The De Morgan negation $\sim a$ is defined as $\neg a^{\smile}$ (that $\neg$ and ${}^{\smile}$ commute is well known in relation algebra). Similar to the syntactic logic case, we make use of the following encodings for $h, k, l \mathrel{\underline{\in}} \mathsf{hlk} \in \mathfrak{C}$:

Formula	De Morgan Encoding	Classical Encoding	Name
$a \xrightarrow{hlk} c$	$\sim(\sim c \overset{khl}{\circ} a)$	$\neg(a^{\smile} \overset{hkl}{\circ} \neg c)$	right residual
$c \xleftarrow{hlk} b$	$\sim(b \overset{lkh}{\circ} \sim c)$	$\neg(\neg c \overset{klh}{\circ} b^{\smile})$	left residual

Similar to the logic, we use $a \in \mathsf{h}$ to denote $a \in D_{\mathsf{h}}$. The tonicity properties of the distributed operators are as follows:

LEMMA 3.1.4. *$\circ$ is monotone on the left and the right, i.e., for $h, k, l \mathrel{\underline{\in}}$ $\mathsf{hlk} \in \mathfrak{C}$,*

$$\frac{b \leq_l \hat{b}, \quad a \in h}{a \overset{hlk}{\circ} b \leq_k a \overset{hlk}{\circ} \hat{b}} \ \circ \text{ Left Monotone}$$

$$\frac{a \leq_h \hat{a}, \quad b \in l}{a \overset{hlk}{\circ} b \leq_k \hat{a} \overset{hlk}{\circ} b} \ \circ \text{ Right Monotone}$$

The tonicities of $\xrightarrow{hlk}$ and $\xleftarrow{hlk}$ are contained in the following derived rules:

$$\frac{a \leq_h \hat{a}, \quad c \in k}{\hat{a} \xrightarrow{hlk} c \leq_l a \xrightarrow{hlk} c} \ \rightarrow \text{ Left Antitone}$$

$$\frac{c \leq_k \hat{c}, \quad a \in h}{a \xrightarrow{hlk} c \leq_l a \xrightarrow{hlk} \hat{c}} \ \rightarrow \text{ Right Monotone}$$

$$\frac{c \leq_k \hat{c}, \quad b \in l}{c \xleftarrow{hlk} b \leq_h \hat{c} \xleftarrow{hlk} b} \ \leftarrow \text{ Left Monotone}$$

$$\frac{b \leq_l \hat{b}, \quad c \in k}{c \xleftarrow{hlk} \hat{b} \leq_h c \xleftarrow{hlk} b} \ \leftarrow \text{ Right Antitone}$$

Following the logic, we can recast residuation as

$$b \leq_l a \xrightarrow{hlk} c \text{ iff } a \overset{hlk}{\circ} b \leq_k c \text{ iff } a \leq_h c \xleftarrow{hlk} b.$$

The algebraic axioms for residuation are equivalent to the following, for $h, k, l \mathrel{\underline{\in}} \mathsf{hlk} \in \mathfrak{C}$:

N8. $a \overset{hlk}{\circ} (a \xrightarrow{hlk} c) \leq_k c$

N8$^{\neg}$. $c \leq_k a \xrightarrow{hkl} (a \overset{hkl}{\circ} c)$

N8$^{\smile}$. $(c \xleftarrow{lhk} a) \overset{lhk}{\circ} a \leq_k c$

N8$^{\smile\neg}$. $c \leq_k (c \overset{khl}{\circ} a) \xleftarrow{khl} a$

The indexing on the cliques has been changed from the logic counterparts of these axioms in order to simplify proofs. Since the h, k, l are variables over the clique hlk, there is no appreciable difference.

THEOREM 3.1.5. *Axioms* M8, M8$^{\smile}$, N8, N8$^{\neg}$, N8$^{\smile}$, *and* N8$^{\smile\neg}$ *are all equivalent.*

The normality of the distributed semigroup operators is used in the representation theorem presented later:

THEOREM 3.1.6. *The distributed semigroup operators are normal: for* $h, l, k \mathrel{\underline{\in}} \mathsf{hlk} \in \mathfrak{C}$, $\overset{hlk}{\circ}$ *distributes over* $\vee$ *from both sides and*

$$a \overset{hlk}{\circ} \bot_l \overset{k}{=} \bot_k = \bot_l \overset{lhk}{\circ} a$$

3.2. Frames

The frames use three-place relations like relevance logic. However, now the relations must be typed. The relations will be morphisms in a multicategory with diagonal relations as the identity morphisms of the category. Following the logic, a type for us will be a node in the underlying graph of a category. We will not use the term "object" but rather the term "node".

DEFINITION 3.2.1. A *local frame* at a node h is a structure $\mathcal{H} = (\mathsf{H}, \mathbb{H}, \mathring{\mathsf{H}}, {}^{\smile})$ such that H is a collection of *points* (also called *worlds*), $\mathring{\mathsf{H}} \subseteq \mathsf{H}$ and $\mathring{\mathsf{H}} \in \mathbb{H}$ where $\mathring{\mathsf{H}}$ is a collection of "zero" worlds, and $\mathbb{H}$ is a subset of the power set of H required to be closed under the Boolean operations and under the operation ${}^{\smile} : \mathsf{H} \to \mathsf{H}$ extended to sets in $\mathbb{H}$ by:

$$C^{\smile} \overset{\text{def}}{=} \{x^{\smile} \mid x \in C\}.$$

It is allowable for $\mathring{\mathsf{H}} = \emptyset$, this occurs if $\mathsf{hhh} \notin \mathfrak{C}$. The distributed relations will be used to interpret the distributed connectives $\circ$ and the defined distributed connectives $\multimap$ and $\multimapinv$.

Each node in a distributed relation logic's distribution structure has a local logic associated with it. Semantically, that local logic must have a local frame associated with it. We use the locution $x \in \mathsf{h}$ to indicate that $x \in \mathsf{H}$ where $\mathcal{H} = (\mathsf{H}, \mathbb{H}, \mathring{\mathsf{H}}, {}^{\smile})$ is the local frame for node h in a graph of nodes.

DEFINITION 3.2.2. Let $\mathcal{H}$, $\mathcal{L}$, and $\mathcal{K}$ be local frames. A *distributed relation* $\mathcal{R}^{\mathsf{hlk}} \colon \mathsf{h} \times \mathsf{l} \to \mathsf{k}$ as a multicategory morphism is a subset $\mathcal{R}^{\mathsf{hlk}} \subseteq \mathsf{H} \times \mathsf{L} \times \mathsf{K}$.

We collect local frames together into a multicategory whose structure is given by a graph $\mathfrak{G}$ and collection of cliques $\mathfrak{C}$:

Definition 3.2.3. A *distributed relation frame*, $\mathcal{DF}$, has a local frame for every node in $\mathfrak{G}$, the underlying graph of the category. The distributed relations are specified by the collection of cliques $\mathfrak{C}$ (see Frame Condition FG3 below). There is a diagonal relation $\mathcal{I}^{\mathsf{hh}}$ for every node h. A distributed relation frame must satisfy the following conditions:

Frame Conditions G

FG1. A collection of nodes $\mathfrak{G}$

FG2. A set $\mathfrak{C}$ of cliques

FG3. A set $\{\mathcal{R}^{\mathsf{hlk}}, \mathcal{R}^{\mathsf{lkh}}, \mathcal{R}^{\mathsf{khl}}, \mathcal{R}^{\mathsf{lhk}}, \mathcal{R}^{\mathsf{hkl}}, \mathcal{R}^{\mathsf{klh}}\}$ of distributed relations for each $\mathsf{hlk} \in \mathfrak{C}$

Frame Conditions A

FA1. A local frame for each node in $\mathfrak{G}$

FA2. $\breve{}\,: \mathsf{h} \to \mathsf{h}$ is a function on H

FA3. $x\breve{}\,\breve{} = x$

Frame Conditions B. For $x \in \mathsf{h}$, $y \in \mathsf{l}$, $u \in \mathsf{k}$, $v \in \mathsf{o}$, and $z \in \mathsf{m}$:

FB1. $\exists z \in \mathsf{m}(\mathcal{R}^{\mathsf{hlm}}xyz$ and $\mathcal{R}^{\mathsf{mko}}zuv)$ iff
$\exists w \in \mathsf{n}(\mathcal{R}^{\mathsf{hno}}xwv$ and $\mathcal{R}^{\mathsf{lkn}}yuw)$ $\quad \mathsf{hlm}, \mathsf{mko}, \mathsf{hno}, \mathsf{lkn} \in \mathfrak{C}$

FB2. For all $z \in \mathring{\mathsf{H}}$, $\mathcal{R}^{\mathsf{hhh}}xzy$ implies $x = y$ $\quad \mathsf{hhh} \in \mathfrak{C}$

FB3. $\mathcal{R}^{hlk}$ is a three-place relation $\quad h, k, l \in \mathsf{hlk} \in \mathfrak{C}$

FB4. $\mathcal{R}^{hlk}xyz$ iff $\mathcal{R}^{lhk}y\breve{}\,x\breve{}\,z\breve{}$ $\quad h, k, l \in \mathsf{hlk} \in \mathfrak{C}$

FB5. $\mathcal{R}^{hlk}xyz$ iff $\mathcal{R}^{hkl}x\breve{}\,zy$ $\quad h, k, l \in \mathsf{hlk} \in \mathfrak{C}$

A defined permutation can be had by combining the effects of FB4 and FB5.

$\mathcal{R}^{hlk}xyz$ iff $\mathcal{R}^{klh}zy\breve{}\,x$

The conditions for a multicategory with the three place relations and the diagonal relation as morphisms are

FC1. Composition is associative:

$$\mathcal{R}^{\mathsf{n_1n_2o}} \cdot \left[(\mathcal{R}^{\mathsf{h_3l_3n_1}} \times \mathcal{R}^{\mathsf{l_3m_3n_2}}) \cdot \left((\mathcal{R}^{\mathsf{h_1h_2h_3}} \times \mathcal{R}^{\mathsf{l_1l_2l_3}}) \times (\mathcal{R}^{\mathsf{k_1k_2k_3}} \times \mathcal{R}^{\mathsf{m_1m_2m_3}})\right)\right] \text{ iff}$$
$$\left[\mathcal{R}^{\mathsf{n_1n_2o}} \cdot (\mathcal{R}^{\mathsf{h_3l_3n_1}} \times \mathcal{R}^{\mathsf{l_3m_3n_2}})\right] \cdot \left((\mathcal{R}^{\mathsf{h_1h_2h_3}} \times \mathcal{R}^{\mathsf{l_1l_2l_3}}) \times (\mathcal{R}^{\mathsf{k_1k_2k_3}} \times \mathcal{R}^{\mathsf{m_1m_2m_3}})\right)$$

FC2. $\mathcal{I}^{\mathsf{hh}}xy$ iff $x = y$

Note that · in FC1 is multicategory composition and not relational composition; the latter would have its arguments reversed.

LEMMA 3.2.4. *The distributed relation frames are multicategories where the non-identity morphisms are three-place relations, and the diagonal relations are the identity morphisms.*

The proof is just to observe that the diagonal relations satisfy the conditions for identity morphisms and composition of the three place relations shown in the following diagram is associative:

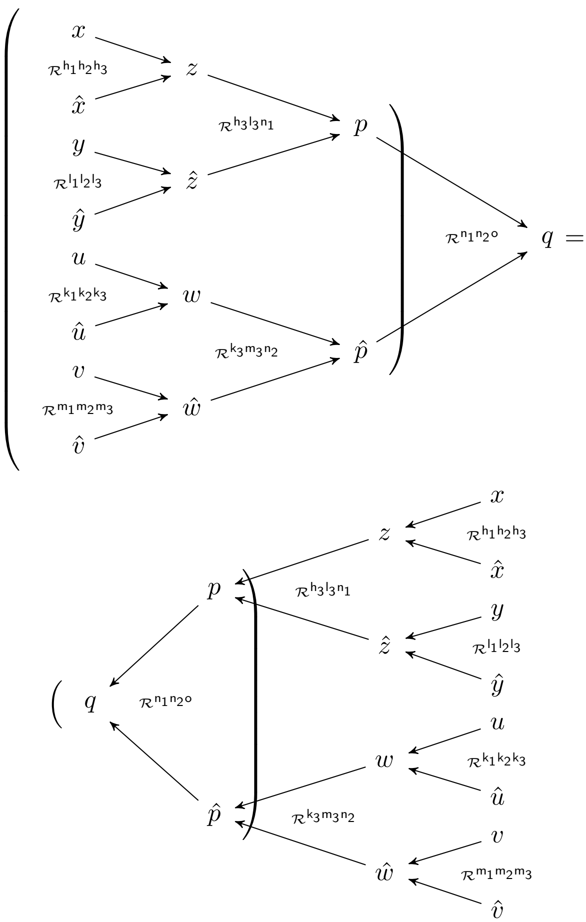


Figure 3. Relation multicategory associativity

3.3. Soundness

Algebraic soundness follows from the Lindenbaum-Tarski heterogeneous algebra (LTA) being free for the class of appropriate algebras for DRel, also termed a DRAlg-free algebra. Note this is not to say that the LTA is a free DRAlg algebra. Free algebras need not be a member of the class of algebras for which they are free. It turns out that the LTA is actually a free DRAlg, but that is fact is used for completeness.

Definition 3.3.1. A DRAlg *appropriate* for a distributed logic DRel is a local converse algebra for each node of the distribution structure, *distributed operators* $\overset{\mathsf{hlk}}{\circ}, \overset{\mathsf{lkh}}{\circ}, \overset{\mathsf{khl}}{\circ}, \overset{\mathsf{lhk}}{\circ}, \overset{\mathsf{hkl}}{\circ}, \overset{\mathsf{klh}}{\circ}$ for each $\mathsf{hlk} \in \mathfrak{C}$ of DRel, and a modal operator $[\iota]$ for every node h. The DRAlg is a multicategory with the morphisms being the distributed and modal operators.

Definition 3.3.2. Let $\mathcal{DA}$ be a DRAlg. An interpretation, $[\![\cdots]\!]$, is a mapping of the propositional variables of DRel into $\mathcal{DA}$ such that for each node h, $[\![T]\!] = \top_{\mathsf{h}}$ and $[\![F]\!] = \bot_{\mathsf{h}}$. If t appears in a local logic at h, then $[\![t]\!] = 1_{\mathsf{h}}$. An interpretation can be extended in the obvious way to preserve the connectives, i.e.,

$$[\![A \vee B]\!] \overset{\text{def}}{=} [\![A]\!] \vee [\![B]\!] \qquad [\![A \wedge B]\!] \overset{\text{def}}{=} [\![A]\!] \wedge [\![B]\!]$$
$$[\![A \supset B]\!] \overset{\text{def}}{=} \neg[\![A]\!] \vee [\![B]\!] \qquad [\![\neg A]\!] \overset{\text{def}}{=} \neg[\![A]\!]$$
$$[\![A \overset{\mathsf{hlk}}{\circ} B]\!] \overset{\text{def}}{=} [\![A]\!] \overset{\mathsf{hlk}}{\circ} [\![B]\!] \qquad [\![A^{\smile}]\!] \overset{\text{def}}{=} [\![A]\!]^{\smile}$$

A DRel sentence A is *true* under an interpretation $[\![\cdots]\!]$ iff $\top_{\mathsf{h}} \leq_{\mathsf{h}} [\![A]\!]$. A sentence is *valid* in $\mathcal{DA}$ iff it is true under all interpretations and it is *DRAlg-valid* iff it is valid in all appropriate DRAlg $\mathcal{DA}$.

A useful feature of Boolean lattices is the following Lemma:

Lemma 3.3.3 (Residuation for Boolean lattices).

$$a \wedge b \leq_{\mathsf{h}} c \text{ iff } a \leq_{\mathsf{h}} b \overset{\mathsf{h}}{\supset} c\,.$$

$A \overset{\mathsf{h}}{\supset} B$ is true in the interpretation $[\![\cdots]\!]$ iff $\top_{\mathsf{h}} \leq_{\mathsf{h}} [\![A \overset{\mathsf{h}}{\supset} B]\!]$ iff $\top_{\mathsf{h}} \leq_{\mathsf{h}} [\![A]\!] \overset{\mathsf{h}}{\supset} [\![B]\!]$. By Lemma 3.3.3, this latter is true iff $\top_{\mathsf{h}} \wedge [\![A]\!] \leq_{\mathsf{h}} [\![B]\!]$ iff $[\![A]\!] \leq_{\mathsf{h}} [\![B]\!]$. Hence $A \overset{\mathsf{h}}{\supset} B$ is true in the interpretation $[\![\cdots]\!]$ iff $[\![A]\!] \leq_{\mathsf{h}} [\![B]\!]$.

From the Replacement Theorem 2.1.2, all the connectives respect bi-equivalence. The LTA is defined in the usual way:

DEFINITION 3.3.4. The elements of the carrier sets are $[A] \overset{\mathsf{h}}{=} \{B \mid A \overset{\mathsf{h}}{\equiv} B\}$ for $A, B \in \mathsf{h}$. The operators are defined inductively: $[A]\bullet[B] = [A\bullet B]$ for $\bullet \in \{\wedge, \vee, \supset\}$, $\neg[A] = [\neg A]$, $[A]^{\smile} = [A^{\smile}]$, and $[A] \overset{\mathsf{hlk}}{\circ} [B] = [A \overset{\mathsf{hlk}}{\circ} B]$ for all $\mathsf{hlk} \in \mathfrak{C}$.

LEMMA 3.3.5. *The* LTA *satisfies all the properties required for an appropriate* DRAlg *algebra under the interpretation* $[\ldots]$.

PROOF SKETCH. All logical axioms easily map to the algebraic axioms under the canonical interpretation $[\cdots]$. The following is an example proving Axiom B5 is true:

1 $[A^{\smile} \overset{hlk}{\circ} \neg(A \overset{hkl}{\circ} B)] \leq_k [\neg B]$ . . . . Identity
2 $[A]^{\smile} \overset{hlk}{\circ} \neg([A] \overset{hkl}{\circ} [B]) \leq_k \neg[B]$ . . . . Def. 3.3.4
3 $[A]^{\smile} \overset{hlk}{\circ} \neg([A] \overset{hkl}{\circ} [B]) \vee \neg[B] \overset{k}{=} \neg[B]$ . . . Lattice properties

The demonstration that each of the rules of inference preserves truth is also routine. The Rule Left $\circ$ Monotonicity is an example. Assume $B \overset{l}{\supset} \hat{B}$, $A \in h$, and $\overset{hlk}{\circ} : h \times l \to k$, then $[B] \leq_l [\hat{B}]$. Therefore $[B] \vee [\hat{B}] \overset{l}{=} [\hat{B}]$. Applying $\overset{hlk}{\circ}$ to both sides yields $[A] \overset{hlk}{\circ} ([B] \vee [\hat{B}]) \overset{k}{=} [A] \overset{hlk}{\circ} [\hat{B}]$. From Theorem 3.1.6, $([A] \overset{hlk}{\circ} [B]) \vee ([A] \overset{hlk}{\circ} [\hat{B}]) \overset{k}{=} [A] \overset{hlk}{\circ} [\hat{B}]$ and from lattice properties, $[A] \overset{hlk}{\circ} [B] \leq_k [A] \overset{hlk}{\circ} [\hat{B}]$.

Theorem 2.3.2 shows that multicategory associativity holds and Axiom C2 provides the category theory identities. □

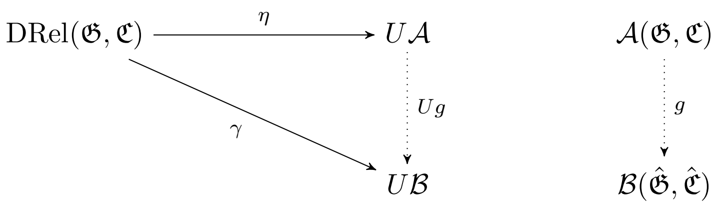
The following lemma allows the transfer of provability of a sentence A in the logic to the condition $\top \leq [\![A]\!]$ for an arbitrary interpretation in a DRAlg.

LEMMA 3.3.6. $\vdash_{\mathsf{h}} A$ *iff* $\vdash_{\mathsf{h}} T \supset A$.

PROOF. For any h in a distribution structure for a DRel, assume $\vdash_{\mathsf{h}} A$. From the axioms, $\vdash_{\mathsf{h}} A \supset (T \supset A)$. From the assumption $\vdash_{\mathsf{h}} A$, and modus ponens, $\vdash_{\mathsf{h}} T \supset A$. For the other half, assume $\vdash_{\mathsf{h}} T \supset A$, then $\vdash_{\mathsf{h}} T$ is provable, and so $\vdash_{\mathsf{h}} A$ follows. □

In the freeness diagram below, the free algebra is $\mathcal{A}$. The algebra $\mathcal{B}$ is some other appropriate distributed algebra, and γ is any interpretation, $U\mathcal{A}$ is the forgetful functor U (from algebras to sets) applied to the algebra $\mathcal{A}$ and returns the carrier sets (types) of $\mathcal{A}$, and similarly for $U\mathcal{B}$. Ug is the underlying set function of the unique homomorphism g

such that the left hand diagram commutes. η injects the atoms of the logic into the proper types of the category $U\mathcal{A}$.



Each node of the distribution structure representing a distinct local logic must be mapped to a distinct local algebra. This informal way of restricting interpretations is the result of treating the hypergraph as not defining everything in a distributed relation logic, but the alternative would make the logic impenetrable. That classes of heterogeneous algebras have free algebras, in our case termed DRAlg-free algebras, is a theorem of [7]. Thus if a formula is provable in DRel, it is DRAlg-valid; this is stated formally as:

Theorem 3.3.7. DRel *is sound with respect to* DRAlg.

The proof is a recognition of the fact that the LTA for a DRel is free for the class of algebras appropriate for the DRel. A provable sentence in the logic is a true expression in the LTA, i.e., $\vdash A$ implies $[\top] \leq [A]$. Any interpretation of the sentence into an algebra in the class factors uniquely into the map from the logic into the LTA and a unique map to the algebra carrying the original interpretation.

Next, we show the soundness for evaluations in DRFrame. Similarly to the algebraic case, each node representing a distinct local logic must be mapped to a distinct frame object in any interpretation. The convention is that the relations that use upper case script letters with appellation hlk, i.e., $\mathcal{R}^{\mathsf{hlk}}$, will interpret $\overset{\mathsf{hlk}}{\circ}$ connectives that use the corresponding types. Each distributed frame interpreting a DRel logic will have conditions matching the axioms.

We assume an initial valuation $[\![\ldots]\!]$ of DRel on all the propositional variables yielding a collection of points in a frame for each variable. The double turnstile $\models$ for the DRel evaluation will mean the set theoretic "element of". Hence $x \overset{\mathsf{h}}{\models} A$ is defined as $x \in_{\mathsf{h}} [\![A]\!]$ where with $[\![\ldots]\!]$, we overload the meaning of these brackets to carry the recursion necessary to extend the interpretation to all nodes h, i.e., $\overset{\mathsf{h}}{\models}$.

DEFINITION 3.3.8. Let $\mathcal{H} = (\mathsf{H}, \mathbb{H}, \mathring{\mathsf{H}}, {}^\smile)$ be a local frame. $\mathcal{H}^* = (\mathbb{H}, \cap, \cup, \neg, \emptyset, \mathsf{H}, {}^\smile)$ is the local algebra of sets (of points) at h where $\mathbb{H}$ is a collection of sets closed under $\cap$, $\cup$, set complement $\neg$, and converse ${}^\smile$, a bottom element $\emptyset$, and a top element H (the set of all points at h). ${}^\smile$ is defined as

$$A^\smile = \{x^\smile \mid x \in A\},$$

LEMMA 3.3.9. *For $\mathcal{H}$ a local DRFrame, $\mathcal{H}^*$ is a local converse algebra of sets.*

PROOF. Converse clearly satisfies $(A \cup B)^\smile \stackrel{\mathsf{h}}{=} A^\smile \cup B^\smile$ and $A^{\smile\smile} \stackrel{\mathsf{h}}{=} A$. The rest of the proof follows from [13, 20]. See also [8, 19, 21]. □

DEFINITION 3.3.10. For $\mathcal{DF}$ a DRFrame multicategory with graph $\mathfrak{G}$ and clique set $\mathfrak{C}$, the algebraic multicategory $\mathcal{DF}^*$ has an LCAlg algebra of sets for every node $\mathsf{h} \in \mathfrak{G}$, and morphisms defined as

$$[\iota]\, A = \{x \mid \mathcal{I}^{\mathsf{hh}} xx\},$$

$$A \overset{\mathsf{hlk}}{\circ} B = \{z \mid \exists x, y (x \in_{\mathsf{h}} A \text{ and } y \in_{\mathsf{l}} B \text{ and } \mathcal{R}^{\mathsf{hlk}} xyz)\}, \quad \mathsf{hlk} \in \mathfrak{C}.$$

THEOREM 3.3.11. *For $\mathcal{DF}$ a DRFrame multicategory with clique set $\mathfrak{C}$, $\mathcal{DF}^*$ is a DRAlg multicategory.*

PROOF. The axioms are sound. The following is an example using M2: $A^\smile \overset{hlk}{\circ} \neg(A \overset{hkl}{\circ} C) \subseteq_k \neg C$.

1 $z \overset{k}{\models} A^\smile \overset{hlk}{\circ} \neg(A \overset{hkl}{\circ} C)$ . . . . Assume
2 $\mathcal{R}^{hlk} xyz$ and $x \overset{h}{\models} A^\smile$ and . . . . . . Def. of $\models$ for some x, y,
$y \overset{l}{\models} \neg(A \overset{hkl}{\circ} C)$ . . . . line 1
3 $y \overset{l}{\not\models} A \overset{hkl}{\circ} C$ . . . . Def. of $\models$, line 2
4 $\neg\mathcal{R}^{hkl} uvy$ or $u \overset{h}{\not\models} A$ or $v \overset{k}{\not\models} C$ . . . Def. of $\models$ for all u, v, line 3
5 $\neg\mathcal{R}^{hlk} u^\smile yv$ or $u \overset{h}{\not\models} A$ or $v \overset{k}{\not\models} C$. Frame Condition FB5, line 4
6 $\neg\mathcal{R}^{hlk} u^\smile yv$ or $u^\smile \overset{h}{\not\models} A^\smile$ or $v \overset{k}{\not\models} C$ . . . . Def. $\models$, FA3, line 5
7 $\mathcal{R}^{hlk} u^\smile yv$ and $u^\smile \overset{h}{\models} A^\smile$ implies $v \overset{k}{\not\models} C$ . . Classical logic, line 6
8 $\mathcal{R}^{hlk} xyz$ and $x \overset{h}{\models} A^\smile$ . . . . Classical logic, line 2
9 $\mathcal{R}^{hlk} x^{\smile\smile} yz$ and $x^{\smile\smile} \overset{h}{\models} A^\smile$ implies $z \overset{k}{\not\models} C$ $x^\smile$ for u, z for v, line 7
10 $\mathcal{R}^{hlk} xyz$ and $x \overset{h}{\models} A^\smile$ implies $z \overset{k}{\not\models} C$ Frame Condition FA3, line 9
11 $z \overset{k}{\not\models} C$ . . . . Modus Ponens, lines 8,10
12 $z \overset{k}{\models} \neg C$ . . . . Def. $\models$, line 11

For $\mathsf{h_1h_2h_3},\ \mathsf{l_1l_2l_3},\ \mathsf{h_3l_3o} \in \mathfrak{C}$, the composition axiom:

C1. $^{\mathsf{h_3\underset{\circ}{l_3}o}}(^{\mathsf{h_1\underset{\circ}{h_2}h_3}} \times {}^{\mathsf{l_1\underset{\circ}{l_2}l_3}})\langle\langle A, \hat{A}\rangle, \langle B, \hat{B}\rangle\rangle \overset{\circ}{\equiv}$
$\big[^{\mathsf{h_3\underset{\circ}{l_3}o}} \cdot (^{\mathsf{h_1\underset{\circ}{h_2}h_3}} \times {}^{\mathsf{l_1\underset{\circ}{l_2}l_3}})\big]\langle\langle A, \hat{A}\rangle, \langle B, \hat{B}\rangle\rangle$

is sound. In relevance logic, there is a useful syntactic device for expressing relational composition:

$$\mathcal{R}^2(u\hat{u})(vz) \text{ iff } \exists x(\mathcal{R}u\hat{u}x \text{ and } \mathcal{R}xvz).$$

We can utilize something similar:

$$\mathcal{R}^2(u\hat{u})(v\hat{v})z\colon \mathsf{h_1h_2h_3} \times \mathsf{l_1l_2l_3} \rightarrow \mathsf{h_3l_3o} \text{ iff}$$
$$\exists x, y(\mathcal{R}^{\mathsf{h_1h_2h_3}}u\hat{u}x \text{ and } \mathcal{R}^{\mathsf{l_1l_2l_3}}v\hat{v}y \text{ and } \mathcal{R}^{\mathsf{h_3l_3o}}xyz).$$

Now we can show the soundness of the axiom over the frames:

1	$z \overset{\circ}{\models} {}^{\mathsf{h_3\underset{\circ}{l_3}o}}(^{\mathsf{h_1\underset{\circ}{h_2}h_3}} \times {}^{\mathsf{l_1\underset{\circ}{l_2}l_3}})\langle\langle A, \hat{A}\rangle, \langle B, \hat{B}\rangle\rangle$	assumption
2	$\mathcal{R}^{\mathsf{h_3l_3o}}xyz$ and $\langle x, y\rangle \models {}^{\mathsf{h_1\underset{\circ}{h_2}h_3}} \times {}^{\mathsf{l_1\underset{\circ}{l_2}l_3}}\langle\langle A, \hat{A}\rangle, \langle B, \hat{B}\rangle\rangle$	for some x, y
3	$\mathcal{R}^{\mathsf{h_1h_2h_3}}u\hat{u}x$ and $u \overset{\mathsf{h_1}}{\models} A$ and $\hat{u} \overset{\mathsf{h_2}}{\models} \hat{A}$ and $\mathcal{R}^{\mathsf{l_1l_2l_3}}v\hat{v}y$ and $v \overset{\mathsf{l_1}}{\models} B$ and $\hat{v} \overset{\mathsf{l_2}}{\models} \hat{B}$	def. ${}^{\mathsf{h_1\underset{\circ}{h_2}h_3}} \times {}^{\mathsf{l_1\underset{\circ}{l_2}l_3}}$, line 2 for some $u, \hat{u}, v, \hat{v}$
4	$\mathcal{R}^{\mathsf{h_3l_3o}}xyz$ and $\mathcal{R}^{\mathsf{h_1h_2h_3}}u\hat{u}x$ and $\mathcal{R}^{\mathsf{l_1l_2l_3}}v\hat{v}y$	classical logic, lines 2,3
5	$\mathcal{R}^2(u\hat{u})(v\hat{v})z$ and $\langle u, \hat{u}\rangle \overset{\mathsf{h_1}\times\mathsf{h_2}}{\models} \langle A, \hat{A}\rangle$ and $\langle v, \hat{v}\rangle \overset{\mathsf{l_1}\times\mathsf{l_2}}{\models} \langle B, \hat{B}\rangle$	classical logic, lines 3,4
6	$z \overset{\circ}{\models} \big[^{\mathsf{h_3\underset{\circ}{l_3}o}} \cdot (^{\mathsf{h_1\underset{\circ}{h_2}h_3}} \times {}^{\mathsf{l_1\underset{\circ}{l_2}l_3}})\big]\langle\langle A, \hat{A}\rangle, \langle B, \hat{B}\rangle\rangle$	def. ${}^{\mathsf{h_3\underset{\circ}{l_3}o}}$, line 5

The proof reads easily from bottom to top and adjusting the justifications on the proof lines. □

Definition 3.3.12. A *distributed relation model* for a distributed relation logic has local frames in a DRFrame multicategory and has a valuation, called a *local model*, for every local frame. A valuation specifies a collection of points in the local frame where the atomic propositions of the appropriate type are true. If the local logic contains t, then the local algebra of sets contains a non-empty $\overset{\circ}{\mathsf{H}}$. The following table then extends the DRel evaluation scheme taken from an interpretation defined on the DRAlg of sets that the DRFrame provides:

Derivation of the Kripke Semantics

$x \models^{\mathsf{h}} A$	iff	$x \in_{\mathsf{h}} [\![A]\!]$ for A a propositional variable		
$x \models^{\mathsf{h}} T$	iff	$x \in_{\mathsf{h}} [\![T]\!]$	iff	$x \in_{\mathsf{h}} \mathsf{H}$
$x \models^{\mathsf{h}} t$	iff	$x \in_{\mathsf{h}} [\![t]\!]$	iff	$x \in_{\mathsf{h}} \mathring{\mathsf{H}}$
$x \models^{\mathsf{h}} \neg A$	iff	$x \in_{\mathsf{h}} [\![\neg A]\!]$	iff	$x \notin_{\mathsf{h}} [\![A]\!]$
			iff	$x \not\models^{\mathsf{h}} A$
$x \models^{\mathsf{h}} A^{\smile}$	iff	$x \in_{\mathsf{h}} [\![A^{\smile}]\!]$	iff	$x \in_{\mathsf{h}} [\![A]\!]^{\smile}$
			iff	$x^{\smile} \models^{\mathsf{h}} A$
$x \models^{\mathsf{h}} A \overset{\mathsf{h}}{\supset} B$	iff	$x \in_{\mathsf{h}} [\![A \overset{\mathsf{h}}{\supset} B]\!]$	iff	$x \in_{\mathsf{h}} \neg[\![A]\!] \cup [\![B]\!]$
			iff	$x \not\models^{\mathsf{h}} A$ or $x \models^{\mathsf{h}} B$
$z \models^{\mathsf{k}} A \overset{\mathsf{hlk}}{\circ} B$	iff	$z \in_{\mathsf{k}} [\![A \overset{\mathsf{hlk}}{\circ} B]\!]$	iff	$z \in_{\mathsf{k}} [\![A]\!] \overset{\mathsf{hlk}}{\circ} [\![B]\!]$
			iff	$\exists x, y(\mathcal{R}^{\mathsf{hlk}}xyz$ and $x \in_{\mathsf{h}} [\![A]\!]$ and $y \in_{\mathsf{l}} [\![B]\!])$
			iff	$\exists x, y(\mathcal{R}^{\mathsf{hlk}}xyz$ and $x \models^{\mathsf{h}} [\![A]\!]$ and $y \models^{\mathsf{l}} [\![B]\!])$

Notice that the following chain of iffs is valid: $\vdash^{\mathsf{h}} A$ iff $\vdash^{\mathsf{h}} T \overset{\mathsf{h}}{\supset} A$ iff $[\![T]\!] \subseteq_{\mathsf{h}} [\![A]\!]$ iff for all $x \in \mathsf{H}$, $x \models^{\mathsf{h}} T$. Now we can state soundness:

Theorem 3.3.13. DRel *is sound with respect to the DRFrame multicategories.*

Proof. All that needs to be shown for soundness with respect to a Kripke semantics, once it is known that DRAlg is sound with respect DRel, is that every DRFrame yields a DRAlg of sets and that the DRel interpretation conditions arise directly from the definitions yielding this DRAlg of sets. □

3.4. Completeness

From Birkhoff [6], a *variety of algebras* is any class of algebras which is closed under homomorphic images, subalgebras, and products. Varieties always have free algebras and these free algebras reside in the very same varietal class. The free algebra can be formed by the usual Lindenbaum-Tarski construction on the logic for which the class provides the algebraic models. That is, the set of well-formed formulas is "divided" by bi-implication yielding a carrier set of equivalence classes, and operations on those classes are defined via the equivalence class representatives. This carries over into heterogeneous algebras [7] or essentially algebraic theories [1]. The Lindenbaum-Tarski algebra LTA for DRel is a member of DRAlg from Lemma 3.3.5.

Theorem 3.4.1. (1) DRel *is complete with respect to DRAlg.* (2) DRel *is complete over the category of DRFrame.*

Proof. (1) $[\ldots]$ is the LTA interpretation. Completeness with respect to DRAlg follows via a contraposition argument: assume $\not\vdash^{\mathsf{h}} A$, then by the definition of $\leq_{\mathsf{h}}$ in the LTA, $\top \not\leq_{\mathsf{h}} [A]$. Therefore, for all DRAlg and for all interpretations, $[\![\ldots]\!]$, $\top \leq_{\mathsf{h}} [\![A]\!]$ implies $\vdash^{\mathsf{h}} A$.

Note that no formula with $\overset{\mathsf{h}}{\supset}$ spans types, i.e., there are no legal formulas of the form $A \overset{\mathsf{h}}{\supset} B$ where A and B in different local logics.

(2) Completeness over DRFrame follows via a contraposition argument using the LTA provided by a DRAlg and the fact that a representation theorem can be shown. The representation map, β, takes a DRAlg into a DRAlg of sets via a DRFrame that is generated directly from the LTA. This representation map is shown to be a 1–1 homomorphism. Then one argues as follows: suppose that $\not\vdash^{\mathsf{h}} C$, then $\top \not\leq_{\mathsf{h}} [C]$ where $[\ldots]$ denotes the equivalence class of C in the LTA. Since the representation map, β, is 1–1 and a homomorphism, then $\beta\top \not\subseteq_{\mathsf{h}} \beta[C]$. By construction, the map $[\ldots]$ composed with β is itself a valuation and hence by the valuation conditions for DRFrame models, there is some world x such that $x \not\models^{\mathsf{h}} C$. Contraposing the argument yields

$$(\text{for all } x \in X,\ x \models^{\mathsf{h}} C) \text{ implies } \vdash^{\mathsf{h}} C.$$

Since every formula can be in only one local logic (at, say, h), this statement is true for all h. □

The work in this section shows that from any DRAlg, a *canonical* frame can be constructed. In the process, a 1-1 representation homomorphism is constructed into the DRAlg of sets derived from the canonical frame. The following theorem is a recap of similar theorems in Jónsson and Tarski [13] and Dunn [11].

Definition 3.4.2. Let $\mathcal{DA}_{\mathsf{h}}$ be a LCAlg at h. The *local canonical frame* at h is $\mathcal{DF}_{\mathsf{h}*} = (\mathsf{H}, \mathbb{H}, \mathring{\mathsf{H}}, {}^{\smile})$ where H is the collection of all proper, maximal filters (as points), $\mathring{\mathsf{H}}$ is defined with

$$z \in_{\mathsf{h}} \mathring{\mathsf{H}} \text{ iff } 1 \in_{\mathsf{h}} z,$$

and

$$x^{\smile} = \{a^{\smile} \mid a \in x\}.$$

$\mathbb{H}$ is the set of all sets of the form

$$\beta_{\mathsf{h}} a = \{x \mid a \in x \text{ and } x \text{ is a maximal filter}\}.$$

THEOREM 3.4.3. *Let $\mathcal{DA}_{\mathsf{h}}$ be a LCAlg at h, the local canonical frame $\mathcal{DF}_{\mathsf{h}*}$ is a local frame.*

PROOF. The Frame Conditions FA hold. From Stone's Representation Theorem [20], $\mathbb{H}$ is known to be a Boolean algebra, i.e., $\mathbb{H}$ is closed under the set operations $\cap, \vee$, and $\neg$. From Axioms M5 and M6 and Lemma 3 of the Appendix, it is clear that $x^{\smile}$ is well-defined and that $x^{\smile\smile} = x$. By unwinding definitions, it is clear that $\beta_{\mathsf{h}} a^{\smile} = (\beta_{\mathsf{h}} a)^{\smile}$, hence $\mathbb{H}$ is closed under $^{\smile}$:

$$\begin{aligned} \beta_{\mathsf{h}} a^{\smile} &\overset{\mathsf{h}}{=} \{x \mid a^{\smile} \in x\} \\ &\overset{\mathsf{h}}{=} \{x^{\smile} \mid a \in x^{\smile}\} \\ &\overset{\mathsf{h}}{=} \{x^{\smile} \mid x^{\smile} \in \beta a\} \\ &\overset{\mathsf{h}}{=} (\beta_{\mathsf{h}} a)^{\smile}. \end{aligned}$$

□

DEFINITION 3.4.4. Let $\mathcal{DA}$ be a DRAlg, then the canonical frame $\mathcal{DA}_*$ has a local canonical frame for every $\mathsf{h} \in \mathfrak{G}$. The relations and hence multicategory morphisms are defined for $h, l, k \Subset \mathsf{hlk} \in \mathfrak{C}$ via

$$\mathcal{I}^{\mathsf{hh}} xy \text{ iff } [\iota]\, a \in x \text{ implies } a \in y,$$

$$\mathcal{R}^{hlk} xyz \text{ iff } a \in x \text{ and } b \in y \text{ implies } a \overset{hlk}{\circ} b \in z.$$

THEOREM 3.4.5. *Let $\mathcal{DA}$ be a DRAlg, then the canonical frame $\mathcal{DA}_*$ is a DRFrame.*

PROOF. The Frame Conditions FG hold. The distribution structure $\mathfrak{C}$ is provided by the DRel for this canonical frame. Hence Frame Conditions FG1 and FG2 are satisfied. The definition 3.4.4 causes FG3 to hold. From Theorem 3.4.3, the Frame Conditions FA hold.

The Frame Conditions FB hold. We show FB1 as an example. Assume left-associativity,

$$(a \overset{\mathsf{hlm}}{\circ} b) \overset{\mathsf{mno}}{\circ} c \leq_{\circ} a \overset{\mathsf{hno}}{\circ} (b \overset{\mathsf{lkn}}{\circ} c)$$

is present in the algebra. The notation $\lambda p.q \circ p$ refers to a function of p with q previously fixed to some value. $(\lambda p.q \circ p)^{-1}_{q \in u}$ refers to the inverse image of this function with values for q taken from maximal filter u. $(v \overset{\mathsf{lkn}}{\circ} y) \uparrow$ is the upper set defined by applying $\overset{\mathsf{lkn}}{\circ}$ componentwise to the elements of v and y.

Assume there is some x such that

$$\mathcal{R}^{\mathsf{hlm}}uvx \text{ and } \mathcal{R}^{\mathsf{mno}}xyz.$$

Then we create an w such that

$$\mathcal{R}^{\mathsf{hno}}uwz \text{ and } \mathcal{R}^{\mathsf{lkn}}vyw.$$

Define the filter U and the ideal V by

$$U = (v \overset{\mathsf{lkn}}{\circ} y) \uparrow \qquad V = (\lambda p.q \overset{\mathsf{hno}}{\circ} p)^{-1}_{q \in u} \overline{z}$$

where $\overline{z}$ is the set complement of the maximal filter z and thus a maximal ideal. Then $U \cap V = \emptyset$: assume the opposite and let $\hat{d} \in U \cap V$. Then there exists some $d \in v$, $d' \in y$, and $q \in u$ such that $d \overset{\mathsf{lkn}}{\circ} d' \leq^{n} \hat{d}$ and $q \overset{\mathsf{hno}}{\circ} \hat{d} \in \overline{z}$. Using the lattice properties, $(d \overset{\mathsf{lkn}}{\circ} d') \vee \hat{d} \overset{\mathsf{n}}{=} \hat{d}$. Distribution of $\overset{\mathsf{hno}}{\circ}$ over $\vee$ yields

$$q \overset{\mathsf{hno}}{\circ} ((d \overset{\mathsf{lkn}}{\circ} d') \vee \hat{d}) \overset{\mathsf{o}}{=} (q \overset{\mathsf{hno}}{\circ} (d \overset{\mathsf{lkn}}{\circ} d')) \vee (q \overset{\mathsf{hno}}{\circ} \hat{d}).$$

Since $q \overset{hno}{\circ} ((d \overset{\mathsf{lkn}}{\circ} d') \vee \hat{d}) \overset{\mathsf{o}}{=} q \overset{\mathsf{hno}}{\circ} \hat{d}$, $q \overset{\mathsf{hno}}{\circ} \hat{d} \in \overline{z}$, and $\overline{z}$ is an ideal, then $q \overset{\mathsf{hno}}{\circ} (d \overset{\mathsf{lkn}}{\circ} d') \in \overline{z}$.

From associativity, $(q \overset{\mathsf{hlm}}{\circ} d) \overset{\mathsf{mno}}{\circ} d' \in \overline{z}$. $\mathcal{R}^{\mathsf{hlm}}uvx$ tells us $q \overset{\mathsf{hlm}}{\circ} d \in x$ but $\mathcal{R}^{\mathsf{mno}}xyz$ tells us that $(q \overset{\mathsf{hlm}}{\circ} d) \overset{\mathsf{mno}}{\circ} d' \in z$ which is a contradiction. So $U \cap V = \emptyset$. This gives us a disjoint filter-ideal pair (U, V) which we extend to a maximal pair (w_1, w_2). By definition, $\mathcal{R}^{\mathsf{hno}}uwz$ and $\mathcal{R}^{\mathsf{lkn}}vyw$. hold.

The Frame Conditions FC hold. Relational composition is associative. $\mathcal{I}^{\mathsf{hh}}xx$ holds iff $[\iota]\, a \in x$ iff $a \in x$ since $[\iota]\, a = a$. Hence the diagonal relations supply the identity morphisms. □

The following theorem (or at least one close to the following) in conjunction with Theorem 3.4.3 is similar to the one found in Jónsson and Tarski [13], Dunn [10], and Routley-Meyer [17]. We follow Dunn [10].

Theorem 3.4.6. *The function* $\beta_{\mathsf{h}} \colon \mathcal{DA}_{\mathsf{h}} \to (\mathcal{DA}_{\mathsf{h}_*})^*$ *defined by*

$$\beta_{\mathsf{h}} a = \{x \mid a \in x \text{ and } x \textit{ is a maximal filter.}\}$$

is a 1–1 LCAlg homomorphism.

Proof. The proof that β_{h} is 1–1 stems from the Stone's representation theorem for Boolean algebras. That β_{h} is a homomorphism is a result of Stone's theorem, and Theorem 3.4.3. □

Theorem 3.4.7. *The function* β*, which is* β_{h} *extended to all of* $\mathcal{DA}$*, is a faithful functor.*

Proof. β_{h} is already known to be a 1-1 LCAlg lattice homomorphism from Theorem 3.4.6. The distribution graph $\mathfrak{C}$ is shared by the domain of β and the codomain of β because β_{h} and β_{k} for $\mathsf{h} \neq \mathsf{k}$ cannot mix maximal filters from the algebras at h and k.

What is left is to show β preserves the $[i]$ modalities and is a distributed semigroup homomorphism. Combined with β being 1-1 a function, this shows that β is a faithful functor. Note that $\mathcal{I}^{\mathsf{hh}}xx$ always holds as a consequence of $[\iota]\, a = a$ for all $a \in \mathsf{h}$. We then have the following chain of iffs: $x \in \beta([\iota]\, a)$ iff $[\iota]\, a \in x$ iff $a \in x$ iff $x \in \beta a$ iff ($\mathcal{I}^{\mathsf{hh}}xx$ implies $x \in \beta a$) iff $x \in [\iota]\, \beta a$. Now we show β is a distributed semigroup homomorphism. We must show

$$\beta(a \overset{\mathsf{hlk}}{\circ} b) = \beta a \overset{\mathsf{hlk}}{\circ} \beta b.$$

The set containment $\beta a \overset{\mathsf{hlk}}{\circ} \beta b \subseteq \beta(a \overset{\mathsf{hlk}}{\circ} b)$ is a direct result of the definitions.

We show $\beta(a \overset{\mathsf{hlk}}{\circ} b) \subseteq \beta a \overset{\mathsf{hlk}}{\circ} \beta b$. Assume $z \in \beta(a \overset{\mathsf{hlk}}{\circ} b)$, an x and y must be generated such that $\mathcal{R}^{\mathsf{hlk}}xyz$ and $a \in x$ and $b \in y$. Consider the principle filters $a{\uparrow} \subseteq \mathsf{H}$ and $b{\uparrow} \subseteq \mathsf{L}$ where H is the set of points at h and L is the set of points at l. It is clear that $a{\uparrow} \overset{\mathsf{hlk}}{\circ} b{\uparrow} \subseteq z$ from the order properties of $\overset{\mathsf{hlk}}{\circ}$ and assuming the $\overset{\mathsf{hlk}}{\circ}$ is applied pointwise to the elements of $a{\uparrow}$ and $b{\uparrow}$.

The filters $a{\uparrow}$ and $b{\uparrow}$ will be expanded to become prime filters, say, x and y, and will be done so that the relation $x \overset{\mathsf{hlk}}{\circ} y \subseteq z$ is preserved. The following set is nonempty:

$$F = \{(u, v) \mid u, v \text{ are filters and } a \in u, b \in v, u \overset{\mathsf{hlk}}{\circ} v \subseteq z\},$$

since it contains $(a{\uparrow}, b{\uparrow})$. The $\overset{\mathsf{hlk}}{\circ}$ operator is normal (Lemma 3.1.6) so that $c \overset{\mathsf{h}}{=} \bot$ or $d \overset{\mathsf{l}}{=} \bot$ implies $c \overset{\mathsf{hlk}}{\circ} d \overset{\mathsf{k}}{=} \bot$ where $\bot$ is the bottom of the lattice at k. This allows that only proper filters need to be considered.

Define a partial order on F by

$$(u, v) \leq_{\mathsf{h}\times\mathsf{l}} (\hat{u}, \hat{v}) \text{ iff } u \subseteq_{\mathsf{h}} \hat{u} \text{ and } v \subseteq_{\mathsf{l}} \hat{v}.$$

Each chain in the $\leq_{\mathsf{h}\times\mathsf{l}}$ order has an upper bound: let E be a chain in F with E_{h} be the chain of filters in the first position of a tuple, and E_{l} the chain of filters in the second position. Now define

$$\bigvee E \overset{\mathsf{h}\times\mathsf{l}}{=} (\bigcup E_{\mathsf{h}}, \bigcup E_{\mathsf{l}}).$$

It must be shown that $\bigvee E \in F$. The union of any chain of filters is clearly a filter, and $a \in \bigcup E_{\mathsf{h}}$ and $b \in \bigcup E_{\mathsf{l}}$ by definition for membership in F. So the item that needs to be checked is that

$$\bigcup E_{\mathsf{h}} \overset{\mathsf{hlk}}{\circ} \bigcup E_{\mathsf{l}} \subseteq z.$$

Suppose that $c \in \bigcup E_{\mathsf{h}}$ and $d \in \bigcup E_{\mathsf{l}}$, then there is some i, j such that $c \in u_i \in E_{\mathsf{h}}$ and $d \in v_j \in E_{\mathsf{l}}$. Without loss of generality, assume $j \leq i$, then $(u_i, v_i) \in F$. Since $(u_i, v_i) \in F$, $u_i \overset{\mathsf{hlk}}{\circ} v_i \subseteq z$, each chain has an upper bound. From Zorn's Lemma, F has a maximal element, call it (x, y).

Now x and y must be shown to be prime. Suppose $c \vee \hat{c} \in x$ (the argument for y is the same). For reductio, suppose $c \notin x$ and $\hat{c} \notin x$. Let $f(x, c)$ and $f(x, \hat{c})$ be the filters generated by x and the elements c and $\hat{c}$. That is, $f(x, c) = \{d \wedge c \mid d \in x\}$. Since $x \subseteq f(x, c), f(x, \hat{c})$, then $(x, y) \subseteq (f(x, c), y), (f(x, \hat{c}), y)$. Each of $(f(x, c), y)$ and $(f(x, \hat{c}), y)$ must fail to be in F since they each contain an element, c or $\hat{c}$, which is not in the maximal element x. This can only occur if there are elements c_1, d_1 and c_2, d_2 such that

$$c_1 \in f(x, c) \text{ and } d_1 \in y \text{ and } c_1 \overset{\mathsf{hlk}}{\circ} d_1 \notin z,$$

and

$$c_2 \in f(x, \hat{c}) \text{ and } d_2 \in y \text{ and } c_2 \overset{\mathsf{hlk}}{\circ} d_2 \notin z.$$

Now let $p = c_1 \vee c_2$ and $q = d_1 \wedge d_2$. It is clear that $p \in f(x, c), f(x, \hat{c})$ since $c_1, c_2 \leq p$ and $f(x, c), f(x, \hat{c})$ are both filters. Also note that $q = d_1 \wedge d_2 \in y$.

Since $(x, y) \in F$, $x \circ y \subseteq z$. This implies that $p \overset{\mathsf{hlk}}{\circ} q \in z$. Using the definition of p,

$$(c_1 \vee c_2) \overset{\mathsf{hlk}}{\circ} q \in z \text{ iff } (c_1 \overset{\mathsf{hlk}}{\circ} q) \vee (c_2 \overset{\mathsf{hlk}}{\circ} q) \in z,$$

and so either $c_1 \overset{\mathsf{hlk}}{\circ} q \in z$ or $c_2 \overset{\mathsf{hlk}}{\circ} q \in z$ since z is a prime filter. Assume the former. Since $q \leq d_1$ and the fact that $\overset{\mathsf{hlk}}{\circ}$ is monotone in each position, then $c_1 \overset{\mathsf{hlk}}{\circ} d_1 \in z$ contradicting $c_1 \overset{\mathsf{hlk}}{\circ} d_1 \notin z$ above. Similarly, $c_2 \overset{\mathsf{hlk}}{\circ} q \in z$ yields a contradiction. The reductio is complete and x is prime. A similar argument shows y is also prime. Prime filters in a Boolean lattice are maximal since for any prime filter x, $c \vee \neg c = \top \in x$ and hence either $c \in x$ or $\neg c \in x$. $\square$

4. Conclusions

The notion of *distribution* has wide applicability as shown in [4] and the current research. Distribution is orthogonal to many constructions in logic in that it does not prevent them. The distribution structure generally is not some abstract, unmotivated structure but rather comes about because the logic is to be applied to some specific domain of discourse. It is for this reason that we designed the distribution structure to be parametric to the logic. For the authors, one of the primary domains is System-on-a-Chip (SoC) architectures. The subcomponents on a chip form a distribution structure. The relations used in a distributed logic over a domain of this kind come from operational behavior and interaction among the subcomponents. Paper length prevents us from going into this here. A relatively complicated SoC will require both modal and two-place intensional connectives in the logic.

Category theory is a good theory of typing. It also is a convenient model of logical morphisms when the logical morphisms are deemed to be intensional connectives in a logic. Modal distributed logic uses a category as opposed to a multicategory. The jump in arity from the modal case seems to indicate that more complicated category theory might be required for more sophisticated logical connectives. One can go the other way around and attempt to discover new logical connections from higher category theory.

Domains of discourse which are naturally distributed should have their distribution structure lifted directly into a logic over those domains. As logicians, we should be interested in making our logics more expressive to increase the utility of those logics. The typing structure also tends to bring about a certain discipline to one's reasoning in the same way that category theory brings about a discipline in reasoning about mathematics, or that type systems in programming languages enforces discipline in reasoning about programs. One cannot merely apply connectives without taking the typing into account.

In the semantics of logic, morphisms are frequently used. As logicians, we should be interested in abstracting these morphisms into the logic and not leave them as meta-logical furniture of the semantics. In modal logic, one can abstract similarity relations into a distributed modal logic [3]. Lifting the morphisms into the logic allows them to be assigned properties with logical axioms and rules. This represents an extension to logic.

Distributed logic is also not restricted to normal connectives; consequently, neighborhoods are employed in the semantics of modal distributed logic. Neighborhood morphisms between logics can be lifted into the logic. The result is very close to Markov transition systems save for the probabilistic aspect. This gives us the possibility that one can analyze a system logically and then by changing the interpretation, treat the morphisms as measurable relations and assigning measurements to logical formulas. This is a very seductive notion for applied logics where the real world is never black and white but admits shades of grey. We have not done so in this paper owing to its length but it opens up new possibilities for future research. We would like to allow the 2-place distributed connectives to be non-normal and thus requiring something like neighborhood maps, as opposed to relations, in the semantics. We would also like a measurement interpretation to widen the utility of distributed logics.

Another area we are beginning to explore is distributed epistemic logics where one can associate a local logic with an agent. The relations between logic are then used to interpret distributed epistemic connectives. This has a ready application in security for System-on-a-Chip architectures. There are several logics for security that are epistemic. When an SoC is under security attack, it is important to know which subcomponents might be compromised. In that situation, it is important to know what a subcomponent can "know" about another. In a more human realm, people can be taken to constitute a distributed system. The distributed connectives describe what one person can know about another, or using higher-arity relations, what information is available to an individual about groupings of individuals.

Appendix

Logic Toolbox

The statements of Lemma 1 easy to prove in DRel.

Lemma 1.

$\vdash T^{\smile} \equiv T$
$\vdash (A \vee B)^{\smile} \equiv A^{\smile} \vee B^{\smile}$
$\vdash F^{\smile} \equiv F$
$\vdash T \supset A^{\smile} \vee (\neg A)^{\smile}$
$\vdash A^{\smile} \wedge (\neg A)^{\smile} \supset F$

Logic Proofs

Most uses of classical logic and theorems 2.1.2 and 2.2.1 in the proofs of this section are omitted, generally only the axioms and rules of the current paper are cited.

LEMMA 2.1.1. *The rule*

$$\frac{A \overset{h}{\supset} \hat{A}, \quad B \in l, \quad h, l, k \Subset \mathsf{hlk} \in \mathfrak{C}}{A \overset{hlk}{\circ} B \supset \hat{A} \overset{hlk}{\circ} B} \text{ Right} \circ \text{Monotonicity}$$

is derivable.

PROOF.

1	$A^{\smile} \overset{h}{\supset} \hat{A}^{\smile}$	Instance of the premise
2	$B^{\smile} \overset{lhk}{\circ} A^{\smile} \overset{k}{\supset} B^{\smile} \overset{lhk}{\circ} \hat{A}^{\smile}$	Rule Left $\circ$ Monotonicity, line 1
3	$(B^{\smile} \overset{lhk}{\circ} A^{\smile})^{\smile} \overset{k}{\supset} (B^{\smile} \overset{lhk}{\circ} \hat{A}^{\smile})^{\smile}$	Rule $^{\smile}$ Monotonicity, line 2
4	$A^{\smile\smile} \overset{hlk}{\circ} B^{\smile\smile} \overset{k}{\supset} \hat{A}^{\smile\smile} \overset{hlk}{\circ} B^{\smile\smile}$	Axiom B4, line 3
5	$A \overset{hlk}{\circ} B \overset{k}{\supset} \hat{A} \overset{hlk}{\circ} B$	Axiom A3, line 4

□

THEOREM 2.2.1. $(\neg A)^{\smile} \equiv \neg(A^{\smile})$.

PROOF. It is a theorem of classical logic that if $\vdash T \supset C \vee D$ and $\vdash C \wedge D \supset F$, then $\vdash C \equiv \neg D$. Using this fact and Lemmas 1 and 1, it follows that $\vdash (\neg A)^{\smile} \equiv \neg(A^{\smile})$. □

THEOREM 2.2.2. *The Axiom B5 and Axiom B5$^{\smile}$ are inter-derivable:*

PROOF.

1	$A^{\smile\smile} \overset{hlk}{\circ} \neg(A^{\smile} \overset{hkl}{\circ} C^{\smile}) \overset{k}{\supset} \neg C^{\smile}$	Instance of Axiom B5
2	$A \overset{hlk}{\circ} \neg(A^{\smile} \overset{hkl}{\circ} C^{\smile}) \overset{k}{\supset} \neg C^{\smile}$	Axiom A3, line 1
3	$(A \overset{hlk}{\circ} \neg(A^{\smile} \overset{hkl}{\circ} C^{\smile}))^{\smile} \overset{k}{\supset} \neg C^{\smile\smile}$	Rule $^{\smile}$ Monotonicity, line 2
4	$(A \overset{hlk}{\circ} \neg(A^{\smile} \overset{hkl}{\circ} C^{\smile}))^{\smile} \overset{k}{\supset} \neg C$	Axiom A3, line 3
5	$\neg(A^{\smile} \overset{hkl}{\circ} C^{\smile})^{\smile} \overset{lhk}{\circ} A^{\smile} \overset{k}{\supset} \neg C$	Axiom B4, line 4
6	$\neg(C^{\smile\smile} \overset{khl}{\circ} A^{\smile\smile}) \overset{lhk}{\circ} A^{\smile} \overset{k}{\supset} \neg C$	Axiom B4, line 5
7	$\neg(C \overset{khl}{\circ} A) \overset{lhk}{\circ} A^{\smile} \overset{k}{\supset} \neg C$	Axiom A3, line 6

The other direction is similar. □

Theorem 2.2.4. *Rule* $\rightarrow$ Residuation *is derivable from Axiom* B5, *and Rule* $\leftarrow$ Residuation *is derivable from Axiom* $\text{B5}^{\smile}$.

Proof.

1	$A \underset{\bigcirc}{hlk} B \overset{k}{\supset} C$	Assume
2	$\neg C \overset{k}{\supset} \neg(A \underset{\bigcirc}{hlk} B)$	$\overset{k}{\supset}$ contraposition, line 1
3	$A^{\smile} \underset{\bigcirc}{hkl} \neg C \overset{l}{\supset} A^{\smile} \underset{\bigcirc}{hkl} \neg(A \underset{\bigcirc}{hlk} B)$	Rule Left $\circ$ Monotonicity, line 2
4	$A^{\smile} \underset{\bigcirc}{hkl} \neg(A \underset{\bigcirc}{hlk} B) \overset{l}{\supset} \neg B$	Axiom B5
5	$A^{\smile} \underset{\bigcirc}{hkl} \neg C \overset{l}{\supset} \neg B$	$\overset{l}{\supset}$ transitivity, lines 3, 4
6	$\neg\neg B \overset{l}{\supset} \neg(A^{\smile} \underset{\bigcirc}{hkl} \neg C)$	$\neg$ contraposition, line 5
7	$B \overset{l}{\supset} \neg(A^{\smile} \underset{\bigcirc}{hkl} \neg C)$	Classical negation, line 6
8	$B \overset{l}{\supset} A \xrightarrow{hlk} C$	Encoding, line 7

It is easy to go back the other way. There are similar proofs showing that Axiom $\text{B5}^{\smile}$ and Rule $\leftarrow$ Residuation are inter-derivable. □

Theorem 2.3.2. *Letting*

$$\vec{X} \overset{\text{def}}{=} \langle\langle A, \hat{A}\rangle, \langle B, \hat{B}\rangle\rangle, \quad \vec{Y} \overset{\text{def}}{=} \langle\langle C, \hat{C}\rangle, \langle D, \hat{D}\rangle\rangle,$$

the following equivalence is provable:

$$\Big[\underset{\bigcirc}{\mathsf{n_1 n_2 o}} \cdot \big[\big(\underset{\bigcirc}{\mathsf{h_3 l_3 n_1}} \times \underset{\bigcirc}{\mathsf{k_3 m_3 n_2}}\big) \cdot \big(\big(\underset{\bigcirc}{\mathsf{h_1 h_2 h_3}} \times \underset{\bigcirc}{\mathsf{l_1 l_2 l_3}}\big) \times \big(\underset{\bigcirc}{\mathsf{k_1 k_2 k_3}} \times \underset{\bigcirc}{\mathsf{m_1 m_2 m_3}}\big)\big)\big]\Big] \langle \vec{X}, \vec{Y}\rangle \overset{\mathsf{o}}{\equiv}$$
$$\Big[\big[\underset{\bigcirc}{\mathsf{n_1 n_2 o}} \cdot \big(\underset{\bigcirc}{\mathsf{h_3 l_3 n_1}} \times \underset{\bigcirc}{\mathsf{k_3 m_3 n_2}}\big)\big] \cdot \big(\big(\underset{\bigcirc}{\mathsf{h_1 h_2 h_3}} \times \underset{\bigcirc}{\mathsf{l_1 l_2 l_3}}\big) \times \big(\underset{\bigcirc}{\mathsf{k_1 k_2 k_3}} \times \underset{\bigcirc}{\mathsf{m_1 m_2 m_3}}\big)\big)\Big] \langle \vec{X}, \vec{Y}\rangle$$

for $\mathsf{h_1 h_2 h_3}$, $\mathsf{l_1 l_2 l_3}$, $\mathsf{k_1 k_2 k_3}$, $\mathsf{m_1 m_2 m_3}$, $\mathsf{h_3 l_3 n_1}$, $\mathsf{k_3 m_3 n_2}$, $\mathsf{n_1 n_2 o}, \in \mathfrak{C}$.

Proof. Let $\circ_1 = \underset{\bigcirc}{\mathsf{n_1 n_2 o}}$, $\circ_2 = \underset{\bigcirc}{\mathsf{h_3 l_3 n_1}}$, $\circ_3 = \underset{\bigcirc}{\mathsf{k_3 m_3 n_2}}$, $\circ_4 = \underset{\bigcirc}{\mathsf{h_1 h_2 h_3}}$, $\circ_6 = \underset{\bigcirc}{\mathsf{l_1 l_2 l_3}}$, $\circ_5 = \underset{\bigcirc}{\mathsf{k_1 k_2 k_3}}$, $\circ_7 = \underset{\bigcirc}{\mathsf{m_1 m_2 m_3}}$.

1	$\circ_3(\circ_5 \times \circ_7)\vec{Y} \overset{\mathsf{n_2}}{\equiv} [\circ_3 \cdot (\circ_5 \times \circ_7)]\vec{Y}$	Axiom C1
2	$\big(\circ_2(\circ_4 \times \circ_6)\vec{X}\big) \circ_1 \big(\circ_3(\circ_5 \times \circ_7)\vec{Y}\big) \overset{\mathsf{o}}{\equiv}$ $\big(\circ_2(\circ_4 \times \circ_6)\vec{X}\big) \circ_1 \big([\circ_3 \cdot (\circ_5 \times \circ_7)]\vec{Y}\big)$	Rule Left $\circ$ Monotonicity, line 1
3	$\circ_2(\circ_4 \times \circ_6)\vec{X} \overset{\mathsf{o}}{\equiv} [\circ_2 \cdot (\circ_4 \times \circ_6)]\vec{X}$	Axiom C1
4	$\big(\circ_2(\circ_4 \times \circ_6)\vec{X}\big) \circ_1 \big([\circ_3 \cdot (\circ_5 \times \circ_7)]\vec{Y}\big) \overset{\mathsf{o}}{\equiv}$ $\big([\circ_2 \cdot (\circ_4 \times \circ_6)]\vec{X}\big) \circ_1 \big([\circ_3 \cdot (\circ_5 \times \circ_7)]\vec{Y}\big)$	Lemma 2.1.1, line 3
5	$\big(\circ_2(\circ_4 \times \circ_6)\vec{X}\big) \circ_1 \big(\circ_3(\circ_5 \times \circ_7)\vec{Y}\big) \overset{\mathsf{o}}{\equiv}$ $\big([\circ_2 \cdot (\circ_4 \times \circ_6)]\vec{X}\big) \circ_1 \big([\circ_3 \cdot (\circ_5 \times \circ_7)]\vec{Y}\big)$	Transitivity of $\overset{\mathsf{o}}{\equiv}$, lines 2,4

6 $\circ_1\langle\circ_2(\circ_4\times\circ_6)\vec{X},\circ_3(\circ_5\times\circ_7)\vec{Y}\rangle \overset{\circ}{\equiv} \Big([\circ_2\cdot(\circ_4\times\circ_6)]\vec{X}\Big)\circ_1\Big([\circ_3\cdot(\circ_5\times\circ_7)]\vec{Y}\Big)$ — Def. 2.3.1: Base Case, line 5

7 $\circ_1(\circ_2\times\circ_3)\langle(\circ_4\times\circ_6)\vec{X},(\circ_5\times\circ_7)\vec{Y}\rangle \overset{\circ}{\equiv} \Big([\circ_2\cdot(\circ_4\times\circ_6)]\vec{X}\Big)\circ_1\Big([\circ_3\cdot(\circ_5\times\circ_7)]\vec{Y}\Big)$ — Def. 2.3.1: App Reorder, line 6

8 $\circ_1(\circ_2\times\circ_3)((\circ_4\times\circ_6)\times(\circ_5\times\circ_7))\langle\vec{X},\vec{Y}\rangle \overset{\circ}{\equiv} \Big([\circ_2\cdot(\circ_4\times\circ_6)]\vec{X}\Big)\circ_1\Big([\circ_3\cdot(\circ_5\times\circ_7)]\vec{Y}\Big)$ — Def. 2.3.1: App Reorder, line 7

9 $\circ_1(\circ_2\times\circ_3)((\circ_4\times\circ_6)\times(\circ_5\times\circ_7))\langle\vec{X},\vec{Y}\rangle \overset{\circ}{\equiv} \circ_1\langle[\circ_2\cdot(\circ_4\times\circ_6)]\vec{X},[\circ_3\cdot(\circ_5\times\circ_7)]\vec{Y}\rangle$ — Def. 2.3.1: Base Case, line 8

10 $\circ_1(\circ_2\times\circ_3)((\circ_4\times\circ_6)\times(\circ_5\times\circ_7))\langle\vec{X},\vec{Y}\rangle \overset{\circ}{\equiv} \circ_1([\circ_2\cdot(\circ_4\times\circ_6)]\times[\circ_3\cdot(\circ_5\times\circ_7)])\langle\vec{X},\vec{Y}\rangle$ — Def. 2.3.1: App Reorder, line 9

11 $\circ_1([\circ_2\cdot(\circ_4\times\circ_6)]\times[\circ_3\cdot(\circ_5\times\circ_7)])\langle\vec{X},\vec{Y}\rangle \overset{\circ}{\equiv} \circ_1(\circ_2\times\circ_3)\cdot((\circ_4\times\circ_6)\times(\circ_5\times\circ_7))\langle\vec{X},\vec{Y}\rangle$ — Def. 2.3.1: Arr Reorder, line 10

12 $\circ_1(\circ_2\times\circ_3)((\circ_4\times\circ_6)\times(\circ_5\times\circ_7))\langle\vec{X},\vec{Y}\rangle \overset{\circ}{\equiv} \circ_1(\circ_2\times\circ_3)\cdot((\circ_4\times\circ_6)\times(\circ_5\times\circ_7))\langle\vec{X},\vec{Y}\rangle$ — $\overset{\circ}{\equiv}$ Transitivity, lines 10, 11

13 $\circ_1(\circ_2\times\circ_3)\cdot((\circ_4\times\circ_6)\times(\circ_5\times\circ_7))\langle\vec{X},\vec{Y}\rangle \overset{\circ}{\equiv} [\circ_1\cdot[(\circ_2\times\circ_3)\cdot((\circ_4\times\circ_6)\times(\circ_5\times\circ_7))]]\langle\vec{X},\vec{Y}\rangle$ — Axiom C1

14 $\circ_1(\circ_2\times\circ_3)((\circ_4\times\circ_6)\times(\circ_5\times\circ_7))\langle\vec{X},\vec{Y}\rangle \overset{\circ}{\equiv} [\circ_1\cdot[(\circ_2\times\circ_3)\cdot((\circ_4\times\circ_6)\times(\circ_5\times\circ_7))]]\langle\vec{X},\vec{Y}\rangle$ — Transitivity of $\overset{\circ}{\equiv}$, lines 12, 13

We also have

1 $\circ_1(\circ_2\times\circ_3)(\langle\circ_4,\circ_6\rangle\times(\circ_5\times\circ_7))\langle\vec{X},\vec{Y}\rangle \overset{\circ}{\equiv} [\circ_1\cdot(\circ_2\times\circ_3)]((\circ_4\times\circ_6)\times(\circ_5\times\circ_7))\langle\vec{X},\vec{Y}\rangle$ — Axiom C1

2 $[\circ_1\cdot(\circ_2\times\circ_3)]((\circ_4\times\circ_6)\times(\circ_5\times\circ_7))\langle\vec{X},\vec{Y}\rangle \overset{\circ}{\equiv} \Big[[\circ_1\cdot(\circ_2\times\circ_3)]\cdot((\circ_4\times\circ_6)\times(\circ_5\times\circ_7))\Big]\langle\vec{X},\vec{Y}\rangle$ — Axiom C1

3 $\circ_1(\circ_2\times\circ_3)((\circ_4\times\circ_6)\times(\circ_5\times\circ_7))\langle\vec{X},\vec{Y}\rangle \overset{\circ}{\equiv} \Big[[\circ_1\cdot(\circ_2\times\circ_3)]\cdot((\circ_4\times\circ_6)\times(\circ_5\times\circ_7))\Big]\langle\vec{X},\vec{Y}\rangle$ — $\overset{\circ}{\equiv}$ transitivity, lines 2, 3

Tying the two proofs together:

$$\Big[\circ_1\cdot[(\circ_2\times\circ_3)\cdot((\circ_4\times\circ_6)\times(\circ_5\times\circ_7))]\Big]\langle\vec{X},\vec{Y}\rangle \overset{\circ}{\equiv}$$
$$\circ_1\,(\circ_2\times\circ_3)((\circ_4\times\circ_6)\times(\circ_5\times\circ_7))\langle\vec{X},\vec{Y}\rangle \overset{\circ}{\equiv}$$
$$\Big[[\circ_1\cdot(\circ_2\times\circ_3)]\cdot((\circ_4\times\circ_6)\times(\circ_5\times\circ_7))\Big]\langle\vec{X},\vec{Y}\rangle \quad \square$$

Theorem 2.4.1. *$\overset{\mathsf{h}}{\models} A \xrightarrow{\mathsf{hhh}} \hat{A}$ implies $\overset{\mathsf{h}}{\models} A \overset{\mathsf{h}}{\supset} \hat{A}$.*

Proof.

1	$\overset{\mathsf{h}}{\models} A \xrightarrow{\mathsf{hhh}} \hat{A}$	Assume
2	$\overset{\mathsf{h}}{\models} T \overset{\mathsf{h}}{\supset} (A \xrightarrow{\mathsf{hhh}} \hat{A})$	Classical logic, line 1
3	$\overset{\mathsf{h}}{\models} t \overset{\mathsf{h}}{\supset} T$	For any A, $A \overset{\mathsf{h}}{\supset} T$
4	$\overset{\mathsf{h}}{\models} t \overset{\mathsf{h}}{\supset} (A \xrightarrow{\mathsf{hhh}} \hat{A})$	Transitivity of $\overset{\mathsf{h}}{\supset}$, lines 2, 3
5	$\overset{\mathsf{h}}{\models} A \overset{\mathsf{hhh}}{\circ} t \overset{\mathsf{h}}{\supset} \hat{A}$	Residuation, line 4
6	$\overset{\mathsf{h}}{\models} A \overset{\mathsf{hhh}}{\circ} t \overset{\mathsf{h}}{\equiv} A$	Axiom B2
7	$\overset{\mathsf{h}}{\models} A \overset{\mathsf{h}}{\supset} \hat{A}$	Replacement, lines 5, 6

□

Algebraic Proofs

Algebraic Toolbox

The items from Lemma 2 through Lemma 6 are used in the succeeding Section 4.

Lemma 2. *From Boolean algebras it is known that $a \vee b = \top$ and $a \wedge b = \bot$ implies $a = \neg b$.*

Lemma 3. *From relation algebras it is known that*

$$\top^{\smile} = \top$$
$$\bot^{\smile} = \bot$$
$$a \leq b \text{ implies } a^{\smile} \leq b^{\smile}$$
$$(a \wedge b)^{\smile} = a^{\smile} \wedge b^{\smile}$$
$$(\neg a)^{\smile} = \neg(a^{\smile})$$

Corollary 4. *$\sim$ is a period 2 operator.*

The proof is easy applications of classical negation and the lemma.

Lemma 5. *If $\mathsf{hhh} \in \mathfrak{C}$, then $1_{\mathsf{h}}^{\smile} \overset{\mathsf{h}}{=} 1_{\mathsf{h}}$.*

Proof.

$1_{\mathsf{h}}^{\smile} \overset{\mathsf{hhh}}{\circ} 1_{\mathsf{h}} \overset{\mathsf{h}}{=} 1_{\mathsf{h}}^{\smile}$	Axiom M4
$(1_{\mathsf{h}}^{\smile} \overset{\mathsf{hhh}}{\circ} 1_{\mathsf{h}})^{\smile} \overset{\mathsf{h}}{=} 1_{\mathsf{h}}^{\smile\smile}$	Apply $^{\smile}$
$1_{\mathsf{h}}^{\smile} \overset{\mathsf{hhh}}{\circ} 1_{\mathsf{h}}^{\smile\smile} \overset{\mathsf{h}}{=} 1_{\mathsf{h}}$	Axiom M7

$$1_{\mathsf{h}}^{\smile} \overset{\mathsf{hhh}}{\circ} 1_{\mathsf{h}} \overset{\mathsf{h}}{=} 1_{\mathsf{h}} \qquad \text{Axiom M5}$$
$$1_{\mathsf{h}}^{\smile} \overset{\mathsf{h}}{=} 1_{\mathsf{h}} \qquad \text{Axiom M4} \qquad \square$$

LEMMA 6. *If* $\mathsf{hhh} \in \mathfrak{C}$, *then* 1_{h} *is a two-sided identity.*

PROOF.

$$(1_{\mathsf{h}} \overset{\mathsf{hhh}}{\circ} a)^{\smile} \overset{\mathsf{h}}{=} (1_{\mathsf{h}} \overset{\mathsf{hhh}}{\circ} a)^{\smile} \qquad \text{Equality}$$
$$(1_{\mathsf{h}} \overset{\mathsf{hhh}}{\circ} a)^{\smile} \overset{\mathsf{h}}{=} a^{\smile} \overset{\mathsf{hhh}}{\circ} 1_{\mathsf{h}}^{\smile} \qquad \text{Axiom M7}$$
$$(1_{\mathsf{h}} \overset{\mathsf{hhh}}{\circ} a)^{\smile} \overset{\mathsf{h}}{=} a^{\smile} \overset{\mathsf{hhh}}{\circ} 1_{\mathsf{h}} \qquad \text{Lemma 5}$$
$$(1_{\mathsf{h}} \overset{\mathsf{hhh}}{\circ} a)^{\smile} \overset{\mathsf{h}}{=} a^{\smile} \qquad \text{Axiom M4}$$
$$(1_{\mathsf{h}} \overset{\mathsf{hhh}}{\circ} a)^{\smile\smile} \overset{\mathsf{h}}{=} a^{\smile\smile} \qquad \text{Apply } ^{\smile}$$
$$1_{\mathsf{h}} \overset{\mathsf{hhh}}{\circ} a \overset{\mathsf{h}}{=} a \qquad \text{Axiom M5}$$

The other side is similar but uses this proof in place of Axiom M4. □

Proofs for the Algebras and Frames (Section 3)

Note that the numbers on the Lemmas and Theorems match the numbers from the Algebras and Frames Section 3.

LEMMA 3.1.4.

$$\frac{b \leq_l \hat{b}, \quad a \in h}{a \overset{hlk}{\circ} b \leq_k a \overset{hlk}{\circ} \hat{b}} \; \circ \text{ Left Monotone}$$

$$\frac{a \leq_h \hat{a}, \quad b \in l}{a \overset{hlk}{\circ} b \leq_k \hat{a} \overset{hlk}{\circ} b} \; \circ \text{ Right Monotone}$$

The tonicity of $\xrightarrow{hlk}$ *and* $\xleftarrow{hlk}$ *is contained in the following derived rules:*

$$\frac{a \leq_h \hat{a}, \quad c \in k}{\hat{a} \xrightarrow{hlk} c \leq_l a \xrightarrow{hlk} c} \; \rightarrow \text{ Left Antitone}$$

$$\frac{c \leq_k \hat{c}, \quad a \in h}{a \xrightarrow{hlk} c \leq_l a \xrightarrow{hlk} \hat{c}} \; \rightarrow \text{ Right Monotone}$$

$$\frac{c \leq_k \hat{c}, \quad b \in l}{c \xleftarrow{hlk} b \leq_h \hat{c} \xleftarrow{hlk} b} \; \leftarrow \text{ Left Monotone}$$

$$\frac{b \leq_l \hat{b}, \quad c \in k}{c \xleftarrow{hlk} \hat{b} \leq_h c \xleftarrow{hlk} b} \; \leftarrow \text{ Right Antitone}$$

Proof.

1 $b \leq_l \hat{b}$ and $a \in h$ . . . Assume
2 $b \vee \hat{b} \overset{l}{=} \hat{b}$ . . . Lattice properties
3 $a \overset{hlk}{\circ} (b \vee \hat{b}) \overset{k}{=} a \overset{hlk}{\circ} \hat{b}$ . . . apply $\overset{hlk}{\circ}$, line 2
4 $(a \overset{hlk}{\circ} (b \vee \hat{b}))^{\smile} \overset{k}{=} (a \overset{hlk}{\circ} \hat{b})^{\smile}$ . . . Apply $^{\smile}$, line 3
5 $(b \vee \hat{b})^{\smile} \overset{lhk}{\circ} a^{\smile} \overset{k}{=} (a \overset{hlk}{\circ} \hat{b})^{\smile}$ . . . Axiom M7, line 4
6 $(b^{\smile} \vee \hat{b}^{\smile}) \overset{lhk}{\circ} a^{\smile} \overset{k}{=} (a \overset{hlk}{\circ} \hat{b})^{\smile}$ . . . Axiom M6, line 5
7 $(b^{\smile} \overset{lhk}{\circ} a^{\smile}) \vee (\hat{b}^{\smile} \overset{lhk}{\circ} a^{\smile}) \overset{k}{=} (a \overset{hlk}{\circ} \hat{b})^{\smile}$ . . . Axiom M3, line 6
8 $((b^{\smile} \overset{lhk}{\circ} a^{\smile}) \vee (\hat{b}^{\smile} \overset{lhk}{\circ} a^{\smile}))^{\smile} \overset{k}{=} (a \overset{hlk}{\circ} \hat{b})^{\smile\smile}$ . . . Apply $^{\smile}$, line 7
9 $(b^{\smile} \overset{lhk}{\circ} a^{\smile})^{\smile} \vee (\hat{b}^{\smile} \overset{lhk}{\circ} a^{\smile})^{\smile} \overset{k}{=} (a \overset{hlk}{\circ} \hat{b})^{\smile\smile}$ . . . Axiom M6, line 8
10 $(b^{\smile} \overset{lhk}{\circ} a^{\smile})^{\smile} \vee (\hat{b}^{\smile} \overset{lhk}{\circ} a^{\smile})^{\smile} \overset{k}{=} a \overset{hlk}{\circ} \hat{b}$ . . . Axiom M5, line 9
11 $(a^{\smile\smile} \overset{hlk}{\circ} b^{\smile\smile}) \vee (a^{\smile\smile} \overset{hlk}{\circ} \hat{b}^{\smile\smile}) \overset{k}{=} a \overset{hlk}{\circ} \hat{b}$ . . . Axiom M7, line 10
12 $(a \overset{hlk}{\circ} b) \vee (a \overset{hlk}{\circ} \hat{b}) \overset{k}{=} a \overset{hlk}{\circ} \hat{b}$ . . . Axiom M5, line 11
13 $a \overset{hlk}{\circ} b \leq_k a \overset{hlk}{\circ} \hat{b}$ . . . Lattice properties, line 12

and

1 $a \leq_h \hat{a}$ and $b \in l$ . . . Assume
2 $a \vee \hat{a} \overset{h}{=} \hat{a}$ . . . Lattice properties
3 $(a \vee \hat{a}) \overset{hlk}{\circ} b \overset{k}{=} \hat{a} \overset{hlk}{\circ} b$ . . . Apply $\overset{hlk}{\circ}$, line 2
4 $(a \overset{hlk}{\circ} b) \vee (\hat{a} \overset{hlk}{\circ} b) \overset{k}{=} \hat{a} \overset{hlk}{\circ} b$ . . . Axiom M6, line 3
5 $a \overset{hlk}{\circ} b \leq_k \hat{a} \overset{hlk}{\circ} b$ . . . Lattice properties, line 4

and

1 $a^{\smile} \leq_h \hat{a}^{\smile}$ and $c \in k$ . . . Assume
2 $a^{\smile} \overset{hkl}{\circ} \neg c \leq_l \hat{a}^{\smile} \overset{hkl}{\circ} \neg c$ . . . proof $\circ$ Right Monotone (above), line 1
3 $\neg(\hat{a}^{\smile} \overset{hkl}{\circ} \neg c) \leq_l \neg(a^{\smile} \overset{hkl}{\circ} \neg c)$ . . . Boolean negation, line 2
4 $\hat{a} \xrightarrow{hlk} c \leq_l a \xrightarrow{hlk} c$ . . . Encoding, line 3

and

1 $c \leq_k \hat{c}$ and $a \in h$ . . . Assume
2 $\neg\hat{c} \leq_k \neg c$ . . . Boolean negation, line 1
3 $a^{\smile} \overset{hkl}{\circ} \neg\hat{c} \leq_l a^{\smile} \overset{hkl}{\circ} \neg c$ . . . proof $\circ$ Left Monotone (above), , line 2
4 $\neg(a^{\smile} \overset{hkl}{\circ} \neg c) \leq_l \neg(a^{\smile} \overset{hkl}{\circ} \neg\hat{c})$ . . . Boolean negation, line 3
5 $a \xrightarrow{hlk} c \overset{l}{=} a \xrightarrow{hlk} \hat{c}$ . . . Encoding, line 4

The proofs involving $\leftarrow$ are similar. □

LEMMA 3.1.3. *Axioms M8 and* $M8^{\smile}$ *are equivalent.*

PROOF.

1 $a^{\smile\smile} \overset{hlk}{\circ} \neg(a^{\smile} \overset{hkl}{\circ} c^{\smile}) \leq_k \neg c^{\smile}$ . . . . Instance of Axiom M8
2 $a \overset{hlk}{\circ} \neg(a^{\smile} \overset{hkl}{\circ} c^{\smile}) \leq_k \neg c^{\smile}$ . . . . Axiom M5, line 1
3 $(a \overset{hlk}{\circ} \neg(a^{\smile} \overset{hkl}{\circ} c^{\smile}))^{\smile} \leq_k \neg c^{\smile\smile}$ . . . . Lemma 3, line 2
4 $(a \overset{hlk}{\circ} \neg(a^{\smile} \overset{hkl}{\circ} c^{\smile}))^{\smile} \leq_k \neg c$ . . . . Axiom M5, line 3
5 $\neg(a^{\smile} \overset{hkl}{\circ} c^{\smile})^{\smile} \overset{lhk}{\circ} a^{\smile} \leq_k \neg c$ . . . . Axiom M7, line 4
6 $\neg(c^{\smile\smile} \overset{khl}{\circ} a^{\smile\smile}) \overset{lhk}{\circ} a^{\smile} \leq_k \neg c$ . . . . Axiom M7, line 5
7 $\neg(c \overset{khl}{\circ} a) \overset{lhk}{\circ} a^{\smile} \leq_k \neg c$ . . . . Axiom M5, line 6

That Axiom $M8^{\smile}$ implies Axiom M8 is similar. □

THEOREM 3.1.5. *Axioms M8,* $M8^{\smile}$*, N8,* $N8^{\neg}$*,* $N8^{\smile}$*, and* $N8^{\smile\neg}$ *are all equivalent.*

PROOF. Axioms M8 and N8 are equivalent.

1 $a^{\smile\smile} \overset{hlk}{\circ} \neg(a^{\smile} \overset{hkl}{\circ} \neg c) \leq_k \neg\neg c$ . . . . Instance of Axiom M8
2 $a \overset{hlk}{\circ} \neg(a^{\smile} \overset{hkl}{\circ} \neg c) \leq_k \neg\neg c$ . . . . Axiom M5, line 1
3 $a \overset{hlk}{\circ} \neg(a^{\smile} \overset{hkl}{\circ} \neg c) \leq_k c$ . . . . Boolean negation, line 2
4 $a \overset{hlk}{\circ} (a \xrightarrow{hlk} c) \leq_k c$ . . . . Encoding, line 3

and

1 $a^{\smile} \overset{hlk}{\circ} (a^{\smile} \xrightarrow{hlk} \neg c) \leq_k \neg c$ . . . . Instance of Axiom N8
2 $a^{\smile} \overset{hlk}{\circ} \neg(a^{\smile\smile} \overset{hkl}{\circ} \neg\neg c) \leq_k \neg c$ . . . . Encoding, line 1
3 $a^{\smile} \overset{hlk}{\circ} \neg(a^{\smile\smile} \overset{hkl}{\circ} c) \leq_k \neg c$ . . . . Boolean negation, line 2
4 $a^{\smile} \overset{hlk}{\circ} \neg(a \overset{hkl}{\circ} c) \leq_k \neg c$ . . . . Axiom M5, line 3

The other proofs are similar. □

THEOREM 3.1.6. *The distributed semigroup operators are normal: for* $h, l, k \in \mathsf{hlk} \in \mathfrak{C}$*,* $\overset{hlk}{\circ}$ *distributes over* $\vee$ *from both sides and*

$$a \overset{hlk}{\circ} \perp_l \overset{k}{=} \perp_k = \perp_l \overset{lhk}{\circ} a.$$

PROOF. Distribution of $\circ$ over $\vee$ from the right is Axiom M3, from the left is easily proven given the rest of the axioms.

Since $\perp_l \leq_l a \xrightarrow{hlk} \perp_k$ ($\perp_l \leq_l b$ for all $b \in l$), $a \overset{hlk}{\circ} \perp_l \leq_k \perp_k$. So $a \overset{hlk}{\circ} \perp_l \overset{k}{=} \perp_k$. Similarly, $\perp_l \leq_l \perp_k \xleftarrow{lhk} a$, so $\perp_l \overset{lhk}{\circ} a \leq_k \perp_k$ and $\perp_l \overset{lhk}{\circ} a \overset{k}{=} \perp_k$. □

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