infinitely many natural numbers.

2. Wansing's C and Nelson's N4

Although the primary target of this paper is Heinrich Wansing's connexive logic C, a satisfactory introduction to the system must take a detour through David Nelson's logics of *constructible falsity*. The first of these systems — N3[3] — was introduced by Nelson in [8] as a revision of intuitionistic practice in which *refutation* is taken to be constructive as well as proof.

Nelson motivates the system by observing that negation is anomalous among the intuitionistic connectives insofar as it is not constructive in an important way. This is made apparent by Nelson in the case of proving a negated universally quantified formula:

> [J]ust as in the case of an existential proposition, we may, in the case of a generality statement $\sim\forall x A(x)$, distinguish two methods of proof. In one there is presented an effective method of constructing an n such that $\sim A(n)$ is true, in the other there is presented a demonstration that $\forall x A(x)$ implies an absurdity. From the viewpoint of constructibility, this distinction in method of demonstration affords the opportunity of a distinction in meaning of the statements $\sim\forall x A(x)$ and $\forall x A(x) \to F$, where F is false. [8, p. 16–17]

[3] This system is sometimes known as N or as CF.

Paraconsistent variants of $\mathsf{N3}$ were introduced in [9] and [10] in which the principle of explosion fails, that is, there exist formulae φ such that the logical closure of the set $\{\varphi, \sim\varphi\}$ is not the entire language.

Following [11], we use $\mathcal{L}$ to describe a recursively constructed first-order language and $CT_{\mathcal{L}}$ to describe the set of closed $\mathcal{L}$ terms.

DEFINITION 1. The Hilbert-style calculus for $\mathsf{QN4}$ includes axioms and axiom schema:

(Int)	Axioms of intuitionistic positive logic
(NN)	$\sim\sim\varphi \leftrightarrow \varphi$
(NA)	$\sim(\varphi \vee \psi) \leftrightarrow (\sim\varphi \wedge \sim\psi)$
(NK)	$\sim(\varphi \wedge \psi) \leftrightarrow (\sim\varphi \vee \sim\psi)$
($\mathrm{NC}_{\mathsf{N4}}$)	$\sim(\varphi \to \psi) \leftrightarrow (\varphi \wedge \sim\psi)$
(NΣ)	$\sim\exists x\varphi \leftrightarrow \forall x \sim\varphi$
(NΠ)	$\sim\forall x\varphi \leftrightarrow \exists x \sim\varphi$
(UI)	$\forall x\varphi(x) \to \varphi(t)$, where t is free for x in φ
(EG)	$\varphi(t) \to \exists x\varphi(x)$, where t is free for x in φ

and rules:

$$\frac{\varphi \to \psi \qquad \varphi}{\psi}$$

$$\frac{\varphi \to \psi(x)}{\varphi \to \forall x\psi(x)} \qquad (x \text{ not free in } \varphi)$$

$$\frac{\varphi(x) \to \psi}{\exists x\varphi(x) \to \psi} \qquad (x \text{ not free in } \psi)$$

The connexive logic C, first introduced in [17], essentially modifies the refutation conditions for a conditional from Nelson-style refutation for something more connexive in flavor, *i.e.*, one that guarantees that $\sim(\varphi \to \sim\varphi)$ is a theorem. Proof-theoretically, then, defining QC requires only a modest change to the axiom system for $\mathsf{QN4}$.

DEFINITION 2. The Hilbert-style calculus for QC is identical to the calculus for $\mathsf{QN4}$ except that it replaces the axiom ($\mathrm{NC}_{\mathsf{N4}}$) with the axiom:

(NC_{C}) $\quad \sim(\varphi \to \psi) \leftrightarrow (\varphi \to \sim\psi)$

The semantics for $\mathsf{QN4}$ and QC we will employ follow the presentation of [11] and [17], respectively.[4] The semantics roughly follow the

[4] Though see [4] for another presentation of Kripke-style semantics for $\mathsf{QN4}$.

lead of Thomason's semantics for N3 from [16] as variants of Kripke-style semantics for intuitionistic logic in which a falsification relation complements the standard verification relation.

As in the proof-theoretic case, the semantics for QN4 and QC are virtually identical, disagreeing only on the point of how to interpret a negated implication.

Definition 3. A QN4 or QC model is a structure $\langle W, \leq, \Delta, D, v^+, v^- \rangle$ for which:

- $\langle W, \leq \rangle$ is a partial order,
- Δ is a set of terms of $\mathcal{L}$ such that $CT_{\mathcal{L}} \subseteq \Delta \subseteq T_{\mathcal{L}}$,
- $D \colon W \to \mathcal{P}(\Delta)$ is a function such that $D(u) \subseteq D(v)$ when $u \leq v$,
- v^+ and v^- are functions from $At_{\mathcal{L}}$ to $\mathcal{P}(W)$.

Note that the definition provided by Wansing includes an increasing domain in a slightly unusual sense. Insofar as the condition is concerned with the preservation of the interpretations of terms rather than the semantical objects in the model, the condition holds for models whose domains might be in fact *decreasing*, so long as the interpretations of terms preserve the verification or falsification of literals. We will rely heavily on this fact in Section 4.

These verification and falsification conditions can be recursively defined as follows:

Definition 4. The QN4 and QC forcing relations are recursively defined by the following common conditions:

- $\mathfrak{M}, w \Vdash^+ R(\vec{t})$ if $w \in v^+(R(\vec{t}))$
- $\mathfrak{M}, w \Vdash^+ {\sim}\varphi$ if $\mathfrak{M}, w \Vdash^- \varphi$
- $\mathfrak{M}, w \Vdash^+ \varphi \wedge \psi$ if $\mathfrak{M}, w \Vdash^+ \varphi$ and $\mathfrak{M}, w \Vdash^+ \psi$
- $\mathfrak{M}, w \Vdash^+ \varphi \vee \psi$ if $\mathfrak{M}, w \Vdash^+ \varphi$ or $\mathfrak{M}, w \Vdash^+ \psi$
- $\mathfrak{M}, w \Vdash^+ \varphi \to \psi$ if $\forall w'$ s.t. $w \leq w'$ & $\mathfrak{M}, w' \Vdash^+ \varphi$, also $\mathfrak{M}, w' \Vdash^+ \psi$
- $\mathfrak{M}, w \Vdash^+ \exists x \varphi(x)$ if for some $t \in D(w)$, $\mathfrak{M}, w \Vdash^+ \varphi(t)$
- $\mathfrak{M}, w \Vdash^+ \forall x \varphi(x)$ if $\forall w'$ s.t. $w \leq w'$ & $\forall t \in D(w')$, $\mathfrak{M}, w' \Vdash^+ \varphi(t)$
- $\mathfrak{M}, w \Vdash^- R(\vec{t})$ if $w \in v^-(R(\vec{t}))$
- $\mathfrak{M}, w \Vdash^- {\sim}\varphi$ if $\mathfrak{M}, w \Vdash^+ \varphi$
- $\mathfrak{M}, w \Vdash^- \varphi \wedge \psi$ if $\mathfrak{M}, w \Vdash^- \varphi$ or $\mathfrak{M}, w \Vdash^- \psi$
- $\mathfrak{M}, w \Vdash^- \varphi \vee \psi$ if $\mathfrak{M}, w \Vdash^- \varphi$ and $\mathfrak{M}, w \Vdash^- \psi$
- $\mathfrak{M}, w \Vdash^- \exists x \varphi(x)$ if $\forall w'$ s.t. $w \leq w'$ & $\forall t \in D(w')$, $\mathfrak{M}, w' \Vdash^- \varphi(t)$
- $\mathfrak{M}, w \Vdash^- \forall x \varphi(x)$ if for some $t \in D(w)$, $\mathfrak{M}, w \Vdash^- \varphi(t)$

The two logics differ in that QN4 has the following negative condition for the conditional:

- $\mathfrak{M}, w \Vdash^{-} \varphi \to \psi$ if $\mathfrak{M}, w \Vdash^{+} \varphi$ and $\mathfrak{M}, w \Vdash^{-} \psi$

while QC includes the following interpretation:

- $\mathfrak{M}, w \Vdash^{-} \varphi \to \psi$ if $\forall w'$ s.t. $w \leq w'$ & $\mathfrak{M}, w' \Vdash^{+} \varphi$, also $\mathfrak{M}, w' \Vdash^{-} \psi$

It is important to note that we treat identity as a relation like any other, with the proviso that self-identity is always verified (though it might also be falsified). We will, of course, prove that identity has the requisite properties in the models of arithmetic to be described. With this in mind, let us proceed to consider arithmetic.

A couple of comments about the status of the conditional in C are in order. For one, C is one of the few systems in the literature on non-classical logics that is authentically *dialethic* in that it has inconsistent theorems, *e.g.*,

- $(\varphi \wedge {\sim}\varphi) \to {\sim}(\varphi \wedge {\sim}\varphi)$, and
- ${\sim}((\varphi \wedge {\sim}\varphi) \to {\sim}(\varphi \wedge {\sim}\varphi))$

are both provable. This can be seen by considering the falsification conditions for the conditional in C. In constructivist terms, to verify $\varphi \to \psi$ is to provide a construction turning any proof of φ into a proof of ψ while to falsify the conditional is to possess a construction turning proofs of φ into *refutations* of ψ. Hence, a proof of $\varphi \wedge {\sim}\varphi$ — a pair of a proof of φ and a refutation of φ can be used to yield a proof of φ (whence the validity of $(\varphi \wedge {\sim}\varphi) \to \varphi$) as well as a refutation of φ (whence the validity of ${\sim}((\varphi \wedge {\sim}\varphi) \to \varphi)$).

It is also worth mentioning that the semantic falsity condition for the conditional in C has a correlate in the Brouwer-Heyting-Kolmogorov interpretation of constructive logic. Where Nelson's refutation condition for a conditional $\varphi \to \psi$ consists of a pair of a proof of φ coupled with a refutation of ψ, Wansing's interpretation of negated conditionals more closely resembles the more dynamic BHK conditions. While the shared BHK account considers a proof of $\varphi \to \psi$ to be a function that can be applied to any proof of φ to yield of proof of ψ, refutations of the connexive $\varphi \to \psi$ are functions that when applied to proofs of φ yield *refutations* of ψ.

3. $\mathsf{C}^\sharp$: Arithmetic in C

Because we are primarily concerned with arithmetic in what follows, we will assume that we are working in the language of arithmetic in the sequel. Notably, $\mathcal{L}_{\mathsf{PA}}$ is the language including only equality as a relation, $\mathbf{0}$ as a constant, and unary function $\_'$ (successor) and binary functions $+$ (addition) and $\cdot$ (multiplication).

The representation of the Peano axioms that we will adopt in the sequel is described below:

Definition 5. The Peano axioms are the following six axioms (PA1)–(PA6) and the induction scheme (Ind):

(PA1) $\sim\exists x(x' = \mathbf{0})$
(PA2) $\forall x(x + \mathbf{0} = x)$
(PA3) $\forall x\forall y(x + y') = (x + y)'$
(PA4) $\forall x(x \cdot \mathbf{0}) = \mathbf{0}$
(PA5) $\forall x\forall y(x \cdot y') = (x \cdot y) + x$
(PA6) $\forall x\forall y(x' = y' \to x = y)$
(Ind) $(\varphi(\mathbf{0}) \wedge \forall(\varphi(x) \to \varphi(x'))) \to \forall x\varphi(x)$

We follow the convention of Robert Meyer in [6] by using the nomenclature $\mathsf{L}^\sharp$ to denote the theory of Peano arithmetic in a logic L. One possible stumbling block is the fact that every One observation about the above representation of the Peano axioms will be useful in what follows. Note that in axioms (PA1)–(PA5) there are no instances of the intensional implication connective. For this reason, these axioms may be thought of as the *extensional* axioms while (PA6) and (Ind) may be thought of as the *intensional* axiom schema. A useful way of thinking of this distinction is that the evaluation of the former axioms at a point, when considered in a $\mathsf{QN4}$ or QC model, only takes features of that point into account.

Wansing has provided a translation of C into intuitionistic logic, motivating a conjecture in [2] about $\mathsf{C}^\sharp$ (*i.e.*, the Peano axioms evaluated against C). Clearly, because C is negation inconsistent, $\mathsf{C}^\sharp$ cannot be negation consistent. But this does not rule out the *Post consistency* of $\mathsf{C}^\sharp$, leading to the aforementioned conjecture in [2]:

Conjecture 1 ([2]). *$\mathsf{C}^\sharp$ is Post consistent if HA (i.e., Heyting Arithmetic) is Post consistent.*

To look deeper into this conjecture, we first take a detour through earlier techniques for proving the Post consistency of paraconsistent arithmetics.

A common strategy for proving the nontriviality of relevant and other inconsistent arithmetics is to provide a finite model that satisfies all Peano axioms in which $\mathbf{0} = \mathbf{0}'$ (*i.e.*, $\mathbf{0} = \mathbf{1}$) is unprovable. Robert Meyer and Chris Mortensen, for example, demonstrated that $\mathbb{Z}/n\mathbb{Z}$ is a model of the Peano axioms in the three-valued logic RM_3 in the paper [7]. Because RM_3 is an extension of the relevant logic R, Meyer and Mortensen were able to demonstrate that $\mathsf{R}^\sharp$ is Post consistent and, in particular, fails to prove $\mathbf{0} = \mathbf{1}$.

The tractability of these finite models has a number of attractive consequences. The theory of a finite model is *decidable*, for example, permitting a proponent of some logic to easily demonstrate the Post consistency of arithmetic in that logic. Indeed, given the paraconsistency of $\mathsf{N4}$ and C, the existence of a single relation in the language, and the finiteness of the domain, the question of whether one of these models makes true a set of sentences reduces to propositional logic. Furthermore, as Meyer frequently pointed out, because these proofs often follow from the existence of a finite model, the Post consistency of these arithmetics can be shown by *a priori* finitistic means. On one reading of Hilbert's program, such systems are therefore quite attractive.

First, we will look at Graham Priest's finite models of arithmetic, using these as a foundational building block for our own intensional models. These models are evaluated as models of the *logic of paradox* LP:

DEFINITION 6. The paraconsistent logic LP is the 4-tuple $\langle \mathscr{V}_{\mathsf{LP}}, \mathscr{D}_{\mathsf{LP}}, \mathbf{S}, I_{\mathsf{LP}} \rangle$, where

- $\mathscr{V}_{\mathsf{LP}} = \{\mathfrak{t}, \mathfrak{b}, \mathfrak{f}\}$ is a set of truth values,
- $\mathscr{D}_{\mathsf{LP}} = \{\mathfrak{t}, \mathfrak{b}\}$ is a set of designated values.

The function I_{LP} interprets connectives $\sim$ and $\wedge$ and the quantifier $\forall$:

$f^{\sim}_{\mathsf{LP}}$	
$\mathfrak{t}$	$\mathfrak{f}$
$\mathfrak{b}$	$\mathfrak{b}$
$\mathfrak{f}$	$\mathfrak{t}$

$f^{\wedge}_{\mathsf{LP}}$	$\mathfrak{t}$	$\mathfrak{b}$	$\mathfrak{f}$
$\mathfrak{t}$	$\mathfrak{t}$	$\mathfrak{b}$	$\mathfrak{f}$
$\mathfrak{b}$	$\mathfrak{b}$	$\mathfrak{b}$	$\mathfrak{f}$
$\mathfrak{f}$	$\mathfrak{f}$	$\mathfrak{f}$	$\mathfrak{f}$

$$f^{\forall}_{\mathsf{LP}}(X) = \begin{cases} \mathfrak{t} & \text{if } \mathfrak{b} \notin X \text{ and } \mathfrak{f} \notin X \\ \mathfrak{b} & \text{if } \mathfrak{b} \in X \text{ and } \mathfrak{f} \notin X \\ \mathfrak{f} & \text{if } \mathfrak{f} \in X \end{cases}$$

Interpretations of disjunction and the existential quantifier follow from the typical definitions. First-order models for LP are defined as follows:

DEFINITION 7. An LP-*model* $\mathfrak{M}$ is a 4-tuple $\langle M, \mathbf{C}^{\mathfrak{M}}, \mathbf{F}^{\mathfrak{M}}, \mathbf{R}^{\mathfrak{M}} \rangle$ where

- M is a set of elements,
- for each $c \in \mathbf{C}$, $c^{\mathfrak{M}} \in M$,
- for each n-ary $f \in \mathbf{F}$, $f^{\mathfrak{M}} \colon M^n \to M$,
- for each n-ary $R \in \mathbf{R}$, $R^{\mathfrak{M}} \colon M^n \to \mathscr{V}_{\mathsf{LP}}$.

In the sequel, we assume that every model is a *Henkin model*, in other words, we assume that every element in a domain has at least one term $t \in CT_{\mathcal{L}}$ that counts that element as its interpretation.

DEFINITION 8. For an LP-model $\mathfrak{M}$, the valuation function $v_{\mathfrak{M}} \colon \mathscr{L}^0_\sigma \to \mathscr{V}_{\mathsf{LP}}$ is defined so that for all n-ary connectives and quantifiers:

- for atomic sentences $\psi = R(t_0, ..., t_{n-1})$, $v_{\mathfrak{M}}(\psi) = R^{\mathfrak{M}}(t_0^{\mathfrak{M}}, ..., t_{n-1}^{\mathfrak{M}})$,
- for sentences of the form $\psi = \odot(\varphi_0, \ldots, \varphi_{n-1})$, $v_{\mathfrak{M}}(\psi) = f_{\mathsf{L}}^{\odot}(v_{\mathfrak{M}}(\varphi_0), \ldots, v_{\mathfrak{M}}(\varphi_{n-1}))$,
- for sentences $\psi = \mathsf{Q}x\varphi$, $v_{\mathfrak{M}}(\psi) = f_{\mathsf{L}}^{\mathsf{Q}}(\{v_{\mathfrak{M}}(\varphi(\underline{a}/x)) \mid a \in M\})$.

The valuation function allows us to define validity in LP as the preservation of designated values in all models.

Priest's finite models of arithmetic rely on a construction of *collapsing* a classical or LP model by effectively taking a quotient of that model modulo a congruence relation.

DEFINITION 9. Let $\mathfrak{M}$ be a model and let $\sim$ be a congruence relation on the domain M where $[a]$ is the equivalence class of $a \in M$ modulo $\sim$. Then a *collapsed model* $\mathfrak{M}^\sim$ is a model where:

- The domain $M^\sim$ is the quotient $\{[a] \mid a \in M\}$.
- For each $t \in \mathbf{C}$, $t^{\mathfrak{M}^\sim} = [t^{\mathfrak{M}}]$.
- For each function $f \in \mathbf{F}$, $f^{\mathfrak{M}^\sim}([a_0], ..., [a_{n-1}]) = [f^{\mathfrak{M}}(a_0, ..., a_{n-1})]$.
- For each relation $R \in \mathbf{R}$ (including identity), we have the following:

$$R^{\mathfrak{M}^\sim}([a_0], ..., [a_{n-1}]) \begin{cases} \in \{\mathfrak{t}, \mathfrak{b}\} & \text{if } \exists a'_0 \in [a_0], ..., a'_{n-1} \in [a_{n-1}] \\ & \qquad \text{s.t. } R^{\mathfrak{M}}(a'_0, ..., a'_{n-1}) \in \{\mathfrak{t}, \mathfrak{b}\} \\ \in \{\mathfrak{b}, \mathfrak{f}\} & \text{if } \exists a'_0 \in [a_0], ..., a'_{n-1} \in [a_{n-1}] \\ & \qquad \text{s.t. } R^{\mathfrak{M}}(a'_0, ..., a'_{n-1}) \in \{\mathfrak{b}, \mathfrak{f}\} \end{cases}$$

In [12], Priest described the *Collapsing Lemma* that assures the preservation of a model's truths under these collapses. (Priest's result is dual to Dunn's *Theorem in Three-Valued Model Theory* introduced in [1]; discussion of the duality can be found in [3].) The lemma guarantees for any classical structure $\mathfrak{M}$ with a congruence relation $\sim$ on M (*i.e.*, an equivalence relation that respects functions) that the following holds:

Lemma 1 (Collapsing Lemma). *Let $\mathfrak{M}$ be a classical first-order model and let $\sim$ be a congruence relation on M inducing a collapsed model $\mathfrak{M}^{\sim}$. Then for any φ: if $\mathfrak{M} \vDash \varphi$ then $\mathfrak{M}^{\sim} \vDash \varphi$.*

Alternatively, one could describe the Collapsing Lemma as the thesis that $Th(\mathfrak{M}) \subseteq Th(\mathfrak{M}^{\sim})$. Note also that although Priest has used the Collapsing Lemma to produce LP models from classical models, the lemma also turns LP models into other LP models with the same preservation properties.

Now, we can define the particular finite models of arithmetic, beginning with an appropriate type of congruence relation. Fix i and n and define $\sim_{i,n}$ as

$$j \sim_{i,n} k \text{ iff } \begin{cases} j = k & \text{if } j < i \text{ and } k < i \\ j \equiv k (\operatorname{mod} n) & \text{if } j \geq i \text{ and } k \geq i \end{cases}$$

and let $[j]_{i,n}$ denote the equivalence class of a natural number j modulo the congruence relation $\sim_{i,n}$. Then:

Definition 10. A *collapsed model of arithmetic* $\mathbb{N}_{i,n}$ is the collapse of the natural numbers modulo $\sim_{i,n}$.

Because $\mathcal{L}_{\mathsf{PA}}$ has only $=$ as a primitive relation, all we need to know is:

- if $j \sim_{i,n} k$ then $\mathbb{N}_{i,n} \vDash [j] = [k]$,
- if there is an $l \in [k]$ such that $j \neq l$, then $\mathbb{N}_{i,n} \vDash {\sim}([j] = [k])$.

N.b. that $\mathbb{N}_{i,n} \vDash {\sim}\varphi$ is *not* the same as $\mathbb{N}_{i,n} \nvDash \varphi$; many of the models discussed in this paper are semantically inconsistent and make true both φ and ${\sim}\varphi$.

Because $\sim_{i,n}$ is a congruence relation, we are able to state the following facts:

- if $j' = k$ then $\mathbb{N}_{i,n} \vDash [j]' = [k]$,
- if $j + k = l$ then $\mathbb{N}_{i,n} \vDash [j] + [k] = [l]$,
- if $j \cdot k = l$ then $\mathbb{N}_{i,n} \vDash [j] \cdot [k] = [l]$.

Because of the properties of each of these models $\mathbb{N}_{i,n}$, they serve to define appropriate possible worlds in the Kripke-style semantics for QC and QN4. In particular, we can define models in the following fashion:

Theorem 1. *Consider a collapsed model of* $\mathbb{N}_{i,n}$ *and a* QN4/QC *model*

$$\mathfrak{N}_{i,n} = \langle \{w_{i,n}\}, \subseteq, CT_{\mathcal{L}_{\mathsf{PA}}}, v^+, v^- \rangle,$$

where:

- $v^+(s = t) = \{w_{i,n}\}$ *iff* $\mathbb{N}_{i,n} \vDash s = t$,
- $v^-(s = t) = \{w_{i,n}\}$ *iff* $\mathbb{N}_{i,n} \vDash \sim(s = t)$.

Then the resulting model is a model of $\mathsf{N4}^\sharp$ *or* $\mathsf{C}^\sharp$, *respectively.*

Proof. We consider first the case of the purely extensional Peano axioms (PA1)–(PA5). In virtue of these axioms' lacking any propositional connectives, we immediately get each of these from Priest's results concerning collapsed models of arithmetic. Having established that these axioms hold, we must next consider the further cases of the axiom and axiom scheme in which the implication connective appears.

(PA6) may be proven quite simply. Take two arbitrary closed terms m and n and suppose that at the single point in the model it is true that $m' = n'$. Then for any elements of the equivalence classes $[m']$ and $[n']$, their predecessors are each in the equivalence classes $[m]$ and $[n]$. Hence, at this world, $m = n$ also holds. Because m and n were chosen arbitrarily, this holds for all x and y.

The induction axiom scheme (Ind) can be established by similar means. Suppose that $\varphi(\mathbf{0})$ and $\forall x(\varphi(x) \to \varphi(x'))$ holds for a formula φ. Then as $\forall x(\varphi(x) \to \varphi(x'))$ is true *ex hypothesi*, also $\varphi(\mathbf{0}) \to \varphi(\mathbf{0'})$. Since the single point is accessible from itself, it follows that $\varphi(\mathbf{0'})$ holds as well. By a *second* application of this procedure, we can establish that $\varphi(\mathbf{0'})$, and, then, that $\varphi(\mathbf{0'''})$, and so forth. Because the model has finitely many elements, eventually we exhaust the domain. Since this establishes that for any element $[m]$, $\varphi(m)$ holds, we conclude that $\forall x \varphi(x)$ holds as well.

A referee has also suggested that it ought to be shown that typical axioms concerning identity hold in the model as well. Given how we have populated v^+, the reflexivity of identity in LP ensures that identity is reflexive in the theory of $\mathbb{N}_{i,n}$. The transitivity of identity — captured by the axiom $\forall x \forall y \forall z(x = y \to (y = z \to x = z))$ is straightforward to establish as well. If $w_{i,n} \Vdash s = t$, then $\mathbb{N}_{i,n} \vDash s = t$. Suppose, moreover, that $w_{i,n} \Vdash t = r$; then $\mathbb{N}_{i,n} \vDash t = r$. The transitivity of identity in LP means that $\mathbb{N}_{i,n} \vDash s = r$ and, by the way that v^+ is defined, entails also that $w_{i,n} \Vdash s = r$. $\square$

The existence of these models is sufficient to establish the primary goal of this paper:

Corollary 1. $\mathsf{C}^\sharp$ *is Post consistent.*

Nelson himself had shown through realizability semantics that $\mathsf{N4}^\sharp$ has a model in [9]. Hence, although the Post consistency of $\mathsf{N4}^\sharp$ could be derived as a corollary of Theorem 1, this fact is already established. Indeed, one could view the Post consistency of $\mathsf{C}^\sharp$ as a corollary of Nelson's results in [9].

Although Theorem 1 could have been demonstrated by a small modification to Nelson's bilateral realizability semantics, there is worth in having presented the proof by appeal to the Kripke-style semantics. In particular, the Kripke semantics will be essential in the next section insofar as they allow us to construct *infinite models*.

4. Infinite Models: Improving on Nelson

Despite Nelson's already having provided a proof of the Post consistency of $\mathsf{N4}^\sharp$, the foregoing observations about $\mathsf{N4}^\sharp$ are not entirely redundant to the extent that they allow us to provide a remedy to Nelson's skepticism regarding certain aspects of $\mathsf{N4}^\sharp$. Regarding the inconsistent-yet-nontrivial arithmetic $\mathsf{N4}^\sharp$, Nelson writes:

> Does the system have any practical interest? I should not want to claim much in this direction; however, the system might be of some interest to a mathematician who cannot make up his mind as to whether there are an infinite number of natural numbers or not. [9, p. 224]

One application of the types of models we are employing in this paper is that we can easily generate *infinite models* that might show that "the system" might be of interest to mathematicians who unequivocally accept an infinitude of natural numbers.

To show this, let us first define a congruence relation on the domains of collapsed models of arithmetic $\mathbb{N}_{i,n}$.

Definition 11. Let $[j]_{i,n}$ and $[k]_{i,n}$ be equivalence classes modulo $\sim_{i,n}$. Then we define the congruence relation $\sim^\star_{i,m}$ as follows:

$$[j]_{i,n} \sim^\star_{i,m} [k]_{i,n} \text{ iff}$$
$$\exists j' \in [j]_{i,n} \;\&\; \exists k' \in [k]_{i,n} \text{ s.t. } \begin{cases} j' = k' & \text{if } j < i \text{ or } k < i \\ j' \equiv k' (\operatorname{mod} m) & \text{otherwise} \end{cases}$$

In preparation, we use the notation $m \mid n$ to represent that the natural number m divides the natural number n. Now, let us also review a few obvious facts.

FACT 1. *If* $j \equiv k (\operatorname{mod} n)$ *and* $m \mid n$ *then* $j \equiv k (\operatorname{mod} m)$.

FACT 2. *In a collapsed model of* $\mathbb{N}_{i,n}$, *if* $j < i$ *then* $[j]_{i,n} = \{j\}$.

FACT 3. *In a collapsed model of* $\mathbb{N}_{i,n}$, *if* $s^{\mathbb{N}_{i,n}} \geq i$ *then* $\mathbb{N}_{i,n} \models \sim(s = t)$ *for any* $t \in CT_{\mathcal{L}_{\mathsf{PA}}}$.

Facts 1–3 allow us to establish a few less trivial lemmas.

LEMMA 2. *Suppose that* $m \mid n$ *and that* $\mathbb{N}_{i,n}$ *is a collapsed model of arithmetic. Then the following are equivalent*:

(a) $\exists j' \in [j]_{i,n}, \exists k' \in [k]_{i,n}$ *s.t.* $j' < i$, $k' < i$, *and* $j' = k'$,
(b) $j < i$, $k < i$, *and* $j = k$.

PROOF. For (a) ⇒ (b): Suppose that all of (a) holds. Then by Fact 2, $[j]_{i,n} = j$ and $[k]_{i,n} = k$, meaning that $j = j'$ and $k = k'$. Hence, $j < i$, $k < i$, and $j = k$ hold, as required.

For (b) ⇒ (a): Suppose that each of (b) holds, *i.e.*, suppose that $j < i$, $k < i$, and $j = k$. Then because $j \in [j]_{i,n}$ and $k \in [k]_{i,n}$, j and k themselves can serve as the elements $j' \in [j]_{i,n}$ and $k' \in [k]_{i,n}$ for which $j' < i$, $k' < i$, and $j' = k'$. Hence, j and k may serve as witnesses for the existential quantifiers in (a). □

LEMMA 3. *Suppose that* $m \mid n$ *and that* $\mathbb{N}_{i,n}$ *is a collapsed model of arithmetic. Then the following are equivalent*:

(a) $\exists j' \in [j]_{i,n}, \exists k' \in [k]_{i,n}$ *s.t.* $j' \geq i$, $k' \geq i$, *and* $j' \equiv k' (\operatorname{mod} m)$,
(b) $j \geq i$, $k \geq i$, *and* $j \equiv k (\operatorname{mod} m)$.

PROOF. For (a) ⇒ (b): Suppose that each element of (a) holds. Then we have assumed that $j \equiv j' (\operatorname{mod} n)$ and $k \equiv k' (\operatorname{mod} n)$. But because $m \mid n$, Fact 1 entails that $j \equiv j' (\operatorname{mod} m)$ and $k \equiv k' (\operatorname{mod} m)$ likewise hold. But by the transitivity of congruence mod m, that $j' \equiv k' (\operatorname{mod} m)$ — together with the congruences between j and j' on the one hand and k and k' on the other — we also may infer that $j \equiv k (\operatorname{mod} m)$.

For (b) ⇒ (a): Suppose that each of (b) holds. Then because $j \in [j]_{i,n}$ and $k \in [k]_{i,n}$, j and k themselves can serve as the j' and k' needed in (a). □

LEMMA 4. *Let* $m \mid n$. *Then*:

$$[j]_{i,n} \sim^{\star}_{i,m} [k]_{i,n} \text{ iff } j \sim_{i,m} k\,.$$

PROOF. The left hand side by definition is equivalent to the existence of $j' \in [j]_{i,n}$ and $k' \in [k]_{i,n}$ to which one of two the following cases applies:
(1) $j' < i$, $k' < i$, and $j' = k'$, or
(2) $j' \geq i$, $k' \geq i$, and $j' \equiv k' (\operatorname{mod} m)$.

By applying lemmas 2 and 3 to the respective cases, we find that these cases are equivalent to the following:

(1′) $j = k$, or
(2′) $j \equiv k (\operatorname{mod} m)$.

But (1′) and (2′) are the two possible cases that are together equivalent to $j \sim_{i,m} k$. □

With these lemmas, we are able to establish a crucial property for the construction of infinite models.

OBSERVATION 1. *Let* $\mathbb{N}_{i,m}$ *and* $\mathbb{N}_{i,n}$ *be two collapsed models of arithmetic such that* $m \mid n$. *Then for every* φ *in the language of arithmetic:*

$$\mathbb{N}_{i,m} \vDash \varphi \text{ iff } \mathbb{N}_{i,n}^{\sim^{\star}_{i,m}} \vDash \varphi\,.$$

PROOF. This follows by induction on complexity of formulae. Lemma 4 and Fact 3 together ensure the property holds for every atom and negated atom. A simple induction over the connectives and quantifiers extends this to the whole of $\mathcal{L}_{\mathsf{PA}}$. □

Given that the models $\mathbb{N}_{i,n}$ are finite, this effectively means that $\mathbb{N}_{i,m}$ and $\mathbb{N}_{i,n}^{\sim^{\star}_{i,m}}$ are interchangeable when $m \mid n$, allowing us to apply the Collapsing Lemma to establish the following corollary, which we obtain from Observation 1 and the Collapsing Lemma:

COROLLARY 2. *Let* $\mathbb{N}_{i,m}$ *and* $\mathbb{N}_{i,n}$ *be two collapsed models of arithmetic such that* $m \mid n$. *Then if* $\mathbb{N}_{i,n} \vDash \varphi$, *also* $\mathbb{N}_{i,m} \vDash \varphi$.

The upshot of this is simple. Suppose the diagram of each world in a $\mathsf{QN4}$ or QC model is induced by a collapsed model of arithmetic such that for worlds $w_{i,n}$ and $w_{i,m}$ induced by models $\mathbb{N}_{i,m}$ and $\mathbb{N}_{i,n}$, $w_{i,n} \leq w_{i,m}$ holds only if $m \mid n$. Then we are guaranteed the type of

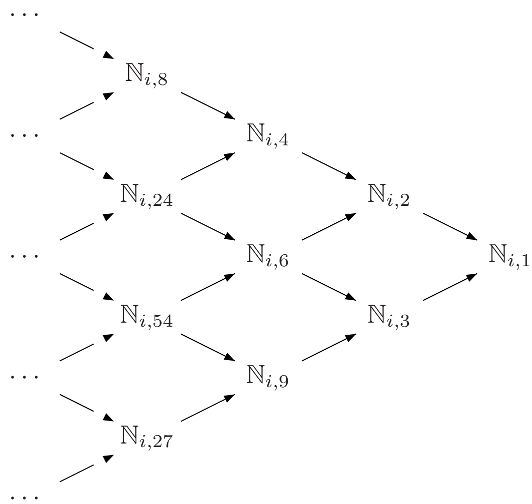

Figure 1. Infinite model of $\mathsf{N4}^\sharp$ and $\mathsf{C}^\sharp$

positive and negative heredity properties between worlds demanded by the definition of a $\mathsf{QN4}$ or a QC model. This licenses us to countenance models such as the one represented in Figure 1.

There is one further wrinkle, however, before we can use this fact to produce infinite models of arithmetic in $\mathsf{N4}^\sharp$ and $\mathsf{C}^\sharp$ if we are to be true to Nelson. Nelson assumes as an axiom of arithmetic an extra-Peano thesis that does not uniformly hold in collapsed models of arithmetic.

(Nelson) $\quad \forall x, y({\sim}(x' = y') \to {\sim}(x = y))$

If $i \neq 0$ in the construction described in Theorem 1, then this axiom can easily be seen to fail. Let $i = 2$. Then despite the fact that $w_{2,n} \Vdash^+ {\sim}(\mathbf{0}''' = \mathbf{0}''')$ holds, we also may observe that $w_{2,n} \nVdash^+ {\sim}(\mathbf{0}'' = \mathbf{0}'')$.

In any case in which $i \neq 0$, the axiom (Nelson) fails. Hence, we must restrict ourselves to models whose worlds are induced by collapsed models for which $i = 0$.

THEOREM 2. *Consider a collapsed model of arithmetic* $\mathbb{N}_{i,n}$ *and consider the* $\mathsf{QN4}/\mathsf{QC}$ *model*

$$\mathfrak{N}^\star_{0,n} = \langle \{w_{0,n} \mid n \geq 1\}, \leq, CT_{\mathcal{L}_{\mathsf{PA}}}, v^+, v^- \rangle,$$

where:

- $w_{0,n} \in v^+(s = t)$ *iff* $\mathbb{N}_{0,n} \vDash s = t$,
- $w_{0,n} \in v^-(s = t)$ *iff* $\mathbb{N}_{0,n} \vDash \sim(s = t)$,
- $w_{0,n} \leq w_{0,m}$ *iff* $m \mid n$.

Then the resulting model is an infinite and non-trivial model of $\mathsf{N4}^\sharp$+(Nelson) *or* $\mathsf{C}^\sharp$+(Nelson), *respectively.*

Proof. The proof of non-triviality runs identically to the proof of Theorem 1, taking into account Corollary 2. Because in any model $\mathbb{N}_{0,n}$ and *any* two terms s and t, $\mathbb{N}_{0,n} \vDash \sim(s = t)$, every instance of the consequent of (Nelson) is true in each point, getting us (Nelson) trivially.

That the model has infinitely many elements can be proven by contradiction. Suppose otherwise, *i.e.*, suppose that there were only j many elements in the domain of the model for a finite j. Then by definition, the world $w_{0,j+1}$ is in the model, and its domain is that of the collapsed model $\mathbb{N}_{0,j+1}$, implying that there exist more than j elements in the model and contradicting the assumption. □

Again, the existence of the model described in Theorem 2 immediately yields an appropriate corollary, one that serves to resolve some of Nelson's concerns from [9]:

Corollary 3. $\mathsf{N4}^\sharp$+(Nelson) *and* $\mathsf{C}^\sharp$+(Nelson) *have models including an infinite number of natural numbers.*

5. Concluding Remarks

The conclusions of [2] were largely pessimistic, suggesting that connexive arithmetic has little hope of working. The observations in this paper clearly ameliorate this cynicism to some degree by showing that the state of $\mathsf{C}^\sharp$ is, if nothing else, no more hopeless than other paraconsistent arithmetics. At this point, $\mathsf{C}^\sharp$ appears to be the most reasonable of all connexive arithmetics built on connexive logics in the literature, but it is not free of difficulties.

One possible stumbling block is the fact that *every* model of $\mathsf{C}^\sharp$ is inconsistent. The relevant logician may think the inconsistency of particular models can be resolved as merely features of a *device* and not reflective of arithmetic proper; to a connexive logician embracing $\mathsf{C}^\sharp$, the inconsistency is an inseverable fact of life. It seems that if $\mathsf{C}^\sharp$ is to be a viable arithmetic, this inconsistency must receive some explanation.

I think that such an explanation should fall out of a deeper philosophical analysis of C, but this avenue is outside of the scope of this paper. Nevertheless, it is still worth considering to *some* degree in the special case of arithmetic.

So let us ask why we would want to reject, *e.g.*, a sentence $\varphi \to \sim\varphi$ in the context of arithmetic. Some of the remarks in [19] on the motivations for connexive logic bear on this this question. If we follow a broadly Brouwer-Heyting-Kolmogorov line concerning implication — and insofar as C is based on the constructive logic $\mathsf{N4}$, this is a reasonable line to take — then we read the demonstration of a conditional $\varphi \to \psi$ as a construction that turns demonstrations of φ into demonstrations of ψ. Thus, two cases appear: One in which φ is provable and another in which φ is not provable.

Now, on a naïve level, it seems intuitive that if φ is in fact *provable*, then there should be no way of converting a valid *proof* of φ into a valid *refutation* of φ. After all, the *purpose* of giving a proof is arguably to guarantee that no such refutation exists. On the other hand, if φ is *not provable*, then *there exists no proof* of φ available that one can convert into a refutation of φ. Although this case counts as a satisfying instance of the BHK interpretation of the conditional, this state of affairs satisfies the BHK condition only *vacuously* by allowing what [5] calls an "empty promise conversion." If one is troubled by the vacuous satisfaction of conditionals — and this seems to be the type of thing that *ought* to trouble a constructivist — then one might then wish to reject all instances of the sentence $\varphi \to \sim\varphi$.

Now, this type of explanation has some deficiencies that stop me from actually endorsing it. Both C and $\mathsf{N4}$ are *paraconsistent*, so the naïve assertion that a proof of φ precludes the existence of a refutation of φ does not extend to this domain. For example, a proof that $\mathbf{2} = \mathbf{2}$ does not rule out discovering a proof that $\sim(\mathbf{2} = \mathbf{2})$ holds. Nor, I will concede, does the argument from the vacuity of the satisfaction of the BHK condition cleanly align with the implicit BHK reading of the negated conditional for C (also implicit in the typed λ-calculus for biconnexive logic described in [18]). But the intuition, at least, shows that there might exist some reason that one might take a line that mirrors connexive principles in arithmetic.

Acknowledgments. I appreciate the helpful and challenging comments due to several reviewers for this journal and participants at the *Philosophy of Contra-Classical Logics* workshop at UNAM. The research in this paper was partly done as part of the PAPIIT project IA401117.

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