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Giangiacomo Gerla

VAGUENESS AND FORMAL FUZZY LOGIC: Some criticisms

Abstract. In the common man reasoning the presence of vague predicates is pervasive and under the name "fuzzy logic in narrow sense" or "formal fuzzy logic" there are a series of attempts to formalize such a kind of phenomenon. This paper is devoted to discussing the limits of these attempts both from a technical point of view and with respect the original and principal task: to define a mathematical model of the vagueness. For example, one argues that, since vagueness is necessarily connected with the intuition of the continuum, we have to look at the order-based topology of the interval [0,1] and not at the discrete topology of the set $\{0, 1\}$. In accordance, in switching from classical logic to a logic for the vague predicates, we cannot avoid the use of the basic notions of real analysis as, for example, the ones of "*approximation*", "*convergence*", "*continuity*". In accordance, instead of defining the compactness of the logical consequence operator and of the deduction operator in terms of finiteness, we have to define it in terms of continuity. Also, the effectiveness of the deduction apparatus has to be defined by using the tools of constructive real analysis and not the one of recursive arithmetic. This means that decidability and semi-decidability have to be defined by involving effective limit processes and not by finite steps stopping processes.

Keywords: vagueness; fuzzy logic; approximate reasoning; compactness; effectiveness, locality

1. Introduction

In spite of the important and impressive literature on formal fuzzy logic, several criticisms exist and are possible against the existing proposals. Some of these criticisms arise from misunderstandings. Nevertheless,

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there are also well-based observations suggesting the opportunity of some kind of reformulation of the paradigms dominant in the current researches. Notice that the considerations in this paper do not call into question the mathematical relevance of these researches. Rather, they aim to discuss whether the main objective of this logic has been reached, once we admit that

> fuzzy set theory is a mathematical theory for modeling of the vagueness phenomenon, similarly as probability theory is for uncertainty.
>
> [Novák, 2005]

Notice that the word *vagueness* in this quotation and in literature is used to denote the phenomenon of the presence in the natural language of predicates whose valuation may vary from ‘false’ and ’true’ in a continuous way. This means that vagueness is strictly related to the real number interval $[0, 1]$ since the set of real number is almost universally considered the adequate model of the continuum.

As an example of a misconception, I quote F. J. Pelletier [2000; 2004] who, in answering a paper of C. W. Entemann [2002] and in reviewing the basic book of P. Hájek [1998], disputes Entemann’s claim that fuzzy logic is an extension of the classical one.

More relevant criticisms are related to compactness and effectiveness. Indeed, Pelletier stresses that while classical logic is semantically compact this is not the case of fuzzy logic

> where there can be (for example) an unsatisfiable infinite set where every finite set is satisfiable.

As a consequence,

> since all proofs are by definition finite, there can therefore be no general proof theory for fuzzy logic.

In addition, Pelletier observes that, while the set of valid formulas in classical logic is recursively enumerable, this falls for either the Łukasiewicz or Goguen logics:

> Hájek then shows that no similar result is possible for either the Łukasiewicz or Goguen logics: no recursive axiomatization for either of these logics is possible [. . .].

The importance of this fact is emphasized dramatically (ironically?):

> Many observers might think this should the death knell for fuzzy logic.
>
> [Pelletier, 2000]

This paper is an attempt to give some answers to all these criticisms and to discuss some further ones.

Namely, sections 2, 3, 4 are devoted to recalling some basic notions and results in fuzzy logic. Namely, we concentrate our attention on the two principal proposals. The approach of Hájek's school which is characterized by deduction apparatus which are crisp in nature, and the one based on the ideas of J. A. Goguen [1968] and J. Pavelka [1979a; 1979b; 1979c] in which "approximate reasoning" are admitted. In the first case to obtain completeness, and therefore compactness and effectiveness, one considers a semantics which is not based on a fixed valuation algebra (as in the tradition in logic) but on a class of algebraic structures [see Hájek, 1998, 2006]. In the latter, a Pavelka-style completeness theorem is obtained firstly by Pavelka for propositional calculus [Pavelka, 1979c] and successively for first order logic by Novák [see Novák, 1990a,b and also Turunen, 1999]. In this case, the idea of approximate reasoning plays a crucial role.

Section 5 is devoted to observing that fuzzy logic is not a philosophical alternative to classical logic but an attempt to extend this logic. This is a rather evident fact, but I will emphasize it since is on the basis of useless contrasts and unjustified enthusiasms.

In Sections 6 and 7 one considers the questions of compactness and effectiveness. In particular, I argue that in the graded approach adequate definitions of these notions need to be hunted in recursive analysis and not in recursive arithmetic.

In Section 8 further criticisms are quoted against both the two approaches. Finally, in Section 9 we conclude expressing the opinion for which a mechanical adaptation of the methods used in classical logic is misleading. So, the enterprise of finding a mathematical model for a logic admitting vague predicates has to search new alternatives.

2. Basic notions and notations

Given a nonempty set S and a bounded lattice L, then we call *L-subset* or *fuzzy subset of S* any map $s\colon S \to L$ from S into L. We denote by $cod(s)$ the codomain $\{s(x) : x \in S\}$ of s. Given a positive natural number n, an *n-ary L-relation* or *fuzzy relation in S* is an L-subset of S^n. We denote by L^S the class of all the L-subsets of S and by $P(S)$ the class of all the subsets of S. Given $\lambda \in L$, the *open λ-cut* of s is the set

$\{x \in S : s(x) > \lambda\}$, the *closed* λ*-cut* is the set $\{x \in S : s(x) \geq \lambda\}$. We say that s is *finite* if the open 0-cut $Supp(s) := \{x \in S : s(x) > 0\}$ is finite. Given two L-subsets s_1 and s_2, we set $s_1 \subseteq s_2$ and we say that s_1 *is contained in* s_2 provided that $s_1(x) \leq s_2(x)$, for every $x \in S$. The pair $(L^S, \subseteq)$ is a bounded lattice whose meets and joins operators are named *intersection* and *union*, respectively. We denote by $\cap$ and $\cup$ these operations and therefore we set, for any s_1 and s_2 in L^S and $x \in S$,

$$(s_1 \cup s_2)(x) := s_1(x) \vee s_2(x),$$
$$(s_1 \cap s_2)(x) := s_1(x) \wedge s_2(x).$$

If 0 and 1 are the minimum and maximum bounds in L, we call *crisp* any L-subset whose values are in $\{0, 1\}$. We identify the subsets with the crisp subsets by the lattice embedding $e\colon P(S) \to L^S$ defined by setting $e(X) = c_X$, where c_X is defined by setting $c_X(x) := 1$, if $x \in X$, and $c_X(x) = 0$, otherwise [see, e.g., Zadeh, 1965].

In this paper, S coincides with the set Form of well-formed formulas of a first order language or with the set $\underline{\text{Form}}$ of the closed formulas of this language. Also, we assume that in the lattice L there are operations enabling us to interpret the logical connectives. In this case, we say that L is a *valuation algebra*. An important class of valuation algebras is the class of *standard algebras*, i.e., algebras $([0, 1], *, \to)$ whose domain is the real interval $[0, 1]$ and whose operations are a continuous triangular norm $*$ together with the related residuum $\to$. As we will see after, further valuation algebras are the structures belonging to the class $Var(*)$ of all the algebraic structures in the variety generated by $([0, 1], *, \to)$. We call $*$*-logic* a logic which is based on a triangular norm $*$.

Given a first order language and a valuation algebra L, we call *fuzzy interpretation* a pair (D, I), where

- D is a set we call *the domain* of the interpretation,
- $I(r)$ is an n-ary L-relation in D for any n-ary relation name r,
- $I(c)$ is an element in D for any constant c,
- $I(f)$ in an n-ary function in D for any n-ary function name f.

Usually one assumes that in the language there are two logical constants **0** and **1** to represent the minimum 0 and the maximum 1 of L. Sometimes, one considers also languages in which there is a logical constant $\underline{r}$, for every rational number r. The intended meaning is that $\underline{r}$ is an atomic formula whose truth value is r. In such a case we say that the language is *with rational constants*. This is permitted by the fact that in this paper we are interested only in valuation structures belonging to the class *Var*$(*)$ and therefore containing a copy of the set $[0,1]_Q$ of rational numbers in $[0,1]$.

Then, given a fuzzy interpretation (D,I), we can evaluate the formulas in a truth-functional way as it is usual in multi-valued logic. In addition, in the case the language is with rational constants, we interpret a logical constant as $\underline{r}$ by the rational number r. The existential quantifier is usually interpreted by the last upper bound and this gives rise to some unusual and interesting phenomenon. For example, there is a possibility that a formula as $\exists x\ \alpha(x)$ is true in a given model but that there is no element (witness) in the domain satisfying α. Again, in the case L is not complete, there is the possibility that some quantified formulas cannot be evaluated.

DEFINITION 2.1. Given an interpretation (D,I) and a formula α whose free or bounded variables are among $x_1,\ldots,x_n$, we denote by $f_\alpha\colon (E-\{Q\})^n \to [0,1]^*$ the function such that $f_\alpha(d_1,\ldots,d_n)$ is the truth-value of α once the variables $x_1,\ldots,x_n$ are interpreted by $d_1,\ldots,d_n$ in D. As usual, if $\alpha \in \underline{\mathrm{Form}}$ is a closed formula, f_α is a constant we call *the truth value of* α. Also, we denote by $m_{(D,I)}\colon \underline{\mathrm{Form}} \to L$ the fuzzy subset such that, for every $\alpha \in \underline{\mathrm{Form}}$, $m_{(D,I)}(\alpha)$ is the truth value of α.

DEFINITION 2.2. We say that (D,I) is a *model of* a closed formula α provided that $m_{(D,I)}(\alpha) = 1$. Given a set T of closed formulas, we say that (D,I) is *a model of* T if (D,I) is a model of all the formulas in T.

3. Ungraded approach

As anticipated in Introduction, two main approaches to fuzzy logic exist [see Belohlavek et al., 2017]. We call *ungraded* the one of Hájek's school, and *graded* the one originates by Goguen and Pavelka (in brief, *U-approach* and *G-approach*). I prefer such a terminology to the usual one "with classical syntax" and "with evaluated syntax" since the difference is mainly in the way one represents the information. We initiate by giving the basic notions of the ungraded approach.

DEFINITION 3.1. Given a standard algebra $([0,1],*,\to)$, we call $*$-*interpretation* an interpretation in $([0,1],*,\to)$ and *Var*$(*)$-*interpretation* an interpretation in a valuation algebra in *Var*$(*)$. Given $T \subseteq \underline{\mathrm{Form}}$, the meaning of the expressions, $*$-*model of* T and *Var*$(*)$-*model of* T is evident. We write:

- $T \models_* \alpha$ provided that every $*$-model of T is a model of α,
- $T \models_{Var(*)} \alpha$ provided that every linear and safe $Var(*)$-model of T is a model of α.

We say that $\models_*$ and $\models_{Var(*)}$ are the *standard consequence relation* and the *general consequence relation* associated with $*$, respectively.

These relations are associated with two different notions of tautology.

Definition 3.2. We say that a formula α is a *standard* $*$*-tautology* if $\varnothing \models_* \alpha$, that α is a *general* $*$*-tautology* if $\varnothing \models_{Var(*)} \alpha$, respectively.

Then a standard $*$-tautology is a formula satisfied in all the $*$-interpretations and a general $*$-tautology is a formula satisfied in all the $Var(*)$-interpretations.

Examples. To illustrate the just given notions, we list some examples in which the language is with rational constants and contains a sequence $P_1, P_2, \ldots$ of constants and a monadic predicate symbol C. The intended meaning is that these constants represent points of the Euclidean plane and C the fuzzy predicate "to be close to Q", where Q is a fixed point. Also, we consider the theory

$$T = \{q(1) \to C(P_1), q(2) \to C(P_2), \ldots, q(n) \to C(P_n,), \ldots\},$$

where $(q(n))_{n\in N}$ is a strictly increasing sequence of rational numbers such that $\lim_{n\to\infty} = 1$. It is evident that every $*$-model of T satisfies $\exists x(C(x))$, i.e. that $T \models_? \exists x(C(x))$.

Example 1. Denote by M_1 the interpretation whose domain is the set E of points of the Euclidean plane and where, once a point $Q \in E$ is fixed, assume that

- C is interpreted by the fuzzy subset $c\colon E \to [0,1]$ such that, for every $P \in E$, $c(P) = 2^{-d(P,Q)}$,
- $P_1, P_2, \ldots$ are interpreted by a sequence $(P_n)_{n\in N}$ of points in $E - \{Q\}$ such that $d(P_n, Q) \leq -\log_2(q(n))$.

Then, for every $n \in N$, $c(P_n) \geq q(n)$ and therefore M_1 is a model of T. Also, since $c(Q) = 1$, in this model there is a witness for the formula $\exists x(C(x))$.

Example 2. Denote by M_2 the interpretation obtained by modifying M_1 only in assuming that the domain is $E - \{Q\}$. In a sense, we interpret Q as a small hole and $C(P)$ as the closeness of P to this hole. It is immediate that M_2 is a model of T satisfying $\exists x(C(x))$ in spite of the fact that no witness exists for this formula.

EXAMPLE 3. Denote by M_3 an interpretation coinciding with M_2 but whose valuation structure is an extension of the standard algebra $([0,1], *, \rightarrow)$ of M_2, namely is a non-standard extension $([0,1]^*, *, \rightarrow)$ of $([0,1], *, \rightarrow)$. This extension belongs to $Var(*)$, obviously. Then, we obtain a model of T in which it is not possible to assign a truth value to $\exists x(Close(x))$. This since the set of the upper bounds of $\{c(P) : P \in E - \{Q\}\}$ coincides with the set of elements in $[0,1]^*$ infinitely close to 1 and in this set there is no minimum. So, no last upper bound of $\{c(P) : P \in E - \{Q\}\}$ exists and M_3 is a model of T which is not safe.

EXAMPLE 4. Let δ be an upper bound of the set $\{d(P_i, Q) : i \in N\}$, let $\delta_1, \ldots, \delta_h$ be positive real numbers such that $0 < \delta_1 < \delta_2 < \cdots < \delta_h = \delta$ and let $u_h < \cdots < u_1$ be elements of $[0,1]^*$ infinitely close to 1. Then an interpretation M_4 is obtained in the domain $E - \{Q\}$ by setting

- $c(P) = 0$, if $d(P,Q) > \delta_h$,
- $c(P) = u_i$, if $\delta_{i-1} < d(P,Q) \leq \delta_i$,
- $c(P) = u_1$, if $0 < d(P,Q) \leq \delta_1$.

In other words, $c \colon E - \{Q\} \rightarrow [0,1]^*$ is the fuzzy circle whose closed cut at level u_i is the circle $\{P : d(P,Q) \leq \delta_i\}$. In particular, the points internal to the circle $\{P : d(P,Q) \leq \delta_h\}$ are considered infinitely close to the Q while the remaining points are not considered close to Q at all. In particular, all the points in the sequence $(P_n)_{n \in N}$ are infinitely close to 1 and therefore $c(P_n) \geq q(n)$, for every $n \in N$. This means that that M_4 is a model of T. I claim that

(a) For any formula α and any $d_1, \ldots, d_n$ in $E - Q$, $f_\alpha(d_1, \ldots, d_n)$ is well defined and $cod(f_\alpha)$ is finite.

Indeed, (a) is trivial in the case α is an atomic formula. Assume that (a) is satisfied by α and β, then $f_{\alpha\wedge\beta}(d_1, \ldots, d_n)$ is well defined and, since $cod(f_{\alpha\wedge\beta})$ is contained in the finite set $\{xy : x \in cod(f_\alpha),\ y \in cod(f_\beta)\}$, $cod(f_{\alpha\wedge\beta})$ is finite. The same argument holds true for $f_{\alpha\rightarrow\beta}$. Assume that (a) is satisfied by α, then since $cod(f_\alpha)$ is finite, there is no difficulty to calculate the truth value of $\exists x_i\ \alpha$ (which is a maximum) and it is evident that $cod(f_{\exists xi\alpha})$ is finite. So, $\exists x_i\ \alpha$ satisfies (a), too.

As a consequence of (a), M_4 is a safe model of T such that the valuation of $\exists x(Close(x))$ is $u_1 \neq 1$ and this shows that $T \models_{Var(*)} \exists x(Close(x))$ is false.

The notion of deduction apparatus in U-approach is defined by adopting the same paradigm of classical logic, i.e., by fixing a suitable set of logical axioms and suitable inference rules. Also, the structure of a proof is the same as in classical logic. Unfortunately, an adequate deduction system for the standard semantics exists only in the case $*$ is the minimum. Then, to obtain a completeness theorem for all the main triangular norms, in the U-approach we have to consider the relation $\models_{Var(*)}$. Indeed, the following theorem holds true.

THEOREM 3.3 (Completeness theorem for the U-approach). *Let $*$ be any of the basic triangular norms, then there is a deduction system defining a relation $\vdash_{Var(*)}$ such that for every set T of formulas and every formula α we have*

$$T \vdash_{Var(*)} \alpha \iff T \models_{Var(*)} \alpha,$$

4. The graded approach

In accordance with the ideas of Pavelka, in the G-approach, fuzzy logic is defined in an abstract way [Pavelka, 1979a,b,c]. Notice that both the semantics and the deduction apparatus are defined by assuming that $[0, 1]$ is fixed as the set of truth values.

DEFINITION 4.1. An *abstract fuzzy semantics* is a class M of fuzzy subsets of Form. The elements in M are named *interpretations* or *models*.

The point of view is that we can identify a model with the valuation of the formulas it determines, i.e., with a particular fuzzy subset of formulas. The truth-functional semantics given in Section 2 is a particular case. Indeed, given a valuation structure, we obtain an abstract fuzzy semantics M by setting $M = \{m_{(D,I)} : (D, I) \text{ is an interpretation}\}$.

Notice that very interesting semantics exist which are not truth-functional. The following definition is on the basis of the G-approach.

DEFINITION 4.2. A *fuzzy theory* is a fuzzy subset of formulas τ, $m \in M$ is a *model of* τ, in brief $m \models \tau$, provided that $m \supseteq \tau$. The *logical* consequence operator $L_f : [0, 1]^{\text{Form}} \to [0, 1]^{\text{Form}}$ is defined by setting, for every $\tau \in [0, 1]^{\text{Form}}$,

$$L_f(\tau)(\alpha) := \inf\{m(\alpha) : m \in M, m \models \tau\}.$$

The *fuzzy subset of tautologies* Tau: Form $\to [0,1]$ is defined by putting $Tau := L_f(\varnothing)$, i.e.,

$$Tau(\alpha) = \inf\{m(\alpha) : m \in M, m \vDash \tau\}.$$

Observe that the value $\tau(\alpha)$ is not interpreted as the truth value of α but as a piece of information on this value. Namely, it represents the constraint $[\tau(\alpha), 1]$ on the possible truth value of α. In accordance, $L_f(\tau)(\alpha)$ is the best constraint on the truth value of α we can obtain given τ. Notice that while in the U-approach one defines the set of the $*$-tautologies and the set of general $*$-tautologies in the G-approach one defines only the fuzzy subset of tautologies. This fact remarks the strong difference between the two approaches.

DEFINITION 4.3. A *fuzzy Hilbert system* is a pair (la, R), where la : Form $\to [0,1]$ is a fuzzy subset of formulas, the *fuzzy subset of logical axioms*, and R is a set of fuzzy inference rules. In turn, *a fuzzy inference rule* is a pair $r = (r', r'')$, where

- r' is a partial n-ary operation on Form,
- r'' is an n-ary operation on $[0,1]$ preserving the least upper bounds (continuity hypothesis).

We indicate an application of an inference rule r by a picture as

$$\frac{\alpha_1, \ldots, \alpha_n}{r'(\alpha_1, \ldots, \alpha_n)} \ldots ; \ldots \frac{\lambda_1, \ldots, \lambda_n}{r''(\lambda_1, \ldots, \lambda_n)}$$

whose meaning is that: IF you know that $\alpha_1, \ldots, \alpha_n$ are true (at least) to the degree $\lambda_1, \ldots, \lambda_n$, respectively, THEN the formula $r'(\alpha_1, \ldots, \alpha_n)$ is true (at least) to the degree $r''(\lambda_1, \ldots, \lambda_n)$.

An important example of an inference rule is the *graded modus ponens* in which the firs component is the usual modus ponens and the second component is the triangular norm.

DEFINITION 4.4. A *proof* π *of a formula* α is a sequence $\alpha_1, \ldots, \alpha_m$ of formulas such that $\alpha_m = \alpha$, together with the related "*justifications*". This means that, for any formula α_i, we must specify whether

(i) α_i is assumed as a logical axiom; or
(ii) α_i is assumed as an hypothesis; or
(iii) α_i is obtained by the first component of a rule (in such a case we have to specify the rule and the formulas in the list $\alpha_1, \ldots, \alpha_{i-1}$ used by the rule).

Definition 4.5. Let τ be a fuzzy theory (the available information) and π a proof $\alpha_1, \dots, \alpha_m$. Then the *valuation* $Val(\pi, \tau)$ *of* π *with respect to* τ is defined by induction on the length m of π as follows:

- $Val(\pi, \tau) = la(\alpha_m)$, if α_m is assumed as a logical axiom,
- $Val(\pi, \tau) = \tau(\alpha_m)$, if α_m is assumed as an hypothesis,
- $Val(\pi, \tau) = r''(Val(\pi(i(1)), \tau), \dots, Val(\pi(i(n)), \tau))$, if $\alpha_m = r'(\alpha_{i(1)}, \dots, \alpha_{i(n)})$, where $i(1), \dots, i(n)$ are in $\{1, \dots, m-1\}$.

If $\alpha = \alpha_m$ is the formula proven by π, the meaning of $Val(\pi, \tau)$ is that *given the information* τ, *the proof* π *ensures that* α *holds true at least at level* $Val(\pi, \tau)$. Let me note that $Val(\pi, \tau)$ is not a truth value but a constraint on a truth value, i.e., the constraint represented by the (crisp) interval $[Val(\pi, \tau), 1]$. Otherwise should be contradictory the existence of two proofs π_1 and π_2 of the same formula α such that $Val(\pi_1, \tau) \neq Val(\pi_2, \tau)$. Instead, we can interpret this existence by the constraint $[\sup\{Val(\pi_1, \tau), Val(\pi_2, \tau)\}, 1]$ and therefore by the fusion of the information given by the two proofs. This means that expression as "α is a theorem in the degree $Val(\pi, \tau)$" or "α is provable in the degree $Val(\pi, \tau)$" are misleading. This interpretation gives also a justification to the maximality principle which is on the basis of the next definition.

Definition 4.6. Given a fuzzy Hilbert's system (a, R), we define the function $D_f \colon [0,1]^{\mathrm{Form}} \to [0,1]^{\mathrm{Form}}$ by setting

$$D_f(\tau)(\alpha) := \mathrm{Sup}\{ Val(\pi, \tau) : \pi \text{ is a proof of } \alpha\}. \tag{$\dagger$}$$

We say that D_f is the *deduction operator of* (a, R).

Observe that the choice of fusing the pieces of information furnished by the proofs of α by (†) is imposed by the fact that

$$\bigcap\{[Val(\pi, \tau), 1] : \pi \text{ is a proof of } \alpha\} = \sup\{D_f(\tau)(\alpha), 1\}.$$

Definition 4.7. A fuzzy semantics *is axiomatizable* provided that there is a fuzzy Hilbert system such that $L_f = D_f$. In such a case, we say also that a *completeness theorem* holds true.

Unfortunately, a completeness theorem looks to be possible only if the conjunction connective $*$ is interpreted by Łukasiewicz's product.

Theorem 4.8 (Completeness theorem for the G-approach). *Given the fuzzy semantics associated with Łukasiewicz first order logic with rational constants, a completeness theorem holds true.*

Note. Usually the deduction apparatus is defined by "*evaluated formulas*", i.e., pairs as $\langle \lambda, \alpha \rangle$, where λ is a truth value and α is a formula. I prefer a clear distinction between the linguistic level in a deduction and the valuation level. This since the linguistic level is conscious, explicit, and plays its main role in the communication. The valuation level, to the extent that vagueness is involved, is not conscious and it is the result of a rather obscure process developed by an individual. Moreover, it is continuously adapted in a pragmatic way. Perhaps, this distinction facilitates an extension of the spirit of fuzzy control to the whole fuzzy logic. In fact, if we introduce parameters in the valuation part of the inference rules and in the available information τ, then (†) becomes a parameterized function and this would make possible tuning, learning, negotiation and so on (see also Section 9).

5. Examples of misunderstanding

We start this section by quoting a criticism in [Pelletier, 2000, 2004]:

> Most logicians think that one logic is an extension of another if it contains all the theorems of the other [...]. But this is not a sense in which fuzzy logic is an extension of classical logic; for, $(A \vee \neg A)$ is a theorem of classical logic but not of fuzzy logic. Indeed, it can be shown that there is no theorem of fuzzy logic (in $\wedge$, $\vee$, $\neg$) which is not already a theorem of classical logic. So classical logic in fact is an extension of fuzzy logic, in the usual use of the term 'extension', and not the other way around.

Moreover, in speaking about the solution of the Heap paradox proposed by fuzzy logic [see Goguen, 1968; Hájek and Novák, 2003; Sorensen, 2001] claims

> I am a logical conservative in that I deny that vagueness provides any reason to reject any theorem or inference rule of standard logic.

In accordance

> instead of changing logic, we should change our opinions about how language works.

The expression used is "*changing logic*" and not "*extending logic*" and this emphasizes author's idea about the alternative nature of fuzzy logic with respect to the classical one. It is evident that this idea is a misunderstanding of the nature of fuzzy logic. Nevertheless I will spend some words on this question since this misunderstanding originates several prejudices and excessive enthusiasms. In fact, a look at the just given definitions shows that every fuzzy logic is completely built up inside second order classical logic. Indeed, in this logic we can define the real numbers structure as a model of the theory of ordered complete fields and therefore the interval $[0, 1]$. Moreover, we can define the operations usually used to interpret the logical connectives and, in the G-approach, the operations associated with the inference rules. This leads to the following (trivial) claim.

Claim 1. *Since second order classical logic is an adequate meta-theory for fuzzy logic, fuzzy logic is "included" in classical logic.*

At the same time, there is also a sense in claiming that all the fuzzy logics are an extension of classical logic.

Claim 2. *Since in a fuzzy logic the logical connectives and the quantifiers are interpreted extending the classical interpretations, classical logic is "included" in this fuzzy logic.*

The fact that there are classical tautologies which are not a theorem of a fuzzy logic is not surprising since if we enlarge the class of interpretations it is natural to restrict the class of tautologies.

In conclusion a fuzzy logic cannot be alternative to classical logic since it is a construct of this logic and, at the same time fuzzy logic is an attempt to extend classical logic. Should this construct a contribution to the understanding of the phenomenon of the vagueness, that would be a further success of classical logic and therefore even a "*logical conservative*" person should be happy.

Obviously, there are several further misunderstandings. For example, Hájek, in the very interesting paper [Hájek, 1999] quoted the following argument of the philosopher R. Parikh [1991]:

> If fuzzy logic says that there is an x such that President de Klerk is x-African, then it must tell us how to measure x and how to resolve the conflict between one person who says that de Klerk is 0.8-African and another that he is 0.2-African. It must also tell us how the correct value x such that he is x-African is related to these two conflicting reports and what it means to say that x is the correct value.

An immediate answer to the first question is that the truth degree of the atomic formula "*President de Klerk is African*" is not a problem of fuzzy logic.

> Is there something like the "real" interpretation (actual possible world)? This question is asked about the classical logic and about the fuzzy logic as well — and for both the answer is outside the scope of logic.
>
> Question 5 in [Hájek, 1999]

Indeed, in a logic:
- the aim of the semantics is to define the truth value of a composed formula from the truth values of the atomic components,
- the aim of the deduction apparatus is to calculate the information we can derive from the available information.

How we can obtain the truth value of the atomic components or the information expressed by the theory is out of the scope of logic.

We conclude this section by emphasizing that in [Belohlavek et al., 2009] one exhibits an impressive list of misunderstandings on fuzzy logic appearing in the literature on the psychology of concepts.

6. Compactness

To give an answer to Pelletier's observations on compactness and effectiveness, we consider at first the answer given by the U-approach. To do this, we have to distinguish the standard semantics from the general semantics.

Definition 6.1. We say that the logic associated with a standard algebra $([0,1],\otimes,\rightarrow)$ is *compact in standard sense*, provided that, given a set T of formulas and a formula α,

$$T \models_* \alpha \Longrightarrow \text{ there is a finite part } T_f \text{ of } T \text{ such that } T_f \models_* \alpha.$$

We say that a logic is *compact in a general sense*, provided that

$$T \models_{Var(*)} \alpha \Longrightarrow \text{ there is a finite part } T_f \text{ of } T \text{ such that } T_f \models_{Var(*)} \alpha.$$

Then we can avoid Pelletier's criticisms by referring to compactness in general sense. Indeed, in accordance with the fact that axiomatizability entails compactness, we have the following very important theorem.

Theorem 6.2. *All the $*$-logics are compact in general sense.*

A different notion of compactness is necessary in the case we refer to the G-approach. Indeed in such a case the compactness is a property of the logical consequence operator $L_f\colon [0,1]^{\mathrm{Form}} \to [0,1]^{\mathrm{Form}}$ and in defining it is necessary to take in account the difference between the topological structures of $\{0,1\}^{\mathrm{Form}}$ and the one of $[0,1]^{\mathrm{Form}}$. This suggests to look at a continuity property and not at a finiteness property [see Gerla, 2000]. Recall the limit of an upward directed class C in an ordered set is defined as its least upper bound of this class, i.e., $\lim C = \sup C$. Also, observe that the imagine $H(C)$ of an upward directed class C by a monotone map H is an upward directed class.

Definition 6.3. For any $(\lambda, \leq)$ a complete lattice, a function $H\colon L \to L$ is *continuous* itf provided that $H(\lim C) = \lim H(C)$, for any upward directed class C of elements in L. Given a nonempty set S, we call *compact* an operator $H\colon L^S \to L^S$ which is continuous in the lattice L^S.

We emphasize that this notion, which is different from the one proposed by Pavelka, is long time known in logic programming (and it is a basic one in domain theory). A useful feature of the continuity is that it enables us to define the least Herbrand model of a program as a fixed point of the immediate consequence operator. There are several reasons to assume the continuity as the correct counterpart of the notion of compactness in fuzzy logic. Firstly, in the case of the lattice of all subsets of a given set, this notion coincides with the usual compactness. Moreover, we can characterize the continuity in the lattice of the fuzzy subsets of a given set in terms of finite fuzzy subsets.

Theorem 6.4. *Define the relation $\prec$ by setting, for any $s_1, s_2 \in L^S$,*

$$S_1 \prec s_2 \iff s_1(x) < s_2(x), \text{ for every } x \in Supp(s_1).$$

Then an operator $H\colon L^S \to L^S$ is continuous if and only if, for every fuzzy subset s of S,

$$H(s) = \bigcup\{H(s_f) : s_f \text{ is finite and } s_f \prec s\}.$$

Once we admit Definition 6.3, the following theorem gives an answer to Pelletier's criticism [see Gerla, 2000].

Theorem 6.5. *The deduction operator of a fuzzy Hilbert system (in a countable language) is compact. In particular, the logical consequence operator of Łukasiewicz logic with evaluated syntax is compact.*

7. Effectiveness

The effectiveness of fuzzy logic is a further crucial question. Again, the answer of Hájek's school is to refer to the entailment relation $\models_{Var(*)}$ and therefore to the notion of general $*$-tautology. In fact, one proves the following very important and surprising result [Hájek, 1995; Montagna, 2001]:

> *While the set of standard tautologies in Łukasiewicz first order logic is not recursively enumerable, the set of general $*$-tautologies is recursively enumerable.*

So, it is sufficient to refer to the notion of general tautology to remove Pelletier's criticism.

A totally different apparatus is necessary if we will consider the G-approach. Indeed, in this case we have to define in a suitable way the notions of *semi-decidable* and *decidable* fuzzy subset. We examine this question in the next subsections.

7.1. Fuzzy Turing machines

Classical computability theory originated from Turing's definition of computing machine. It is thus not surprising that in fuzzy literature there are many papers devoted to extend this notion to the fuzzy framework. The following definition is sufficiently representative of the existing ones.

Definition 7.1. A *nondeterministic fuzzy Turing machine*, in brief a FTM, is a structure $\mathcal{F} = (Q, T, I, b, q_0, q_f, \mu, \otimes)$ such that

- Q, T, I are nonempty finite sets such that $I \subseteq T$,
- $\underline{b} \in T - I$ and $q_0, q_f \in Q$,
- $\mu\colon \Delta \to [0,1]_Q$ is a fuzzy subset of Δ, where $\Delta := Q \times T \times Q \times T \times \{-1, 0, +1\}$,
- $\otimes$ is a t-norm in $[0,1]$ such that the product of two rational numbers is a rational number.

The elements in Q, T, and I are named, *internal states*, *tape symbols*, and *input symbols*, respectively. Also, $\underline{b}$ is the *blank symbol*, q_0, q_f are the *initial* and the *accepting state*, respectively. As it is usual, the intended meaning of -1, 0, $+1$ is "move one cell to the left", "not move", "move one cell to the right". An element $\delta = (q_i, t_i, q_{i+1}, t_{i+1}, d)$ in Δ is called a *transition rule*. Its intended meaning is that when the machine $\mathcal{F}$ is in state q_i and the current symbol has been read is t_i, then this machine is authorized to print t_{i+1} on the current cell, to move the scanning head according with d and to enter in the state q_{i+1}. Then a FTM is like a classical nondeterministic Turing machine in which instead of a subset of the set Δ of transition rules one considers a fuzzy subset of Δ. The role of the operation $\otimes$ will be evident in Definition 7.3. One defines the notions of *configuration* and of *accepting configuration* as usual. Given an input $w \in I^+$, we denote by $C_0(w)$ the configuration in which the characters of w are printed on tape starting from the leftmost cell, the scanning head is placed atop the leftmost cell, and the machine enters state q_0.

Definition 7.2. Given two configurations C and $\underline{C}$, a *computational path from C to $\underline{C}$* is a sequence $Z = (\langle C_0, \delta(1)\rangle, \dots, \langle C_{n-1}, \delta(n)\rangle, C_n)$, where $\delta(i) \in \Delta$, such that $C_0 = C$, $C_n = \underline{C}$ and C_{i+1} is obtained from C_i, by $\delta(i+1)$ for $i = 0, \dots, n-1$. We denote by $\mathrm{PATH}(C, \underline{C})$ the class of all the computational paths from C to $\underline{C}$ and we say that *$\underline{C}$ is reachable from C* provided that $\mathrm{PATH}(C, \underline{C}) \neq \varnothing$. In the case C is an accepting configuration, we say that Z is an *accepting computational path*.

Definition 7.3. Given two configurations C and $\underline{C}$, and a computational path $Z = (\langle C_0, \delta(1)\rangle, \dots, \langle C_{n-1}, \delta(n)\rangle, C_n)$, from C to $\underline{C}$ we put

- $D(Z) := \mu(\delta(1)) \otimes \cdots \otimes \mu(\delta(n))$.

Assume that $\underline{C}$ is reachable from C then we put

- $d(C, \underline{C}) = \sup\{D(Z) : Z \in \mathrm{PATH}(C, \underline{C})\}$.

Definition 7.4. Let $\mathcal{F}$ be a FTM, then the *fuzzy subset $e_{\mathcal{F}} \colon I^+ \to [0,1]$ of I^+ associated with $\mathcal{F}$* is defined by setting, for every $w \in I^+$,

$$E_{\mathcal{F}}(w) = \sup\{d(C_0(w), \underline{C}) : \underline{C} \text{ is an accepting configuration}\}.$$

A fuzzy subset e of I^+ which is associated with a fuzzy Turing machine is called *FT-semidecidable*. We say that e is FT-*decidable* if both e and its complement are FT-semidecidable.

As usual, while this definition refers to a fuzzy subset of the set I^+ of words in an alphabet I, there is no difficulty to extend it to every set admitting a coding, for example the set of natural numbers, the set of formulas of a logic and so on.

Unfortunately, the class of FT-semidecidable subsets has proved inadequate to give a basis for a general theory of effectiveness in fuzzy logic. This fact is a consequence mainly of the following Lemma on the ordered monoids which is rather known in the literature [see, e.g., Gerla, 2016].

Lemma 7.5. *Let* $(A, \otimes, \leq, 0, 1)$ *be a finitely generated totally ordered commutative monoid with minimum* 0 *and maximum* 1. *Then every nonempty subset of* A *admits a maximum.*

Now, it is immediate to see that a FT-machine works with the monoid generated by the codomain $cod(\mu)$ of μ and therefore with a finitely generated commutative monoid. This means that the conditions of this lemma are satisfied and therefore that the following theorem holds true.

Theorem 7.6. *Let* $e\colon I^+ \to [0,1]$ *be a FT-semidecidable fuzzy subset, then* $cod(e)$ *is a set of rational numbers such that every subset of* $cod(e)$ *admits a maximum (equivalently, the dual of the natural order in* $code(e)$ *is a well order).*

An immediate consequence of this theorem, several fuzzy subsets which are decidable from an intuitive point of view cannot be FT-semidecidable. The following theorem gives an example.

Theorem 7.7. *Let* $e\colon I^+ \to [0,1]$ *be a fuzzy subset whose co-domain has no maximum, then* e *is not FT-semidecidable. In particular, denote by* $length(w)$ *the length of a word* w *and consider the fuzzy subset of big world* $big\colon I^+ \to [0,1]$ *defined by setting*

$$big(w) := \frac{length(w)}{length(w)+1}. \tag{‡}$$

Then big *is a Turing-computable fuzzy subset which is not FT-semidecidable.*

Proof. It is evident that big is computable and that, since $cod(big)$ has no maximum, big is not recognizable by a FTM. □

Notice that this theorem is not based on the particular definition of big in (‡). Indeed, it is evident that given any reasonable interpretation $big\colon I^+ \to [0,1]$ of the notion of "big word", we have that $big(w)$ is less than $big(w^{length(w)})$ and therefore that $cod(big)$ has no maximum.

Thus the notion of FT-semi-decidability is not in accordance with our intuition.

Several further criticisms on the fuzzy Turing machines are possible. For example, we list the following ones [see, e.g., Gerla, 2016].

1. The notion of FT-semidecidability completely depends on the choice of the triangular norm.
2. The one of FT-decidability depends also on the choice of the operation to interpret the negation and it is questionable in the case this operation is not involutory.
3. The FT-semidecidability is not compatible with the application of computable linguistic modifiers.
4. There is no universal FT-machine in the case the valuation structure is not finite.

These criticisms suggest a different notion of effectiveness for fuzzy logic.

7.2. Computation as an effective approximation process

In a series of papers, L. Biacino and G. Gerla proposed a limit-based approach to fuzzy computability [see, e.g., Biacino and Gerla, 2002; Gerla, 1982, 2000, 2007]. Their proposal is based on the following observations. In classical logic the effectiveness is based on finite-steps terminating processes as a consequence of the discrete topological structure of the two elements Boolean algebra $\{0,1\}$. Instead, fuzzy logic refers to $[0,1]$ and this suggests basing the effectiveness by endless approximation processes and by the notion of limit. In other words, the effectiveness in fuzzy logic has to be based on the topological structure of the continuum. On the other hand, a limit-based notion of computability is also possible in classical logic provided we refer to the discrete topology of $\{0,1\}$. Indeed, it is easy to prove that a subset X of a set S is *effectively enumerable* (*co-enumerable*) provided that, there is a computable map $h\colon S\times N \to \{0,1\}$ which is increasing (decreasing) with respect to n and such that $\lim_{n\to\infty} h(x,n) = c_X(x)$. If we refer to the natural topology in $[0,1]$, we obtain the following definitions where we prefer the words "*semi-decidable*" and "*dually semi-decidable*" in the place of "effectively enumerable" and "effectively co-enumerable".

Definition 7.8. Let S be a nonempty set with a coding. Then we call *semi-decidable* (dually semi-decidable) a fuzzy subset $s\colon S \to [0,1]$ of S provided that a computable map $h\colon S\times N \to [0,1]_Q$ exists which is order-preserving (order-reversing) with respect to the second variable and such that,

$$s(x) = \lim_{n\to\infty} h(x, n), \tag{$\star$}$$

for every $x \in S$. We say that s is *decidable* if it is both semi-decidable and dually semi-decidable. Then s is decidable provided that for any $x \in S$, $s(x)$ is the limit of a nested effectively computable sequence of intervals with rational bounds.

There are several reasons supporting these definitions. The first one is that they are in accordance with the classical ones for crisp subsets of S. Another reason is that the existing definitions of computability in fuzzy set theory are all in accordance with Definition 7.8. Moreover, differently from the existing definitions, the proposed definition does not depend on the choice of the triangular norm. Obviously, as in the case of Church thesis, it is not possible to give a definitive proof of adequateness.

As a consequence of ($\star$) we have the following theorem.

THEOREM 7.9. *Assume that s is semi-decidable. Then for any $\lambda \in [0, 1]_Q$ the open λ-cut of s is recursively enumerable while the closed λ-cut belongs to the Π_2-level of the arithmetical hierarchy.*

The notion of semi-decidability enables us to extend the classical notion of enumeration operator [see Rogers, 1967] to an operator on the lattice of fuzzy subsets of a given nonempty set [see Gerla, 2007]. We put $SEQ = F_f(S) \times S$, where $F_f(S)$ is the class of finite fuzzy subsets of S.

DEFINITION 7.10. We say that a fuzzy operator $H\colon [0, 1]^S \to [0, 1]^S$ is an *enumeration operator* or a *left computable operator* if a semi-decidable fuzzy subset $w\colon SEQ \to [0, 1]$ exists such that

$$H(s)(x) = \sup\{w(s_f, x) : s_f \prec s\}.$$

This notion coincides with the one of computable operator in effective domains theory.

PROPOSITION 7.11. *Let H be an enumeration operator. Then H is continuous and, for every semi-decidable fuzzy subset s, the fuzzy subset $H(s)$ is semi-decidable.*

In [Gerla, 2000] one proves the following theorem where a fuzzy deduction system is called *effective* provided that the fuzzy subset of logical axioms is decidable and the inference rules are computable.

Theorem 7.12. *The deduction operator of an effective fuzzy Hilbert system (in a countable language) is an enumeration operator. Conversely, given an enumeration operator H, an effective fuzzy Hilbert system exists whose deduction operator coincides with H.*

In the following corollary we call *e complete* a theory τ such that $D_f(\tau)(\neg\alpha) + D_f(\tau)(\alpha) = 1$.

Corollary 7.13. *Given an effective fuzzy Hilbert system, if a fuzzy set of axioms τ is decidable, then the related fuzzy set $D_f(\tau)$ of consequences is semi-decidable. If τ is complete and decidable, then $D_f(\tau)$ is decidable.*

In account of the axiomatizability of Łukasiewicz first order logic with countable language, we obtain the following corollary.

Corollary 7.14. *The logical consequence operator $D_{Ł}$ in Łukasiewicz first order logic with countable evaluated syntax is computable. In particular, the fuzzy subset of tautologies $Tau_{Ł}$ is semi-decidable.*

Observe that the criticized non effectiveness of fuzzy logic is based on the fact that the (classical) set of standard tautologies, i.e., the closed 1-cut $\{\alpha \in \text{Form} : D_{Ł}(\varnothing)(\alpha) = 1\}$, is not recursively enumerable. In accordance with Theorem 7.9, this does not contradict the fact that the fuzzy subset $Tau_{Ł}$ of tautologies is semi-decidable. It means only that, given any formula α,

> while we are able to produce an increasing sequence of rational numbers converging to $Tau_{Ł}(\alpha)$, we are not able to decide if the limit of this sequence is equal to 1 or not.

This phenomenon is not a characteristic of fuzzy logic since it emerges whenever a constructive approach is proposed for a notion involving real numbers. Indeed, in recursive analysis one proves the following proposition:

> *In the class of computable real numbers it is not decidable whether two recursive real numbers are equal or not. In particular, it is not decidable whether a recursive real number is equal to* 1 *or not.*

Theorem 7.9 explains also why in the case we assume the set of designed values is an interval like $(\lambda, 1]$, the set of tautologies is semi-decidable while in the case we assume this set is a closed interval $[\lambda, 1]$, the set of tautologies is not semi-decidable.

We conclude this section by observing that since Corollary 7.15 is a consequence of the axiomatizability of Łukasiewicz logic, it refers to a (countable) language in which there is a logical constant for every element in $[0,1]_Q$. In spite of that, this corollary holds true also if we refer to a language whose unique logical constants are 0 and 1.

COROLLARY 7.15. *Let $L^*_Ł$ be the Łukasiewicz logic with evaluated syntax in a language with only the constants* 0 *and* 1. *Then the related logical consequence operator $D^*_Ł$ is computable. Consequently, this logic is effectively axiomatizable and the fuzzy subset of tautologies is semidecidable.*

PROOF. Taking in account the coincidence of the class of interpretations in the two logics, we have that $D^*_Ł$ is the restriction of $D_Ł$ to the fuzzy subsets of formulas in $L^*_Ł$, i.e., the set of formulas whose unique constants are 0 and 1. Then $D^*_Ł$ is computable. □

My hypothesis is that a large part of the negative results on decidability for fuzzy logic cannot be proved if one accepts the limit-based definition of decidability.

8. Further criticisms

The answers to the criticisms on compactness and effectiveness in both the U-approach and in the G-approach are correct from a technical point of view, obviously. Nevertheless, we can consider these answers satisfactory only if we admit that the related apparatus is adequate to formalize the phenomenon we are interested in: the human everyday rational activity in which the vagueness is constantly involved. As in the case of Church Thesis, a definitive verdict on this question is not possible, differently from Church Thesis, there are again several arguments against this adequateness. In the following subsections we list some of them.

8.1. Criticisms of the ungraded approach

C1. The standard semantics has difficulties with the completeness theorem. Indeed, the unique possible completeness theorem is for a logic in which $*$ is the minimum. Now, while the resulting logic is interesting from a mathematical point of view, it looks to be inadequate to represent the vagueness phenomenon. For example, it is not able to face the heap paradox. Moreover, it is not able to express the transitivity of basic binary vague relations as "*to be close*", "*to be similar*" and so on. This is in contrast with our intuition suggesting that some kind of weak transitivity holds true for these relations.

C2. The general semantics is not intuitive. The general semantics enable us to have a satisfactory completeness theorem for a large class of triangular norms. Unfortunately, this is obtained by involving the class $Var(*)$ in which there are too many unusual structures. In contrast, the notion of vagueness is strictly connected with the idea of the continuum and this idea is universally modelled by the set of real numbers.

C3. The general semantics is not connected with the actual applications of fuzzy set theory. Indeed, far as I am aware, in these applications there is no presence of structures whose domain is different from $[0, 1]$. This departure from the existing technique of fuzzy logic should not be underestimated. In fact, the task of a logic is to give a rigorous justification the existing activities and not to invent a semantics ad hoc to obtain a completeness theorem.

C4. Unsafe interpretations. The simple examples of fuzzy interpretations M_2 and M_3 in Section 3 allow to highlighting further difficulties. Indeed, we have that every time a model interpreted in $[0, 1]$ admits a predicate α such that $\exists x_i(\alpha)$ holds true without a witness, this model is safe if considered in $[0, 1]$ and becomes unsafe if it is considered in a non-standard extension of $[0, 1]$. This is not in contrast with the completeness theorem, obviously, but is rather disturbing. Indeed, there is something wrong in the fact that the valuation of a formula in an interpretation depends on the choice of the valuation structure and not on the intrinsic nature of the interpretation. Again, in referring to $Var(*)$, we have to handle valuation algebras which are not complete and, consequently, we have to exclude unsafe valuations. Are we authorized to make this exclusion? If we admit non-standard truth values, then there is not a reason to exclude a rather natural interpretation as M_3. Again

- *there is a way to decide in advance whether an interpretation originates a safe interpretation or not?*

C5. The questionable meaning of the completeness theorem. I have some doubts about the meaning of the completeness theorem for the general semantics. For example, once we consider the very simple structure M_1 in Section 3, we aspect the possibility of deducing from T a simple formula as $\exists x(Close(x))$. Now, M_4 is a safe model of T in which $\exists x(Close(x))$ is not satisfied. So, since $T \models_{Var(*)} \exists x(Close(x))$ does not hold, no proof of this formula exists. On the other hand, in accordance with Pelletier criticisms related to the compactness, a deduction apparatus cannot exist able to prove this formula from T. Indeed, every finite part of T admits a model in which $\exists x(Close(x))$ assumes a value different from 1.

Instead, observe that in the G-approach, it is possible to substitute T by the fuzzy theory τ defined by setting $\tau(C(P_i)) = q(i)$, for every $i \in N$ and by admitting the formulas $\alpha(x/t) \rightarrow \exists x\ \alpha(x)$ as logic axioms, where t is a closed term. Once we admit a graded modus ponens as inference rule, from $C(P_i)$ and $C(P_i) \rightarrow \exists x(Close(x))$ we obtain $\exists x(Close(x))$ at degree $q(i)$. So, in accordance with (†) of Definition 4.6, we are able to prove $\exists x(Close(x))$ at degree $\sup\{q(i) : i \in N\} = 1$.

C6. Approximate reasoning. What's new in fuzzy logic (with respect to the tradition of multi-valued logic) is the acceptance of approximate reasonings as in Goguen's solution of Heap paradox. In these reasonings the available information is not necessarily a crisp set and the conclusions are not necessarily at degree 1.

C7. The presence of a logical constant for every rational number in $[0, 1]$. This presence is unnatural since no one adopts in its language sentences as $3/4$ or $small(c) \rightarrow 3/4$ or more complex formulas as $(small(c) \rightarrow 3/4) \rightarrow 6/7$. On the other hand, in the U-approach this presence is a necessary one since the absence of rational constants in the language entails that no vague predicate can be managed.

8.2. Criticisms of both the approaches

The G-approach avoids the above criticisms in my opinion. Mainly, it shows that also in the case the valuation structure is fixed and the truth values are real numbers, a logic with a related completeness theorem is possible and that reasonable properties of compactness and effectiveness are obtainable. In addition, it enables us to give a notion of approximate reasoning which is very important from a philosophical point of view. It is also important for applications, as an example for fuzzy logic programming [see Vojtáš, 2001] and fuzzy control once we observe that fuzzy control is only an example of an application of fuzzy logic programming [see Gerla, 2005]. I quote also fuzzy equational and implicational logics, see [Belohlavek, 2002; Belohlavek and Vychodil, 2005] as further fragments of Pavelka-style predicate fuzzy logics and a logic of dependencies in data with fuzzy attributes [see Belohlavek and Vychodil, 2005, 2015].

Unfortunately, there are criticisms of both the approaches. The first two are related to the truth-functionality.

C8. Difficulties for the truth-functionality: the connectives. The linguists claim that there is empirical evidence against truth-functionality in fuzzy logic. For example, U. Sauerland claims:

> When I started interacting with logicians, I was surprised to learn that fuzzy logic is still a big and active field of basic research. This surprise stemmed from my experience with fuzzy logic in my own field, linguistic semantics: In semantics, fuzzy logic ... has been regarded as unsuitable for the analysis of language meaning at least since the influential work of Kamp in 1975. [Sauerland, 2011]

Namely, Sauerland emphasizes that a large series of experiments show that a claim as "Luise is both tall and not tall" sounds not as nonsensical as in classical logic. Indeed, a high percentage of people interpret this claim as a way to communicate that we are in presence of a borderline case. This means that it is not excluded at all that a formula as $\mathrm{A} \wedge \neg \mathrm{A}$ assumes the value 1 and this contrasts with the formalizations proposed by the main fuzzy logics. Indeed, assume that in a fuzzy logic this formula assumes value 1, and therefore that there is $\lambda \in [0, 1]$ such that $\lambda * \sim \lambda = 1$, where $*$ and $\sim$ are the operations interpreting $\wedge$ and $\neg$, respectively. Then, since it is easy to prove that $\lambda * \mu \leq \min\{\lambda, \mu\}$, we have that $\min\{\lambda, \sim \lambda\} \geq \lambda * \sim \lambda \geq 1$ and therefore that $\lambda = 1$ and $\sim \lambda = 1$ and this is impossible.

C9. Further difficulties for the truth-functionality: the quantifiers. The interpretation of the existential quantifier is questionable from an ontological point of view. Indeed, the validity of a formula as $\exists x C(x)$ does not imply the existence of an element in which C is satisfied at degree 1 (see Example 2 in Section 3). In other words, it is possible to claim the existence of non-existent objects. This phenomenon, which is well-known in literature, and it is rather far from the meaning of the word "existence" in a natural language.

C10. Information arising from the structure of a formula. A further criticism is related to the way the information on the truth value of the formulas is stored. In classical logic we can obtain information arising from the structure of the formulas. This is obtained by the notion of tautology and the one of contradiction. In fuzzy logic there are difficulties to make this. For example, consider a propositional logic in $[0,1]$ in which the unique rational constants are 0 and 1, $*$ is the minimum and the negation is interpreted by the function $1-x$. Then, given any formula α whose propositional variable are among $p_1, \ldots, p_n$ the associated interpretation $t_\alpha(x_1,\ldots,x_n)$ has 0.5 as a fixed point, i.e., $t_\alpha(0.5,\ldots,0.5) = 0.5$. This means that in the U-approach no tautology or contradiction exists. In the G-approach $Tau(p \vee \neg p) = 0.5$, i.e., $p \vee \neg p$ is a tautology at degree 0.5 while $Tau(p \wedge \neg p) = 0$ and this gives no information on $p \wedge \neg p$. This is unsatisfactory since the structure of $p \vee \neg p$ (of $p \wedge \neg p$) enables us to claim that this formula satisfies the constraint $[0.5, 1]$ (the constraint $[0, 1\ 0.5]$). Obviously, in the case the logical constant 0.5 is admitted, we can express such a kind of information by claiming that of $\underline{0.5} \to p \vee \neg p$ and $p \wedge \neg p \to \underline{0.5}$ are tautologies. This suggests to extend fuzzy logic in such a way that constraints as "the truth value of α is between 0.4 and 0.6" are admitted [see the attempts in Carotenuto and Gerla, 2013; Genito and Gerla, 2014].

C11. The lack of flexibility. Both the approaches are not sufficiently flexible and this makes the existing formalizations of fuzzy logic incompatible with the role played by the vagueness in our language. This role refers to concepts as "negotiation", "learning", "tuning", "testing" as fuzzy control shows.

For example, in the G-approach one admits a fuzzy subset of hypotheses and therefore precise lower-bound constraints to the truth value of the formulas while it should more natural to assign fuzzy interval constraints. A lack of flexibility is also apparent in the proposed notion of fuzzy model of a fuzzy theory which is based on Zadeh's crisp inclusion. Indeed, let m be a fuzzy model of a fuzzy theory τ. It is evident that both the assignments defined by m and τ cannot be considered definitive and precise. This since m depends on the subjective modeling of the vague predicates and τ depends on the subjective valuation of the truth degree of the formulas. Now assume that either m or τ is subject to a slight variation as a consequence of a tuning process, an essential component in all the applications in fuzzy mathematics. Then it is possible that m ceases completely to be a model of τ while it should be natural to expect m is again a model of τ at some degree. This suggests that it should be opportune to reformulate the notion of fuzzy model of a fuzzy theory by substituting the crisp inclusion with a graded inclusion.

9. Conclusions

My persuasion is that we are rather far from a satisfactory answer to the question of formalizing the inferential processes in which vague notions are involved. This in spite of the increasing number of mathematically sophisticated papers devoted to this enterprise. In this regard I share the following opinion expressed by Belohavek who, by referring mainly to the U-approach, says:

> most of these contributions were in fact developing systems of many-valued logic which were, however mathematically sophisticated, somewhat sterile. [Belohlavek, 2015]

Indeed I am convinced that the main difficulties will be resolved by redefining in a more flexible way the basic formalisms and by abandoning some taboo of Hilbert approach to logic. An example is the Turing's notion of computability and the compactness notion. At this purpose the ideas expressed by M. E. Tabacchi and S. Termini look to be rather interesting. Indeed, in [Tabacchi and Termini, 2017b] one claims that:

> what we are heralding here is that for truly understanding the notion of reasoning, we must have a completely new start, not an adaptation of the technical results of classical mathematical logic. The crucial points of the latter are not motivated by genuine general aspects of the informal notion of reasoning. Its agenda was different, and was different since it was dictated by the needs of Hilbert program [...]. The same can be said also for 'fuzzy mathematical logic' which has modelled itself on the same standards of classical mathematical logic.

and again

> If in a conceptual and programmatic scheme in which there was the hope to reduce semantics to (finitary) syntax the two central notions to control and develop were rightly coherence and completeness, what role can they play in a context in which, uncertainty and imprecision play not only central but also an unavoidable one? At the moment, we can only say that coherence and completeness become surely less important, and certainly not crucial.

We cannot even exclude that the research for a general theory is an impossible task and that we have to be satisfied in fragments of fuzzy logic able to formalize processes in which, for example, the involved pieces of information are not too complicate (a local approach, in a sense).

More in general, perhaps vagueness entails a semantics in which, differently from Tarski's paradigm, notions as:

> "linguistic game", "logic game", "evolutionary meaning", "negotiation", "distance between models", "flexibility", "learning", "tuning"

play a crucial role.

On the other hand, it is not only for technical reasons that these notions are present in all the applications of fuzzy logic and, in particular, in fuzzy control. Again, I quote Hájek:

> I (very fuzzily) imagine a conversation as a game [...] in which both my and your meaning of fuzzy words may change: the cooperative conversation may (possibly) be imagined as a kind of 'tuning' the characteristic functions of fuzzy sets involved. By the way, tuning the characteristic functions is a very important part of building a fuzzy controller: here the good semantics is that which makes the controller to behave well.
> Question 7: How do we communicate? in [Hájek, 1999]

Concluding, I agree with Trillas's idea for which:

> [...] fuzzy logic is, [...] closer to an experimental science than to a formal one. [Trillas, 2006]

Indeed, an experimental science starts from a provisional framework and it is continuously open to amend or to change this framework in accordance with the results of new experiments. So people interested in defining a formal system for a logic of vagueness have to spend their energies not only for "hard technical work inside a well-known and established formal framework", but mainly for "the effort of constructing an innovative formal framework" [Termini, 2002].

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Giangiacomo Gerla
Department of Mathematics
University of Salerno, Italy
gerla@unisa.it