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Daniele Chiffi
Fabien Schang

THE LOGICAL BURDENS OF PROOF
Assertion and hypothesis

Abstract. The paper proposes two logical analyses of (the norms of) justification. In a first, realist-minded case, truth is logically independent from justification and leads to a pragmatic logic LP including two epistemic and pragmatic operators, namely, assertion and hypothesis. In a second, antirealist-minded case, truth is not logically independent from justification and results in two logical systems of information and justification: AR_4 and $AR_{4\blacksquare}$, respectively, provided with a question-answer semantics. The latter proposes many more epistemic agents, each corresponding to a wide variety of epistemic norms. After comparing the different norms of justification involved in these logical systems, two hexagons expressing Aristotelian relations of opposition will be gathered in order to clarify how (a fragment of) pragmatic formulas can be interpreted in a fuzzy-based question-answer semantics.

Keywords: logic for pragmatics; assertion; hypothesis; question-answer semantics; logical opposition

1. Introduction

Justifying an empirical hypothesis is considered to be a key issue in science. In fact, some standards of justification are usually required, in order to accept or reject a hypothesis. In this classical picture, there is a clear distinction between the truth-conditions of the content of a hypothesis and the epistemic acceptance of the hypothesis. And it is following this framework that a pragmatic logic LP for assertions and hypotheses will be presented. Such a pragmatic system is intended to

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serve as a possible logical basis behind the formal representation of the standards of justification of hypotheses. Interestingly, however, in a multi-agent epistemic context agents may hold different standards for the justification and acceptance of hypotheses. We argue, therefore, that an agent-based view on the justification and acceptance conditions for hypotheses can be easily associated with an antirealist epistemology. Two logical systems endowed with a question-answer semantics, AR_4 and $AR_{4\blacksquare}$, will be introduced in order to formalize these antirealist aspects of hypothesis acceptance in the context of agents with different standards of proof and justification. The following sections will expose a thorough analysis of the *pros* and *cons* of the realistic and antirealist epistemologies behind the proposed logical systems. Section 2 explores the pragmatic logic of assertion. In Section 3, the language of the pragmatic logic for hypotheses is presented. Section 4 provides an outline of AR_4 and $AR_{4\blacksquare}$ and the antirealist epistemology on which they are based. Section 5 compares these systems with LP, focusing on their opposition relations. Finally, concluding remarks are discussed in Section 6.

2. Logic for pragmatics: assertions

This section is devoted to introduce a pragmatic logic for assertion (LP) originally developed by Dalla Pozza and Garola [7].[1] The main feature of this logical system is the possibility to distinguish the propositional content of assertions expressed by radical formulas and sentential formulas expressed by asserted propositions such as $\vdash \gamma$, in which "$\vdash$" is the sign for assertion. The former formulas are propositions, while the latter formulas are thus obtained by applying the Fregean assertion sign in front of radical formulas. Coherently with what noticed by Frege, (i) there can be no nested occurrences of the assertion sign and (ii) truth-functional connectives cannot be applied to formulas expressing judgments of assertions. For this reason pragmatic connectives, which are not truth-conditional, are introduced in order to formulate complex sentential formulas expressing assertions. As we will show, such pragmatic connectives have an intuitionistic-like behaviour.

The pragmatic language LP is the union of the set of radical formulae RAD and the set of sentential formulas SENT, which can be defined recursively:

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[1] Some applications of LP to philosophical problems are provided in [4, 5, 6].

RAD $\gamma ::= p|\neg\gamma|\gamma_1 \wedge \gamma_2|\gamma_1 \vee \gamma_2|\gamma_1 \rightarrow \gamma_2|\gamma_1 \leftrightarrow \gamma_2|$

SENT i) Elementary sentential formulas $\theta ::=\vdash \gamma$

ii) Sentential $\delta ::= \theta| \sim \delta|\delta_1 \cap \delta_2|\delta_1 \cup \delta_2|\delta_1 \supset \delta_2|\delta_1 \equiv \delta_2|$.

In particular, a pragmatic language LP is composed of two categories of logical-pragmatic signs:

- the signs of pragmatic illocutionary force ("$\vdash$" assertion);
- the pragmatic connectives: $\sim$ pragmatic negation, $\cap$ pragmatic conjunction, $\cup$ pragmatic disjunction, $\supset$ pragmatic implication, $\equiv$ pragmatic equivalence.

Every radical formula of LP has a truth value (true or false) and every sentential formula has a justification value ('J' justified or 'U' unjustified) that is defined in terms of the intuitive notion of proof and depends on the truth value of its radical sub-formulas. The semantics of LP is the same as for classical logic, and it provides only the interpretation of the radical formulas, by assigning them a truth-value and interpreting propositional connectives as truth functions in the standard way. The semantic rules for radical formulas are the usual classical Tarskian ones and specify the truth-conditions (only for radical formulas) through an assignment function σ, thus regulating the semantic interpretation of LP. Let γ_1, γ_2 be radical formulas and $1 =$ true and $0 =$ false; then:

1. $\sigma(\neg\gamma_1) = 1$ iff $\sigma(\gamma_1) = 0$
2. $\sigma(\gamma_1 \wedge \gamma_2) = 1$ iff $\sigma(\gamma_1) = 1$ and $\sigma(\gamma_2) = 1$
3. $\sigma(\gamma_1 \vee \gamma_2) = 1$ iff $\sigma(\gamma_1) = 1$ or $\sigma(\gamma_2) = 1$
4. $\sigma(\gamma_1 \rightarrow \gamma_2) = 1$ iff $\sigma(\gamma_1) = 0$ or $\sigma(\gamma_2) = 1$
5. $\sigma(\gamma_1 \leftrightarrow \gamma_2) = 1$ iff $\sigma(\gamma_1) = \sigma(\gamma_2)$.

Pragmatic connectives have a meaning which is explicated by the BHK (Brouwer, Heyting, Kolmogorov) intended interpretation of intuitionistic logical constants. The illocutionary force of assertion plays an essential role in determining the *pragmatic* component of the meaning of an elementary expression, together with the *semantic* component expressed in the radical formulas.

Justification rules regulate the pragmatic evaluation π, specifying the justification conditions for the assertive formulas in function of the σ-assignments of truth-values for their radical sub-formulas:

JR1 – Let γ be a radical formula. $\pi(\vdash \gamma) = J$ iff a proof exists that γ is true, i.e. that σ assigns to γ the value 1. $\pi(\vdash \gamma) = U$ iff no proof exists that γ is true.

JR2 – Let δ be an assertive formula. Then, $\pi(\sim \delta) = J$ iff a proof exists that δ is unjustified. i.e., that $\pi(\delta) = U$.

JR3 – Let δ_1 and δ_2 be assertive formulas. Then:
1. $\pi(\delta_1 \cap \delta_2) = J$ iff $\pi(\delta_1) = J$ and $\pi(\delta_2) = J$;
2. $\pi(\delta_1 \cup \delta_2) = J$ iff $\pi(\delta_1) = J$ or $\pi(\delta_2) = J$;
3. $\pi(\delta_1 \supset \delta_2) = J$ iff a proof exists that $\pi(\delta_2) = J$ whenever $\pi(\delta_1) = J$;
4. $\pi(\delta_1 \equiv \delta_2) = J$ iff $\pi(\delta_1 \supset \delta_2) = J$ and $\pi(\delta_2 \supset \delta_1) = J$.

The Soundness criterion (SC) is the following:

Let be $\gamma \in \mathsf{RAD}$, then $\pi(\vdash \gamma) = J$ implies that $\sigma(\gamma) = 1$.

SC states that if an assertion is justified, then the content of assertion is true.

Definition 1. A formula δ is pragmatically valid iff for every Tarskian semantic interpretation σ and for every pragmatic function of justification π, the formula $\delta = J$.

An intuitionistic fragment is obtained by limiting the language of LP to complex formulas that are valid with atomic radicals. This is an intuitionistic fragment ILP, while the classical fragment corresponds to the fragment of sentential formulas without pragmatic connectives [7]. It is worth noting that the justification rules do not always allow determining the justification value of a complex sentential formula when all the justification values of its components are known. For instance,

NR1 $\pi(\delta) = J$ implies $\pi(\sim \delta) = U$,
NR2 $\pi(\delta) = U$ does not imply $\pi(\sim \delta) = J$,
NR3 $\pi(\sim \delta) = J$ implies $\pi(\delta) = U$,
NR4 $\pi(\sim \delta) = U$ does not imply $\pi(\delta) = J$.

There exists a function $(\ )^*$ from assertive formulas to the corresponding modal ones in the modal system S4, where $\Box\gamma$ means that there is an (intuitive) proof (conclusive evidence) for γ:

$(\vdash \gamma)^*$	$\Box\gamma$
$(\sim \delta)^*$	$\Box\neg(\delta)^*$
$(\delta_1 \cap \delta_2)^*$	$(\delta_1)^* \wedge (\delta_2)^*$
$(\delta_1 \cup \delta_2)^*$	$(\delta_1)^* \vee (\delta_2)^*$
$(\delta_1 \supset \delta_2)^*$	$\Box((\delta_1)^* \rightarrow (\delta_2)^*)$
$(\delta_1 \equiv \delta_2)^*$	$\Box((\delta_1)^* \leftrightarrow (\delta_2)^*)$

Connectives for radical and sentential formulas are related in the following way:

(a) $(\vdash \neg\gamma) \supset (\sim\vdash \gamma)$
(b) $((\vdash \gamma_1) \cap (\vdash \gamma_2)) \equiv (\vdash (\gamma_1 \wedge \gamma_2))$
(c) $((\vdash \gamma_1) \cup (\vdash \gamma_2)) \supset (\vdash (\gamma_1 \vee \gamma_2))$
(d) $(\vdash (\gamma_1 \rightarrow \gamma_2)) \supset (\vdash \gamma_1 \supset \vdash \gamma_2)$
(e) $(\vdash (\gamma_1 \leftrightarrow \gamma_2)) \supset (\vdash \gamma_1 \equiv \vdash \gamma_2)$

Bridge principles (a)–(e) show the formal relations between pragmatic connectives and connectives in the radicals. Formula (a) expresses that from the assertion of not-γ the non-assertability of γ can be inferred. (b) states that the conjunction of two assertions is equivalent to the assertion of a conjunction; (c) indicates that from the disjunction of two assertions one can infer the assertion of a disjunction. Formula (d) states the idea that from the assertion of a classical material implication follows the pragmatic implication between two assertions. Finally, (e) states that from the assertion of a biconditional follows the equivalence of assertions.

3. Logic for pragmatics: hypotheses

In this section, we consider hypotheses as a primitive illocutionary force, indicated by $\mathcal{H}$, which is justified by means of a *scintilla of evidence* [4]. What counts as evidence is contextually specified. The language of this hypothetical logic for pragmatics (HLP) is the union of RAD and the set of hypothetical formulas $\mathcal{H}F$.

RAD	$\gamma ::= p \mid \neg\gamma \mid \gamma_1 \wedge \gamma_2 \mid \gamma_1 \vee \gamma_2 \mid \gamma_1 \rightarrow \gamma_2 \mid \gamma_1 \leftrightarrow \gamma_2 \mid$
$\mathcal{H}F$	(i) elementary hypothetical formulas: $\eta ::= \mathcal{H}\gamma$
	(ii) hypothetical formulas:
	$\kappa ::= \eta \mid \frown \kappa \mid \kappa_1 \cap \kappa_2 \mid \kappa_1 \cup \kappa_2 \mid \kappa_1 \sqsupset \kappa_2 \mid \kappa_1 \equiv \kappa_2 \mid$

As in the case of assertions, there are connectives in the radical formulas. Moreover there are hypothetical connectives (which show a certain type of duality with respect to the ones connecting assertions) for hypothetical formulas which formally behave in accordance with the following justification rules. Observe that ε is a function of evidence from hypothetical formulas to justification values.

HJR1 Let γ be a radical formula. $\varepsilon(\mathcal{H}\gamma) = J$ iff there is a *scintilla of evidence* that γ is true, while $\varepsilon(\mathcal{H}\gamma) = U$ iff no scintilla of evidence exists that γ is true.

HJR2 Let κ be a hypothetical formula. Then, $\varepsilon(\frown \kappa) = J$ iff the evidence that $\varepsilon(\kappa) = J$ is smaller than the evidence that $\varepsilon(\frown \kappa) = U$ (that is, iff we are more justified in doubting about k rather than in believing it).

HJR3 Let κ_1 and κ_2 be hypothetical formulas. Then:
- (i) $\varepsilon(\kappa_1 \cap \kappa_2) = J$ iff both $\varepsilon(\kappa_1) = J$ and $\varepsilon(\kappa_2) = J$;
- (ii) $\varepsilon(\kappa_1 \cup \kappa_2) = J$ iff either $\varepsilon(\kappa_1) = J$ or $\varepsilon(\kappa_2) = J$;
- (iii) $\varepsilon(\kappa_1 \sqsupset \kappa_2) = J$ iff there is evidence that κ_2 is justified whenever there is evidence that $\kappa_1$is justified.
- (iv) $\varepsilon(\delta_1 \equiv \delta_2) = J$ iff $\varepsilon(k_1 \sqsupset k_2) = J$ and $\varepsilon(k_2 \sqsupset k_1) = J$.

The soundness criterion for hypotheses is the following one:

> Let be $\gamma \in \mathsf{RAD}$, then $\varepsilon(\mathcal{H}\gamma) = J$ implies that there is a *scintilla of evidence* that $\sigma(\gamma) = 1$.

Let us consider now some notable principles regarding hypothetical negation:

HNR1 $\varepsilon(\kappa) = J$ does not imply that $\varepsilon(\frown \kappa) = U$,
HNR2 $\varepsilon(\kappa) = U$ implies that $\varepsilon(\frown \kappa) = J$,
HNR3 $\varepsilon(\frown \kappa) = J$ implies that $\varepsilon(\kappa) = U$,
HNR4 $\varepsilon(\frown \kappa) = U$ implies that $\varepsilon(\kappa) = J$.

A fuzzy interpretation of the content of hypotheses seems to be quite natural in order to easily handle our pre-theoretical insights concerning them, especially when there exists severe uncertainty[2]. Radical formulas can be also intuitively interpreted in a fuzzy logic, namely a logic whose truth-values, indicated by | |, range in degree between 0 and 1, in the following way:

$$
\begin{aligned}
|\neg\gamma| &= 1 - |\gamma| \\
|\gamma_1 \vee \gamma_2| &= \max(|\gamma_1|, |\gamma_2|) \\
|\gamma_1 \wedge \gamma_2| &= \min(|\gamma_1|, |\gamma_2|) \\
|\gamma_1 \to \gamma_2| &= \begin{cases} 1 & \text{if } |\gamma_1| \leq |\gamma_2| \\ 1 - (|\gamma_1| - |\gamma_2|) & \text{if } |\gamma_1| > |\gamma_2| \end{cases}
\end{aligned}
$$

[2] This might be particularly welcome when we want to deal with notions like the plausibility of hypotheses. Unlike probability, plausibility measures are, for instance, not required to be additive.

$$|\gamma_1 \leftrightarrow \gamma_2| = \begin{cases} 1 & \text{if } |\gamma_1| = |\gamma_2| \\ 1 - (|\gamma_1| - |\gamma_2|) & \text{if } |\gamma_1| \neq |\gamma_2| \end{cases}$$

The following interpretation[3] shows how hypothetical formulas are interpreted in this fuzzy framework.

$\mathcal{H}\gamma_1 = J$	$\|\gamma_1\| \neq 0$
$\mathcal{H}\gamma_1 = U$	$\|\gamma_1\| = 0$
$\frown \mathcal{H}\gamma_1 = J$	$1 - \|\gamma_1\| > \|\gamma_1\|$
$\frown \mathcal{H}\gamma_1 = U$	$\|\gamma_1\| \geq 1 - \|\gamma_1\|$
$(\mathcal{H}\gamma_1 \sqsupset \mathcal{H}\gamma_2) = J$	$\|\gamma_1\| \leq \|\gamma_2\|$
$(\mathcal{H}\gamma_1 \sqsupset \mathcal{H}\gamma_2) = U$	$\|\gamma_1\| > \|\gamma_2\|$
$(\mathcal{H}\gamma_1 \cap \mathcal{H}\gamma_2) = J$	$\|\gamma_1\| \neq 0$ and $\|\gamma_2\| \neq 0$
$(\mathcal{H}\gamma_1 \cap \mathcal{H}\gamma_2) = U$	$\|\gamma_1\| = 0$ or $\|\gamma_2\| = 0$
$(\mathcal{H}\gamma_1 \cup \mathcal{H}\gamma_2) = J$	$\|\gamma_1\| \neq 0$ or $\|\gamma_2\| \neq 0$
$(\mathcal{H}\gamma_1 \cup \mathcal{H}\gamma_2) = U$	$\|\gamma_1\| = 0$ and $\|\gamma_2\| = 0$
$(\mathcal{H}\gamma_1 \equiv \mathcal{H}\gamma_2) = J$	$\|\gamma_1\| = \|\gamma_2\|$
$(\mathcal{H}\gamma_1 \equiv \mathcal{H}\gamma_2) = U$	$\|\gamma_1\| \neq \|\gamma_2\|$

DEFINITION 2 (Pragmatic validity for hypothetical formulas). A formula k is pragmatically valid iff for every $|\cdot|$ and ε, k is justified.

The modal translation in S4 of hypothetical formulas is the following, $(\ )^{**}$ being a function from hypothetical formulas to the corresponding modal ones:[4]

$(\mathcal{H}\gamma)^{**}$	$\Diamond\gamma$
$(\frown \kappa)^{**}$	$\Diamond\neg(\kappa)^{**}$
$(\kappa_1 \cap \kappa_2)^{**}$	$(\kappa_1)^{**} \wedge (\kappa_2)^{**}$
$(\kappa_1 \cup \kappa_2)^{**}$	$(\kappa_1)^{**} \vee (\kappa_2)^{**}$
$(\kappa_1 \sqsupset \kappa_2)^{**}$	$((\kappa_1)^{**} \rightarrow (\kappa_2)^{**})$
$(\kappa_1 \equiv \kappa_2)^{**}$	$((\kappa_1)^{**} \leftrightarrow (\kappa_2)^{**})$

Hypothetical bridge principles between connectives for radical and hypothetical formulas are the following:

(a′) $(\frown \mathcal{H}\gamma) \sqsupset (\mathcal{H}\neg\gamma)$
(b′) $\mathcal{H}(\gamma_1 \wedge \gamma_2) \sqsupset (\mathcal{H}(\gamma_1) \cap \mathcal{H}(\gamma_2))$
(c′) $\mathcal{H}(\gamma_1 \vee \gamma_2) \sqsupset (\mathcal{H}(\gamma_1) \cup \mathcal{H}(\gamma_2))$

[3] Hereafter, when not necessary the indication of the justification function will be omitted.

[4] Similar modal translations have been recently analyzed by Shramko [12, 13].

(d′) $(\mathcal{H}\gamma_1 \sqsupset \mathcal{H}\gamma_2) \sqsupset \mathcal{H}(\gamma_1 \to \gamma_2)$
(e′) $(\mathcal{H}\gamma_1 \equiv \mathcal{H}\gamma_2) \sqsupset \mathcal{H}(\gamma_1 \leftrightarrow \gamma_2)$

General principles connecting assertions and hypotheses are the following:

(GP1): $\vdash \neg\gamma = J$ iff $\mathcal{H}\gamma = U$
and
(GP2): From the justification of $\vdash \gamma$ follows the justification of $\mathcal{H}\gamma$.

(GP1) states that a propositional content cannot be part of a justified hypothesis when the assertion of its negation is justified. (GP2) indicates that the ground justifying an assertion of γ is sufficient to justify the hypothesis of γ. In fuzzy terms, this means that if the $|\gamma| = 1$, then it is certainly different from 0.

4. Justification and truth

In LP there are two main justification conditions that may be interpreted as burdens of proof, based on the different levels of justification for the illocutionary acts of assertion and hypothesis. However, truth-conditions and justification conditions are clearly differentiated, assuming a semantic realist perspective on the nature of truth. On the other hand, the holders of an antirealist epistemology may avoid accepting a transcendent notion of truth whilst favouring a notion of truth that is epistemically constrained. As we will see, this different view on truth may enrich the family of epistemic burdens of proof. At the same time, an epistemically constrained notion of truth might rule out some usual distinctions: between the purely semantic notions of truth and meaning, on the one hand; and, on the other hand, between the epistemic notions of knowledge and justification.

4.1. Modes and modalities

Assertions and hypotheses are two modes of judgment with distinctive commitments. Therefore, the next transition into modal logic may seem anything but consequential. LP relies upon an analogy with modalities. Following the previous translations, assertion corresponds to the strong modality $\Box$ whereas hypothesis corresponds to the weak modality $\Diamond$, all of which nicely match with the interpretations assigned to assertions and hypotheses in LP. Indeed, it has been said that an assertion $\vdash \gamma$ is justified iff there is a conclusive evidence for the truth of γ; and a hypothesis $\mathcal{H}\gamma$ is justified iff there is a *scintilla of evidence* for the truth of γ. Another obvious comparison with quantifiers is feasible, in the sense that any justified assertion is made with sentences that are true in every case, whilst any justified hypothesis is made with sentences that are true in at least one case. However, can anything be said about the burdens of proof associated with the concepts of assertion and hypothesis?

Roughly speaking, the difference between two ways of thinking about epistemic attitudes corresponds to a difference between two ways of handling with philosophical logic, namely: in a modal, or many-valued way. LP coheres with a modal approach towards epistemic attitudes; at the same time, the following logical system supports a many-valued view of evidence including various criteria of justification. However, it is worth mentioning that not every illocutionary act expresses a given degree of force or can be formalized within a modal framework. To shed some light on this pluralist view, we will delve into two levels of discourse in formal epistemology; such a result can be achieved by adopting an antirealist perspective on truth and evidence.

4.2. Some issues about truth

It is taken for granted that any epistemic agent ought to believe what is true. As a matter of fact, the crucial problem lies in the way to warrant the access of these agents to truth, but the classical and tripartite Platonic definition of knowledge does not specifically address this issue. As we have seen, realists and anti-realists have different views on the role of truth and the way in which it may be acknowledged. More specifically, a clear distinction between an *ontic* and an *epistemic* level on truth may contribute to elucidating the dispute.

Starting from an ontic level of discourse (see Section 4.4), agents accept or not the occurrence of given "facts" on the basis of accepted or rejected evidence. In the light of this, we also present an *epistemic* definition of truth-values related to the occurrence of evidence (see Section 4.6).

Definition 3 (Truth and falsity). A given sentence γ is:

- (anti-realistically) true iff there is an evidence for γ;
- (anti-realistically) false iff there is an evidence against γ.

A standard objection to this definition is that the latter gives a too weak view of truth-values: the occurrence of evidence for or against a sentence is clearly not a sufficient condition for its truth or falsity, respectively. After all, some evidence may be misleading and contrary to others. A possible antirealist reply to this objection is that it assumes a *realist* perspective of truth-ascription: no agent can have a direct access to truth without making use of intermediate evidence; correspondingly, any truth-ascription is to be made in the light of evidence made available to agents. In other words, the trouble consists in finding the right justification among a great amount of defeasible evidence. In this respect, the *burden of proof* depicted by LP is an idealization of what any given evidence is supposed to afford in epistemology: to what extent can evidence be "conclusive" in any epistemic context of discourse, and how can the epistemic force of any given evidence facing with opposite data be outweighed? Are there no different strengths of evidence for different agents, thereby requiring a more detailed picture of what a burden of proof may mean? Indeed, the mainstream opposition between classical and intuitionist logicians can be viewed as a typical case of disagreement between two different requirements for truth-ascription. Hereafter, we want to show that such a disagreement may be exemplified in many more ways. But it should be kept in mind that the shift towards an antirealist notion of truth has a price to pay, since justification and semantic conditions wind up being indistinguishable; and for instance, undecidable sentences would turn out to be meaningless even when they seem to express a cognitive meaning. However, in a strongly antirealist framework there is a pluralist view on the norms of acceptability of evidence, which is based on different criteria and norms of justification.

4.3. Question-answer semantics

To give a more comprehensive, pluralist and antirealist formalization of the burdens of proof, let us first consider a logic of information AR_4. It is based on a dialogical formal semantics; that is to say, question-answer semantics, where the logical value of a sentence is a set of ordered answers to previous ordered questions. Questions are about the semantic value of sentences, and they match with a *bilateralist* theory of judgment that departs from Frege's theory of judgment.

DEFINITION 4 (Questions). Questions about γ are formalized by a dyadic function $\mathbf{Q}(\gamma) = \langle \boldsymbol{q}_1(\gamma), \boldsymbol{q}_2(\gamma) \rangle$ such that

- $\boldsymbol{q}_1(\gamma)$: 'Is γ (anti-realistically) true?', i.e., 'Is there evidence for γ?'
- $\boldsymbol{q}_2(\gamma)$: 'Is γ (anti-realistically) false?' i.e., 'Is there evidence against γ?'

DEFINITION 5 (Answers). Answers to questions about γ are combinations of yes- and no-answers $\boldsymbol{A}(\gamma) = \langle \boldsymbol{a}_1(\gamma), \boldsymbol{a}_2(\gamma)\rangle$. Let 1 and 0 be symbols for 'yes' and 'no', respectively. Then:

- $\boldsymbol{A}(\gamma) = \langle 1, 1\rangle$ means that there is evidence for γ and evidence against γ,
- $\boldsymbol{A}(\gamma) = \langle 1, 0\rangle$ means that there is evidence for γ and no evidence against γ,
- $\boldsymbol{A}(\gamma) = \langle 0, 1\rangle$ means that there is no evidence for γ and evidence against γ,
- $\boldsymbol{A}(\gamma) = \langle 0, 0\rangle$ means that there is no evidence for γ and no evidence against γ.

4.4. A logic of information $\boldsymbol{AR_4}$

The system AR_4 is a formalization of what agents may accept or reject, according to ontic norms of truth-preservation that hold for any agent. We take this norm to be a primary burden of truth, assuming that the latter cannot be attained without a preliminary step of justification. $AR_4 = \langle L, f_c, \mathbf{4}\rangle$ is composed of:

- a language L of atomic sentences: p, q, …, and complex sentences γ_1, γ_2, …,
- a set of logical constants $f_c = \{\neg, \wedge, \vee, \rightarrow\}$, i.e., functions mapping on L,
- a set of four logical values $\mathbf{4} = \{11,10,01,00\}$.

Each of the logical constants f_c is characterized by a unique ontic norm about its truth- and falsity-conditions, by means of maximal and minimal answer-functions $\sqcap$ and $\sqcup$ upon their available data. Thus, for any sentence γ interpreted ontically as $\boldsymbol{A}(\gamma) = \langle \boldsymbol{a}_1(\gamma), \boldsymbol{a}_2(\gamma)\rangle$:

- negation: $\boldsymbol{A}(\neg\gamma_1) = \langle \boldsymbol{a}_2(\gamma_1), \boldsymbol{a}_1(\gamma_1)\rangle$
- conjunction: $\boldsymbol{A}(\gamma_1 \wedge \gamma_2) = \langle \boldsymbol{a}_1(\gamma_1) \sqcap \boldsymbol{a}_1(\gamma_2), \boldsymbol{a}_2(\gamma_1) \sqcup \boldsymbol{a}_2(\gamma_2)\rangle$
- disjunction: $\boldsymbol{A}(\gamma_1 \vee \gamma_2) = \langle \boldsymbol{a}_1(\gamma_1) \sqcup \boldsymbol{a}_1(\gamma_2), \boldsymbol{a}_2(\gamma_1) \sqcap \boldsymbol{a}_2(\gamma_2)\rangle$
- implication: $\boldsymbol{A}(\gamma_1 \rightarrow \gamma_2) = \langle \boldsymbol{a}_1(\gamma_1) \sqcap \boldsymbol{a}_1(\gamma_2), \boldsymbol{a}_1(\gamma_1) \sqcap \boldsymbol{a}_2(\gamma_2)\rangle$

The main feature of AR_4 comes from the bilateral definition of strong implication: its (antirealist) truth-conditions are strengthened and correspond to those of conjunction, whilst the falsity-conditions remain the same as those for material implication. This stronger conditional has been studied elsewhere [10], and fulfills the two main inference rules for implication: namely, *modus ponens* and *modus tollens*.

4.5. Consistency

A last and central critical remark to AR_4 is the following: even if one tolerates a perspectival or agent-centered definition of truth-values (cf. Definition 3), there still remains a blatant confusion here above between *information* and *evidence*. The latter is more specific than the former, in the sense that evidence should afford a good reason for agents to make a distinction between data they accept and data they reject. Such a distinction is not made *a priori* in our framework: despite a mainstream view of justification as giving either a sufficient or a conclusive evidence for any given sentence, we explore the variability of this criterion of sufficiency to argue for an extension of the range of burdens of proof. In this respect, the following wants to show that a difference is to be made between several degrees of strength in truth-ascriptions. This leads to an additional distinction between two logical norms of discourse: *consistency* and *coherence*.

DEFINITION 6. [Consistency] A language L is consistent iff for any model M interpreting and any sentence γ true in L: $\boldsymbol{a}_1(\gamma) = 1$ iff $\boldsymbol{a}_2(\gamma) = 0$.

DEFINITION 7. [Coherence] A language L is coherent iff for any model M interpreting and any sentence γ true in L: $\boldsymbol{a}_i(\gamma) = 1$ iff $\boldsymbol{a}_i(\gamma) \neq 0$, for $i = 1, 2$.

Coherence is stronger than consistency: every consistent language is also coherent, whereas the converse need not hold. The language L of AR_4 is coherent and inconsistent.

4.6. A logic of justification $AR_{4\blacksquare}$

The system $AR_{4\blacksquare}$ is a formalization of what agents may accept or reject, according to epistemic norms of truth-preservation. This norm is taken to be a secondary burden of proof, assuming that epistemic agents all agree about the conditions of truth-preservation but may disagree about the justification leading to truth.

DEFINITION 8. [Justification] A sentence γ is (anti-realistically) justified iff the agent is entitled to believe γ.

It is worthwhile to note that the criteria of entitlement are located at the epistemic level of norms, contrary to what is commonly assumed in formal epistemology. This agent-centered approach to justification also departs from the classical evidence-based approach to hypotheses in HLP. According to a strongly antirealist epistemology, it is not only the amount of evidence of an event that justifies its corresponding sentence; rather, the choice is made by agents according to their admitted set of norms of acceptance ruling the relation between 'truth' and 'falsity' claims. Hence, this type of strongly antirealist justification seems adequate when it concerns decisions not requiring to be merely based on an objective view on evidence, but on different standards of acceptability of evidence. These two views serve different purposes. Let us see now what these epistemic norms consist in.

Each of the epistemic attitudes is represented by a unary operator in AR_4. Given that the latter is a four-valued system, there is a total amount of 256 such operators, including these attitudes.

Some of these operators may be used in order to characterize the different ways of dealing with evidence and we analyse a selection of them based on their relevancy in dealing with evidence. Of course, towards almost all the selected epistemic attitudes there is a very robust philosophical discussion. Our operators have the limited purpose of explicating the formal properties of some epistemic attitudes in order to clarify their different burdens of proof.

A basic epistemic attitude is that of rationality, which ought to be followed by any proper epistemic agent. Thus, for any sentence γ to be interpreted epistemically as $A(\blacksquare\gamma) = \langle \boldsymbol{a}_1(\blacksquare\gamma), \boldsymbol{a}_2(\blacksquare\gamma)\rangle$ — about what the agent believes and disbelieves, respectively. The symbol '$\blacksquare$' has to be read as an unspecified operator for epistemic attitudes.

DEFINITION 9. [Rationality] Epistemic agents are *rational* iff they trust the minimal amount of evidence at hand, whether for or against the corresponding sentence γ.

- $\boldsymbol{a}_i(\blacksquare\gamma) \neq 1$, whenever $\boldsymbol{a}_i(\gamma) = 1$ and $\boldsymbol{a}_j(\gamma) = 0$, for $i, j = 1, 2$, where $i \neq j$.

Such a broad definition of rationality helps to encompass a large range of epistemic agents who do not share the same criteria of justification. It may seem queer at first sight, by including an inequality sign. A more intuitive account of rationality would be that an agent believes a sentence rationally whenever there is an argument for it, simpliciter. But this excludes the next characterization of some rational agents, i.e., positivists (Definition 15). A more stringent clause would claim that a rational agent believes a sentence whenever there is an argument for it and no evidence against it. But that also excludes one category of rational agents, viz. eclecticists (Definition 18). A more tolerant version is that rational agents believe a sentence iff there is no evidence against it, simpliciter. But that rules out the categories of negativists (Definition 16) and skeptics (Definition 17).

By contrast, irrationality resorts to agents who do not base their attitude upon the evidence at hand. An analogy can also be made with some unary operators of two-valued logic.

The two extreme operators of tautology and antilogy are operators whose values are the same everywhere: only truth for tautology, only falsehood for antilogy. Similarly, let us define the two irrational attitudes of *relativism* and *nihilism*.

DEFINITION 10. [Relativism ⧄] Epistemic agents are *relativists* iff:

1. they believe that γ, if there is evidence for γ *or* there is no evidence for γ;
2. they disbelieve thatγ (i.e., believe that$\neg\gamma$), if there is evidence against γ *or* there is no evidence against γ.

DEFINITION 11. [Nihilism ⧅] Epistemic agents are *nihilists* iff:

1. they believe that γ if there is evidence for γ *and* there is no evidence for γ;
2. they disbelieve that γ if there is evidence against γ *and* there is no evidence against γ.

Let us consider now the following epistemic attitudes: *credulity* and *incredulity*.

DEFINITION 12. [Credulity ⧆] Epistemic agents are credulous iff:

1. they believe that γ iff there is evidence for γ;
2. they disbelieve that γ iff there is evidence against γ.

DEFINITION 13. [Incredulity ⧈] Epistemic agents are *incredulous* iff:

1. they believe that γ iff there is no evidence for γ;
2. they disbelieve that γ iff there is no evidence against γ.

Credulous agents are rational agents who always trust the evidence at hand, even inconsistently, while incredulous agents are irrational since they always argue against the evidence at hand. Whether rational or irrational, all these agents can be said to be intelligible for their behaviour is always predictable. This is clarified by the following definition:

DEFINITION 14. [Intelligibility] Epistemic agents are intelligible iff their attitude is determined according to the evidence at hand.

- $\boldsymbol{a}_i(\blacksquare\gamma) = f(\boldsymbol{a}_i(\gamma))$, for $i = 1, 2$, where f is any function on $\boldsymbol{a}_i(\blacksquare\gamma)$.

We can compare epistemic attitudes by means of their characteristic matrices, whilst rationality and intelligibility are general attitudes embracing a set of more specific attitudes and therefore do not have any specific characteristic matrix:

γ	$\boxslash\gamma$	$\boxbslash\gamma$	$\boxast\gamma$	$\boxbox\gamma$
11	11	00	11	00
10	11	00	10	01
01	11	00	01	10
00	11	00	00	11

Other more relevant attitudes are to be listed, corresponding to more rational behaviours of epistemic agents.

Hereafter, we propose at least four such attitudes: positivism, negativism, skepticism, and eclecticism. To make sense of the first two attitudes, an analogy with religious beliefs can be made.

On the one hand, believers and atheists embed two opposed attitudes of positivism and negativism: the former favours any evidence for the existence of God, whereas the latter favours any evidence against God. In other words, both favour one sort of evidence against the other one.

DEFINITION 15. [Positivism $\boxbar$] Epistemic agents are *positivists* iff:

1. they believe that γ iff there is evidence for γ *or* no evidence against γ;
2. they disbelieve that γ iff there is no evidence for γ *and* evidence against γ.

DEFINITION 16. [Negativism $\boxminus$] Epistemic agents are *negativists* iff:

1. they believe that γ iff there is evidence for γ *and* no evidence against γ;
2. they disbelieve that γ iff there is no evidence for γ *or* evidence against γ.

On the other hand, skeptics and eclecticists are those agents who do not want to favour positive or negative evidence and take any sort of evidence into account, yet in a divergent way: the skeptic requires complete evidence to believe a sentence, whereas the eclectisist has a weaker criterion for the justification of beliefs.

DEFINITION 17. [Skepticism $\boxtimes$] Epistemic agents are skeptics iff:

1. they believe that γ iff there is evidence for γ *and* no evidence against γ;
2. they disbelieve that γ iff there is no evidence for γ *and* evidence against γ.

DEFINITION 18. [Eclecticism $\boxplus$] Epistemic agents are eclecticists iff:

1. they believe that γ iff there is evidence for γ *or* no evidence against γ;
2. they disbelieve that γ iff there is no evidence for γ *or* evidence against γ.

Here are the characteristic matrices of the last four rational epistemic attitudes in $AR_{4\blacksquare}$.

γ	◫γ	$\boxminus\gamma$	$\boxtimes\gamma$	$\boxplus\gamma$
11	10	01	00	11
10	10	10	10	10
01	01	01	01	01
00	10	01	00	11

These clearly show that both positivists and negativists have a *consistent* behaviour: they always opt for a sentence or its negation, so that the lack of evidence for or against a sentence is taken as evidence against or for it. Then a varying list of theorems may correspond to each of the single epistemic operators. Due to their significance, the aforementioned four operators of positivism, negativism, skepticism and eclecticism have been studied elsewhere. For example, it has been argued in [11] that the asymmetric difference between positivists and negativists echoes with the difference between prosecution and defense in legal epistemology.

5. $AR_{4\blacksquare}$ and LP

In this section we will compare the pragmatic approach to epistemic attitudes with the one presented in $AR_{4\blacksquare}$. So, what is the added value of $AR_{4\blacksquare}$, with respect to LP? LP matches with the modal interpretations of assertion and hypothesis, also paralleling the box and the diamond in modal logic, respectively. AR_4 proposes a more fine-grained taxonomy of epistemic agents, thanks to its many-valued characterization of truth and falsity conditions. Is there any possible translation of one of these systems into the other? This hardly seems possible, following the negative theorem given by Dugundji [8], according to which none of the modal systems S1–S5 can be characterized by a many-valued finite matrix. Furthermore, the occurrence of strong implication[5] in AR_4 should entail some sensible differences in any theorem including this logical constant in $AR_{4\blacksquare}$. Nonetheless, we will show that some methods for making sense of pragmatic formulas in a question-answer semantics are still possible.

The following wants to betray a crucial semantic difference between the modal and many-valued treatments of epistemic attitudes. That is, it is possible to interpret HLP as a system in which there is still a *quantitative* perspective regarding the content of hypotheses due to the fuzzy interpretation of radical formulas, whereas a *qualitative* perspective on justification is associated with the question-answer semantics. Let us see how such a discrepancy can be investigated by a logical perspective (Section 5.1), before proposing another way to fill (at least partially) the gap between modal and many-valued readings of epistemic attitudes (Section 5.2).

5.1. From LP to $AR_\blacksquare$

Let τ be a translation function from LP to $AR_{4\blacksquare}$. Then, it may be used for a comparative analysis of the two modal and many-valued systems. The task consists in translating the logics of assertions and hypotheses into the logic of epistemic agents. We have:[6]

$\tau(\vdash \gamma)$	$\blacksquare\gamma$
$\tau(\sim \delta)$	$\blacksquare\neg(\tau(\delta))$
$\tau(\delta_1 \cap \delta_2)$	$(\tau(\delta_1))\wedge(\tau(\delta_2))$
$\tau(\delta_1 \cup \delta_2)$	$(\tau(\delta_1))\vee((\tau(\delta_2))$

[5] For a study of implication in AR_4, see [10, 11].

[6] Notice, however, that the translation of $(\delta_1 \supset \delta_2)$ is $\blacksquare(\tau(\delta_1) \to \tau(\delta_2))$ and then, by factivity of the operator describing assertion, we get $\tau(\delta_1) \to \tau(\delta_2)$. Moreover, it is easy to understand that $\blacksquare$ cannot be instantiated here with the same epistemic attitude for assertions and hypotheses.

$\tau(\delta_1 \supset \delta_2)$	$(\tau(\delta_1)) \to (\tau(\delta_2))$
$\tau(\delta_1 \equiv \delta_2)$	$(\tau(\delta_1)) \leftrightarrow (\tau(\delta_2))$
$\tau(\mathcal{H}\gamma)$	$\blacksquare\gamma$
$\tau(\frown \kappa)$	$\blacksquare\neg(\tau(\kappa))$
$\tau(\kappa_1 \cap \kappa_2)$	$(\tau(\kappa_1)) \wedge (\tau(\kappa_2))$
$\tau(\kappa_1 \cup \kappa_2)$	$(\tau(\kappa_1)) \vee (\tau(\kappa_2))$
$\tau(\kappa_1 \sqsupset \kappa_2)$	$(\tau(\kappa_1)) \to (\tau(\kappa_2))$
$\tau(\kappa_1 \equiv \kappa_2)$	$(\tau(\kappa_1)) \leftrightarrow (\tau(\kappa_2))$

If so, then which of the epistemic operators of $AR_{4\blacksquare}$ has a behaviour at closest with the acts of assertion and hypothesis? Skeptic agents, on the one hand, are those whose behaviour mostly approximates those of assertive agents in LP[7], which have the highest epistemic standards; on the other hand, eclecticists have the weakest behaviour by accepting the greatest number of possibilities and this approximates the behaviour of hypotheses in HLP.

5.2. Fuzzy questions

Another way to deal with pragmatic hypotheses in question-answer semantics is the following. Instead of the two qualitative questions in AR_4, i.e., as to whether the sentence γ is (anti-realistically) true or false, it is possible to provide a semantics based on a set of four quantitative questions about the fuzzy value $|\cdot|$ of the content of a hypothesis γ in $[0, 1]$. Let us consider the following set of questions:

$\boldsymbol{q}_1(\gamma)$: $|\gamma| = 1$?
$\boldsymbol{q}_2(\gamma)$: $0.5 < |\gamma| < 1$?
$\boldsymbol{q}_3(\gamma)$: $0 < |\gamma| \leq 0.5$?
$\boldsymbol{q}_4(\gamma)$: $|\gamma| = 0$?

The upshot of such a new questioning is a system including 16 single logical values; we call it AR_{16}. Thanks to this fuzzy-based questioning, a more detailed translation of a fragment of LP can be found by means of a translation function $\tau*$ from the following formulas of LP to AR_{16}. As we will show, this fragment is particularly useful for analyzing the opposition relations of some pragmatic formulas (interpreted in a question-answer semantics).

[7] For an antirealist version of LP and for an analysis of other illocutionary acts, see [1, 2, 3].

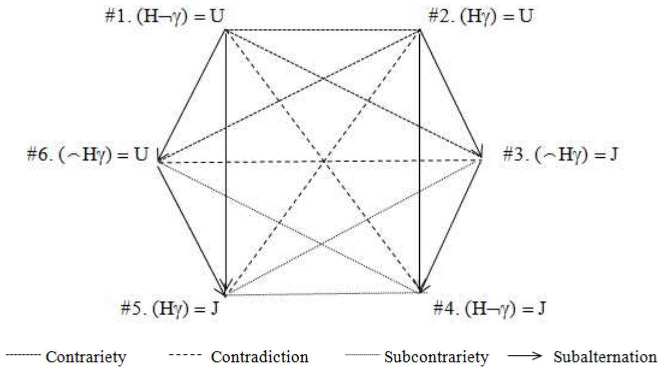

Figure 1. Czeżowski-like hexagon of opposition

$\tau * ((\vdash \gamma) = J)$	$\boldsymbol{A}(\blacksquare\gamma) = 1000$
$\tau * ((\vdash \gamma) = U)$	$\boldsymbol{A}(\blacksquare\gamma) = 0111$
$\tau * ((\vdash \neg\gamma) = J)$	$\boldsymbol{A}(\blacksquare\gamma) = 0001$
$\tau * ((\vdash \neg\gamma) = U)$	$\boldsymbol{A}(\blacksquare\gamma) = 1110$
$\tau * ((\mathcal{H}\gamma) = J)$	$\boldsymbol{A}(\blacksquare\gamma) = 1110$
$\tau * ((\mathcal{H}\gamma) = U)$	$\boldsymbol{A}(\blacksquare\gamma) = 0001$
$\tau * ((\mathcal{H}\neg\gamma) = J)$	$\boldsymbol{A}(\blacksquare\gamma) = 0111$
$\tau * ((\mathcal{H}\neg\gamma) = U)$	$\boldsymbol{A}(\blacksquare\gamma) = 1000$
$\tau * ((\frown \mathcal{H}\gamma) = J)$	$\boldsymbol{A}(\blacksquare\gamma) = 0011$
$\tau * ((\frown \mathcal{H}\gamma) = U)$	$\boldsymbol{A}(\blacksquare\gamma) = 1100$

Note that there is no substantial difference between AR_4 and AR_{16}, with respect to their definitions of logical constants. However, in AR_{16} is thus possible to give a semantics based on quantitative questions. In this way it is also possible to interpret some basic opposition relations, which are particularly relevant in LP given its variety of negations. This task is easily accomplished by considering the Czeżowski-like hexagon of opposition (see Figure 1) relating pragmatic formulas and the question-answer semantics proposed in AR_{16}.

An algebraic proof of the above logical relations can be given in Boolean terms, following some previous results (see [9, 14]). In a nutshell, logical opposition are to be defined by set-theoretical operations of meet and join on bitstrings. It results in the second hexagon of logical opposition (see Figure 2), where the valuations are semantic interpretations of the corresponding pragmatic formulas.

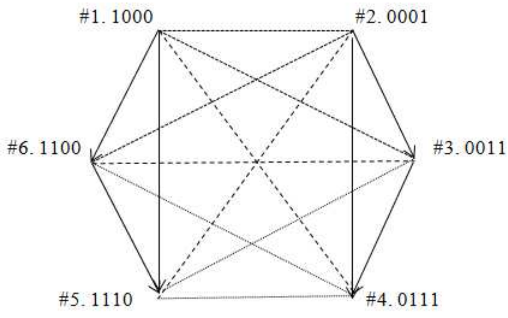

Figure 2. Hexagon of logical opposition

6. Conclusion

The acceptance of hypotheses requires certain epistemic norms that agents are supposed to follow in order to make sound epistemic decisions. Epistemic normativity can be modelled following: (i) a classical view on truth and justification, in which the concept of truth is a semantic notion independent of the epistemic conditions of justification, or (ii) an antirealist view in which the notion of truth is epistemically constrained. Thus, according to (i) there may exist transcendent truth, whereas this is not possible for (ii). First, we have presented a pragmatic logical framework for hypotheses and assertions based on a realistic epistemological ground. Another specific feature of these pragmatic systems is the presence of a variety of negations. Then, we have introduced the systems AR_4 and $AR_{4\blacksquare}$, which are grounded on an antirealist epistemology. An extended version of AR_4, called AR_{16}, has been briefly outlined in order to give a question-answer interpretation of the role of negations in LP, which are relevant for understanding pragmatic opposition relations. We have pointed out that in LP it is possible to express clear distinctions between semantic and justification conditions, even though not many forms of proof burdens can be formalized. On the other hand, AR_4 and $AR_{4\blacksquare}$ may express many burdens of proof regarding the acceptance of hypotheses for agents accepting different norms of epistemic rationality. Finally, we have provided some (partial) translations among the logical systems considered in order to analyze (from a compatibilist perspective) the dialectics between realists and antirealists in logic and logical philosophy.

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References

[1] Bellin, G., "Assertions, hypotheses, conjectures, expectations: Rough-sets semantics and proof-theory", pages 193–241 in *Advances in Natural Deduction: A Celebration of Dag Prawitz's Work*, L. C. Pereira, E. H. Haeusler, V. de Paiva (eds.), "Trends in Logic", vol. 39, Springer, Dordrecht, 2014. DOI: 10.1007/978-94-007-7548-0_10

[2] Bellin, G., M. Carrara, D. Chiffi, D., and A. Menti, "Pragmatic and dialogic interpretations of bi-intuitionism. Part I", *Logic and Logical Philosophy* 23, 4 (2014): 449–480. DOI: 10.12775/LLP.2014.011

[3] Bellin, G., M. Carrara, and D. Chiffi, "On an intuitionistic logic for pragmatics", *Journal of Logic and Computation*, exv036 (2015). DOI: 10.1093/logcom/exv036

[4] Carrara, M., D. Chiffi, and C. De Florio, "Assertions and hypotheses: A logical framework for their opposition relations", *Logic Journal of the IGPL* 25, 2 (2017): 131–144. DOI: 10.1093/jigpal/jzw036

[5] Carrara, M., and D. Chiffi, "The knowability paradox in the light of a logic for pragmatics", pages 47–58 in *Recent Trends in Philosophical Logic*, R. Ciuni, H. Wansing, and C. Willkommen (eds.), "Proceedings of Trends in Logic XI", Studia Logica Library, "*Trends in Logic*", vol. 41, Springer, Berlin, 2014. DOI: 10.1007/978-3-319-06080-4_3

[6] Carrara, M., D. Chiffi, and D. Sergio, "Knowledge and proof: a multimodal pragmatic language", pages 1–13 in *Logica Yearbook 2013*, V. Punčochář and M. Dančák (eds.), College Publication, London, 2014.

[7] Dalla Pozza, C., and C. Garola, "A pragmatic interpretation of intuitionistic propositional logic", *Erkenntnis* 43 (1995): 81–109. DOI: 10.1007/BF01131841

[8] Dugundji, J., "Note on a property of matrices for Lewis and Langford's calculi of propositions", *The Journal of Symbolic Logic* 5, 4 (1940): 150–151. DOI: 10.2307/2268175

[9] Schang, F., “Abstract logic of oppositions”, *Logic and Logical Philosophy* 21 (2012): 415–438. DOI: 10.12775/LLP.2012.019

[10] Schang, F., “A four-valued strong implication”, 2017 (unpublished manuscript).

[11] Schang, F., and A. Costa-Leite, “Une sémantique générale des croyances justifiées”, *CLE* 16, 3 (2016).

[12] Shramko, Y. “Dual intuitionistic logic and a variety of negations: the logic of scientific research”, *Studia Logica* 80, 2–3 (2005): 347–367. DOI: 10.1007/s11225-005-8474-7

[13] Shramko, Y., “A modal translation for dual-intuitionistic logic”, *The Review of Symbolic Logic* 9, 2 (2016): 251–265. DOI: 10.1017/S1755020316000022

[14] Smessaert, H., “On the 3D visualization of the logical relations”, *Logica Universalis* 3 (2009): 212–231. DOI: 10.1007/s11787-009-0010-5

Daniele Chiffi
Tallinn University of Technology
Ragnar Nurkse Department of Innovation and Governance
Akadeemia tee 3
12618 Tallinn, Estonia
chiffidaniele@gmail.com

Fabien Schang
Universidade Estadual de Maringá
Centro de Cięncias Humanas
Letras e Artes
Avenida Colombo
Campus Universitário – CEP 87.020-900 – Maringá – PR, Brazil
schangfabien@gmail.com