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PARTIAL AND PARACONSISTENT THREE-VALUED LOGICS

Abstract. On the sidelines of classical logic, many partial and paraconsistent three-valued logics have been developed. Most of them differ in the notion of logical consequence or in the definition of logical connectives. This article aims, firstly, to provide both a model-theoretic and a proof-theoretic unified framework for these logics and, secondly, to apply these general frameworks to several well-known three-valued logics. The proof-theoretic approach to which we give preference is sequent calculus. In this perspective, several results concerning the properties of functional completeness, cut redundancy, and proof-search procedure are shown. We also provide a general proof for the soundness and the completeness of the three sequent calculi discussed.

Keywords: four-valued logic; three-valued logic; partial logic; paraconsistent logic; sequent calculus; functional completeness; cut redundancy; proof-search procedure

1. Introduction

Three semantic assumptions underlie this discussion: only truth and falsehood exist and they are *a priori* neither exhaustive nor exclusive. This means that four semantic states are possible for a single sentence: ‘true and not false’, ‘false and not true’, ‘neither true nor false’, and ‘both true and false’. A logic is called partial if it does not reject the possibility that a sentence is neither true nor false, and it is called paraconsistent if it does not reject the possibility that a sentence is both true and false.

Yet the fact remains that on the sidelines of classical logic many three-valued logics (which are closely related to these assumptions) have been

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developed. Most of them differ in the notion of logical consequence or in the definition of logical connectives. Indeed, from a semantic point of view, a plurality of three-valued logics can be distinguished by changing the designated truth-values or the truth-functions associated with the logical connectives.

Starting with the aforementioned threefold assumption, this article aims, firstly, to provide both a model-theoretic and a proof-theoretic unified framework for these logics and, secondly, to apply these general frameworks to several well-known three-valued logics. The proof-theoretic approach to which we give preference is sequent calculus. Insofar as it consists in developing a unified sequent calculus for the investigation of three-valued logics (dealing sometimes with truth-value gaps, sometimes with truth-value gluts), this article is faced with two issues: the plurality of interpretations of logical consequence and the plurality of interpretations of logical connectives.

Based on the distinction between the surface structure and the deep structure in the context of transformational grammar, we distinguish between the surface language and semantics on the one hand, and the deep language and semantics on the other. *The central idea of transformational grammar is that they [deep and surface structures] are, in general, distinct and that the surface structure is determined by repeated application of certain formal operations called "grammatical transformations" to objects of a more elementary sort* (see [9]). For our purposes, we borrow from theoretical linguistics this philosophical idea and implement it in the context of three-valued logics. While the surface language is a propositional language whose logical symbols can be interpreted in various ways depending on the many-valued notion of negation, conjunction, disjunction, or implication that they are supposed to reflect, the deep language is assigned a unique interpretation in terms of truth and falsehood. To address the diversity of the three-valued interpretations of logical symbols, we then show that any surface semantics can be embedded in the deep semantics through a suitable translation from the surface language into the deep language.

On the other hand, to address the distinction between the partial and paraconsistent notions of logical consequence, we propose a generalization of the notion of sequent introduced by Gentzen and a set of rules of inference for the deep language. By means of this single set of rules, three sequent calculi for the deep language are defined so that they differ only in the definition of axiomatic sequent.

Besides the fact that the proposed frameworks provide a unified understanding of partial and paraconsistent three-valued logics, more arguments are advanced in favor of this approach. The main arguments can be summarized as follows. First, the deep language (containing only two logical symbols) is functionally complete with regard to the deep semantics. Second, the sequent calculi considered enjoy a uniform proof-search method. Third, several forms of the original cut rule are admissible in these sequent calculi. Fourth, the general sequent calculus is sound and complete for the deep semantics and can be adapted in a straightforward way to three- or four-valued logics modulo a translation from the surface language into the deep language.

In order to illustrate our point, some well-known three-valued logics are discussed, such as Kleene's strong three-valued logic (K_3), Łukasiewicz's three-valued logic ($Ł_3$), Gödel's three-valued logic (G_3), the maximal paracomplete three-valued logic (I^1), Priest's logic of paradox (LP), Dunn's R-mingle three-valued logic (RM_3), Gödel's dual three-valued logic (G_3^*), and the maximal paraconsistent three-valued logic (P^1).

2. Deep language and surface language

This section is devoted to introducing two predicate languages without quantifiers, namely the surface language and the deep language. The main reason why we prefer to use predicate languages without quantifiers rather than propositional languages or full first-order languages is twofold. First, most of the results presented in this article may be extended to first-order logic. Second, not using full predicate languages allows us to preserve the clarity of the text without omitting any essential detail.

A *surface language* $\mathcal{L}$ is composed of an at most countable set of proper symbols including a non-empty set of n-ary relation symbols and a countably infinite set of constant symbols. In addition to these proper symbols, a surface language consists of the logical symbols $\sim$, $\cap$, $\cup$ and $\supset$, which correspond roughly to the usual notions of negation, conjunction, disjunction, and implication.

The *deep language* $\mathcal{L}_\mathcal{D}$ of a surface language $\mathcal{L}$ is composed of the proper symbols of $\mathcal{L}$ plus the unary logical symbol $-$ and the binary logical symbol $\ominus$. Insofar as the meaning of these symbols is not immediately obvious, their interpretation will be given after presenting some preliminary semantic notions.

As for the syntax of these languages, the notion of formula is defined inductively in the usual way. The nullary relation symbols are called propositional symbols. To simplify the notation, the following abbreviations are introduced:

$$\begin{aligned}
\neg A &= (-(A \ominus A) \ominus -(A \ominus A)) \\
\triangleleft A &= (A \ominus A) \\
\triangleright A &= -(- A \ominus - A) \\
(A \vee B) &= ((A \ominus A) \ominus (B \ominus B)) \\
(A \wedge B) &= \neg(\neg A \vee \neg B) \\
(A \rightarrow B) &= (\neg A \vee B) \\
\blacklozenge A &= ((A \vee \triangleright A) \wedge (A \vee - A)) \\
\lozenge A &= ((A \vee \triangleleft A) \wedge (A \vee - A)) \\
\blacksquare A &= ((A \wedge \triangleleft A) \vee (A \wedge - A)) \\
\square A &= ((A \wedge \triangleright A) \vee (A \wedge - A)) \\
\vartriangle A &= \lozenge \blacksquare A \\
\triangledown A &= \blacklozenge \square A
\end{aligned}$$

Remark. The reader who wishes to get an idea of the meaning of these expressions is invited to anticipate their semantic interpretation which is provided in Section 3.1.

Let $P_1, \ldots, P_n$ be a finite sequence of distinct atomic formulas of a surface language $\mathcal{L}$. Then, $F(P_1, \ldots, P_n)$ denotes a formula of $\mathcal{L}_\mathcal{D}$ whose atomic formulas are exactly those occurring in the sequence. Also, if $A_1, \ldots, A_n$ are formulas of $\mathcal{L}_\mathcal{D}$, then $F(P_1, \ldots, P_n)[P_1 := A_1, \ldots, P_n := A_n]$ denotes the formula resulting from the simultaneous substitution of A_i for P_i in $F(P_1, \ldots, P_n)$, for all i $(1 \leqslant i \leqslant n)$. Roughly speaking, a translation from a surface language into its deep language consists in matching each n-ary logical symbol $\sharp$ of the surface language with the structure of a formula $F_\sharp(P_1, \ldots, P_n)$ of its deep language. More precisely, a *translation* from $\mathcal{L}$ into $\mathcal{L}_\mathcal{D}$ is a function τ from the set $\mathcal{F}$ of formulas of $\mathcal{L}$ to the set $\mathcal{F}_\mathcal{D}$ of formulas of $\mathcal{L}_\mathcal{D}$ such that:

$$\begin{aligned}
\tau[Rt_1 \ldots t_n] &= Rt_1 \ldots t_n \\
\tau[\sim A] &= F_\sim(P)[P := \tau[A]] \\
\tau[(A \cap B)] &= F_\cap(P_1, P_2)[P_1 := \tau[A], P_2 := \tau[B]]
\end{aligned}$$

$$\tau[(A \cup B)] = F_{\cup}(P_1, P_2)[P_1 := \tau[A], P_2 := \tau[B]]$$
$$\tau[(A \supset B)] = F_{\supset}(P_1, P_2)[P_1 := \tau[A], P_2 := \tau[B]]$$

Many translations from a surface language into its deep language are possible. For our purposes, four of them are defined here below. They will allow us to focus our discussion on some well-known three-valued logics.

$\tau_1 \colon \mathcal{F} \to \mathcal{F}_{\mathcal{D}}$ such that:

$$\begin{aligned}
\tau_1[Rt_1 \ldots t_n] &= Rt_1 \ldots t_n \\
\tau_1[\sim A] &= \neg\, \tau_1[A] \\
\tau_1[(A \cap B)] &= (\tau_1[A] \wedge \tau_1[B]) \\
\tau_1[(A \cup B)] &= (\tau_1[A] \vee \tau_1[B]) \\
\tau_1[(A \supset B)] &= (\tau_1[A] \to \tau_1[B])
\end{aligned}$$

$\tau_2 \colon \mathcal{F} \to \mathcal{F}_{\mathcal{D}}$ such that:

$$\begin{aligned}
\tau_2[Rt_1 \ldots t_n] &= Rt_1 \ldots t_n \\
\tau_2[\sim A] &= \neg\, \tau_2[A] \\
\tau_2[(A \cap B)] &= (\tau_2[A] \wedge \tau_2[B]) \\
\tau_2[(A \cup B)] &= (\tau_2[A] \vee \tau_2[B]) \\
\tau_2[(A \supset B)] &= ((\triangle\, \tau_2[A] \to \tau_2[B]) \wedge (\tau_2[A] \to \triangledown\, \tau_2[B]))
\end{aligned}$$

$\tau_3 \colon \mathcal{F} \to \mathcal{F}_{\mathcal{D}}$ such that:

$$\begin{aligned}
\tau_3[Rt_1 \ldots t_n] &= Rt_1 \ldots t_n \\
\tau_3[\sim A] &= \neg\, \triangledown\, \tau_3[A] \\
\tau_3[(A \cap B)] &= (\tau_3[A] \wedge \tau_3[B]) \\
\tau_3[(A \cup B)] &= (\tau_3[A] \vee \tau_3[B]) \\
\tau_3[(A \supset B)] &= ((\triangledown\, \tau_3[A] \to \triangledown\, \tau_3[B]) \wedge (\blacksquare\, \tau_3[A] \to \lozenge\, \tau_3[B]))
\end{aligned}$$

$\tau_4 \colon \mathcal{F} \to \mathcal{F}_{\mathcal{D}}$ such that:

$$\begin{aligned}
\tau_4[Rt_1 \ldots t_n] &= Rt_1 \ldots t_n \\
\tau_4[\sim A] &= \neg\, \triangledown\, \tau_4[A] \\
\tau_4[(A \cap B)] &= (\triangle\, \tau_4[A] \wedge \triangle\, \tau_4[B]) \\
\tau_4[(A \cup B)] &= (\triangle\, \tau_4[A] \vee \triangle\, \tau_4[B]) \\
\tau_4[(A \supset B)] &= (\triangle\, \tau_4[A] \to \triangle\, \tau_4[B])
\end{aligned}$$

Remark. For every translation τ_i $(1 \leqslant i \leqslant 4)$, an inverse function τ_i^{-1} restricted to the image of τ_i can be defined such that $\tau_i^{-1}[\tau_i[A]] = A$, for all formulas A of $\mathcal{L}$.

3. Semantics

A *model* $\mathcal{M}$ for a surface language $\mathcal{L}$ and its deep language $\mathcal{L}_{\mathcal{D}}$ is composed of a structure for $\mathcal{L}$ and an interpretation of the proper symbols of $\mathcal{L}$ in this structure.

A *structure* for $\mathcal{L}$ consists of a universe and a set of relations on this universe such that, for every $n \in \mathbb{N}$, if $\mathcal{L}$ has some n-ary relation symbols, the structure must have at least one n-ary relation.

The universe $|\mathcal{M}|$ of a model $\mathcal{M}$ is a non-empty set. An n-ary relation R is an ordered pair of subsets of $|\mathcal{M}|^n$ such that $R = \langle R^+, R^- \rangle$. The first term of the ordered pair denotes the set of n-tuples of elements of the universe that verify the relation R and the second term of the ordered pair denotes the set of n-tuples of elements of the universe that falsify the relation [see 23].

An *interpretation* of $\mathcal{L}$ assigns an object in the universe to every constant symbol and an n-ary relation to every n-ary relation symbol of $\mathcal{L}$. The interpretation of an n-ary relation symbol R of $\mathcal{L}$ in the universe of the model $\mathcal{M}$ is denoted $R_{\mathcal{M}}$ and is equated with the ordered pair $\langle (R^n)^+_{\mathcal{M}}, (R^n)^-_{\mathcal{M}} \rangle$ of subsets of $|\mathcal{M}|^n$. The interpretation of a constant symbol t of $\mathcal{L}$ in $|\mathcal{M}|$ is denoted $t_{\mathcal{M}}$.

3.1. Deep semantics

The deep semantics plays a crucial role in the development of the model-theoretic and the proof-theoretic frameworks that will be discussed later. In contrast to the surface (many-valued) semantics, the deep semantics suggests that partial and paraconsistent three-valued logics should ultimately be regarded as logics that involve only truth and falsehood, but do not assume they are exhaustive or exclusive. While the surface semantics provides a fragmented many-valued understanding of these logics, the deep semantics is intended to provide a philosophically more fundamental and conceptually more unified view. In this sense, we call this semantics ‘deep’.

Moreover, this semantics can also be described as ‘deep’ in a weaker sense. Any surface semantics can be embedded in the deep semantics due to the fact that the deep language is functionally complete with regard to the deep semantics (see Theorem 1). However, the reason why this semantics is deep does not rely directly on the definition of logical connectives. As such, any functionally complete language including other logical connectives could also be described as 'deep'. The two main advantages of the set of connectives we have chosen are that it is more concise and it does not suggest that some notions of negation, conjunction, disjunction, and implication are more fundamental than others.

The *truth* and the *falsehood* of a formula of a deep language $\mathcal{L}_\mathcal{D}$ are defined in a model. Given a model $\mathcal{M}$, the truth (denoted by $\mathcal{M} \vDash^+$) and the falsehood (denoted by $\mathcal{M} \vDash^-$) of the formulas of $\mathcal{L}_\mathcal{D}$ in $\mathcal{M}$ are defined inductively:

$\mathcal{M} \vDash^+ Rt_1 \dots t_n$ if and only if $\langle t_{1\mathcal{M}}, \dots, t_{n\mathcal{M}} \rangle \in R^+_\mathcal{M}$
$\mathcal{M} \vDash^- Rt_1 \dots t_n$ if and only if $\langle t_{1\mathcal{M}}, \dots, t_{n\mathcal{M}} \rangle \in R^-_\mathcal{M}$
$\mathcal{M} \vDash^+ -A$ if and only if $\mathcal{M} \nvDash^- A$
$\mathcal{M} \vDash^- -A$ if and only if $\mathcal{M} \nvDash^+ A$
$\mathcal{M} \vDash^+ (A \ominus B)$ if and only if $\mathcal{M} \vDash^+ A$ or $\mathcal{M} \vDash^+ B$
$\mathcal{M} \vDash^- (A \ominus B)$ if and only if $\mathcal{M} \nvDash^- A$ and $\mathcal{M} \nvDash^- B$

The unary logical symbol $-$ is interpreted as a dualisation connective that inverts truth and non-falsehood on the one hand and falsehood and non-truth on the other hand. In this way, this logical connective corresponding to what Fitting [15] calls conflation highlights the symmetry between the approaches dealing with truth-value gaps and those dealing with truth-value gluts. As for the binary logical symbol denoted $\ominus$ [see 11], its semantic interpretation can be regarded as a combination of two well-known classical connectives, namely the inclusive disjunction (with regard to the definition of truth) and the Sheffer stroke (with regard to the definition of falsehood).

Through the semantic interpretation of the logical symbols of a deep language, it is easy to verify the following properties for the abbreviations previously described:

$\mathcal{M} \vDash^+ \neg A$ if and only if $\mathcal{M} \vDash^- A$
$\mathcal{M} \vDash^- \neg A$ if and only if $\mathcal{M} \vDash^+ A$
$\mathcal{M} \vDash^+ (A \vee B)$ if and only if $\mathcal{M} \vDash^+ A$ or $\mathcal{M} \vDash^+ B$
$\mathcal{M} \vDash^- (A \vee B)$ if and only if $\mathcal{M} \vDash^- A$ and $\mathcal{M} \vDash^- B$
$\mathcal{M} \vDash^+ (A \wedge B)$ if and only if $\mathcal{M} \vDash^+ A$ and $\mathcal{M} \vDash^+ B$
$\mathcal{M} \vDash^- (A \wedge B)$ if and only if $\mathcal{M} \vDash^- A$ or $\mathcal{M} \vDash^- B$

$\mathcal{M} \vDash^{+} (A \rightarrow B)$ if and only if $\mathcal{M} \vDash^{-} A$ or $\mathcal{M} \vDash^{+} B$
$\mathcal{M} \vDash^{-} (A \rightarrow B)$ if and only if $\mathcal{M} \vDash^{+} A$ and $\mathcal{M} \vDash^{-} B$
$\mathcal{M} \vDash^{+} \triangleleft A$ if and only if $\mathcal{M} \vDash^{+} A$
$\mathcal{M} \vDash^{-} \triangleleft A$ if and only if $\mathcal{M} \nvDash^{-} A$
$\mathcal{M} \vDash^{+} \triangleright A$ if and only if $\mathcal{M} \nvDash^{+} A$
$\mathcal{M} \vDash^{-} \triangleright A$ if and only if $\mathcal{M} \vDash^{-} A$
$\mathcal{M} \vDash^{+} \blacklozenge A$ if and only if $\mathcal{M} \vDash^{+} A$ or $\mathcal{M} \nvDash^{-} A$
$\mathcal{M} \vDash^{-} \blacklozenge A$ if and only if $\mathcal{M} \vDash^{-} A$
$\mathcal{M} \vDash^{+} \lozenge A$ if and only if $\mathcal{M} \vDash^{+} A$
$\mathcal{M} \vDash^{-} \lozenge A$ if and only if $\mathcal{M} \nvDash^{+} A$ and $\mathcal{M} \vDash^{-} A$
$\mathcal{M} \vDash^{+} \blacksquare A$ if and only if $\mathcal{M} \vDash^{+} A$
$\mathcal{M} \vDash^{-} \blacksquare A$ if and only if $\mathcal{M} \nvDash^{+} A$ or $\mathcal{M} \vDash^{-} A$
$\mathcal{M} \vDash^{+} \square A$ if and only if $\mathcal{M} \vDash^{+} A$ and $\mathcal{M} \nvDash^{-} A$
$\mathcal{M} \vDash^{-} \square A$ if and only if $\mathcal{M} \vDash^{-} A$
$\mathcal{M} \vDash^{+} \vartriangle A$ if and only if $\mathcal{M} \vDash^{+} A$
$\mathcal{M} \vDash^{-} \vartriangle A$ if and only if $\mathcal{M} \nvDash^{+} A$
$\mathcal{M} \vDash^{+} \triangledown A$ if and only if $\mathcal{M} \nvDash^{-} A$
$\mathcal{M} \vDash^{-} \triangledown A$ if and only if $\mathcal{M} \vDash^{-} A$

The first four abbreviations correspond respectively to the notions of negation, disjunction, conjunction, and implication as defined in Dunn-Belnap's [see 13, 5] four-valued logic. Abbreviations $\triangleleft A$ and $\triangleright A$ can be thought of as reflecting a swap between what Belnap [5] calls the approximation lattice and the logical lattice. The expression $\triangleleft A$ is both true and false if and only if A is true, and it is neither true nor false if and only if A is false. On the other hand, the expression $\triangleright A$ is neither true nor false if and only if A is true, and it is both true and false if and only if A is false. As for the following four abbreviations, they can be regarded as a generalization of the possibility and the necessity logical connectives proposed by Łukasiewicz [20] for his three-valued modal logic. According to this interpretation, the abbreviations $\blacklozenge A$ and $\blacksquare A$ correspond to the gappy notions of possibility and necessity applied to the formula A while the abbreviations $\lozenge A$ and $\square A$ correspond to the glutty ones. Finally, the last two abbreviations reflect the truth and the falsehood conditions of the abbreviations $\lozenge\blacksquare A$ and $\blacklozenge\square A$, respectively. Note that the order of the logical symbols has no effect on the truth nor on the falsehood of these expressions. Indeed, the abbreviations $\lozenge\blacksquare A$ and $\blacksquare\lozenge A$ as well as the abbreviations $\blacklozenge\square A$ and $\square\blacklozenge A$ are semantically equivalent.

3.2. Surface semantics

Starting with the deep semantics, the development of three-valued logics does not simply consist in adding a third truth-value. Instead, it consists in giving new definitions of truth and falsehood. In general, we can say that a formula is true in a many-valued logic if and only if it is both true and not false in the deep semantics. Similarly, we can say that a formula is false in a many-valued logic if and only if it is both false and not true in the deep semantics.

As for the third truth-value, two options are possible from the deep semantics viewpoint. The truth-value gap interpretation suggests that the third truth-value in three-valued logic is assigned to a formula if and only if it is neither true nor false in the deep semantics. On the other hand, the truth-value glut interpretation suggests that the third truth-value in three-valued logic is assigned to a formula if and only if it is both true and false in the deep semantics.

This choice concerning the third truth-value highlights the distinction between glutty and gappy interpretations of three-valued logic. In light of the deep semantics, four new truth-values can therefore be defined.

The *valuation* associated to a model $\mathcal{M}$ for a surface language $\mathcal{L}$, denoted $v_{\mathcal{M}}$, is the function from the set of atomic formulas of $\mathcal{L}$ to the set of truth-values $\{t, f, n, b\}$ such that:

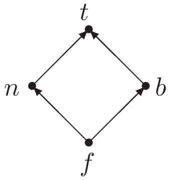
$$v_{\mathcal{M}}[Rt_1...t_n] = t \text{ iff } \langle t_{1\mathcal{M}}, ..., t_{n\mathcal{M}}\rangle \in R^+_{\mathcal{M}} \text{ and } \langle t_{1\mathcal{M}}, ..., t_{n\mathcal{M}}\rangle \notin R^-_{\mathcal{M}}$$
$$v_{\mathcal{M}}[Rt_1...t_n] = f \text{ iff } \langle t_{1\mathcal{M}}, ..., t_{n\mathcal{M}}\rangle \notin R^+_{\mathcal{M}} \text{ and } \langle t_{1\mathcal{M}}, ..., t_{n\mathcal{M}}\rangle \in R^-_{\mathcal{M}}$$
$$v_{\mathcal{M}}[Rt_1...t_n] = n \text{ iff } \langle t_{1\mathcal{M}}, ..., t_{n\mathcal{M}}\rangle \notin R^+_{\mathcal{M}} \text{ and } \langle t_{1\mathcal{M}}, ..., t_{n\mathcal{M}}\rangle \notin R^-_{\mathcal{M}}$$
$$v_{\mathcal{M}}[Rt_1...t_n] = b \text{ iff } \langle t_{1\mathcal{M}}, ..., t_{n\mathcal{M}}\rangle \in R^+_{\mathcal{M}} \text{ and } \langle t_{1\mathcal{M}}, ..., t_{n\mathcal{M}}\rangle \in R^-_{\mathcal{M}}$$

This definition can be extended inductively to all formulas of the language by assigning an n-ary function $f_\sharp$ defined on the set of truth-values to every n-ary logical symbol $\sharp$. A *many-valued assignment* for a surface language $\mathcal{L}$ is a function that maps each model $\mathcal{M}$ for $\mathcal{L}$ to a function $\mathcal{M}[\ .\ ]$ from the set of formulas of $\mathcal{L}$ to the set $\{t, f, n, b\}$ such that:

$$\begin{aligned}
\mathcal{M}[Rt_1 \dots t_n] &= v_{\mathcal{M}}[Rt_1 \dots t_n] \\
\mathcal{M}[\sim A] &= f_\sim(\mathcal{M}[A]) \\
\mathcal{M}[(A \cap B)] &= f_\cap(\mathcal{M}[A], \mathcal{M}[B]) \\
\mathcal{M}[(A \cup B)] &= f_\cup(\mathcal{M}[A], \mathcal{M}[B]) \\
\mathcal{M}[(A \supset B)] &= f_\supset(\mathcal{M}[A], \mathcal{M}[B])
\end{aligned}$$

In this way, the meaning of the logical connectives can be specified in different ways. Indeed, it is possible to define a wide variety of many-valued assignments that differ only in their interpretation of the logical symbols. Insofar as this article intends to address specific gappy and glutty semantics, an order relation and some operations suitable for these semantics are defined on the set of truth-values.

Let $\langle\{t, f, n, b\}, \leq\rangle$ be the lattice such that t is the greatest element, f is the least element, and the values n and b are two intermediate elements that are incomparable:



Then, let $\langle\{t, f, n, b\}, \vee, \wedge, \blacklozenge, \lozenge, \blacksquare, \square, \vartriangle, \triangledown, \overline{\cdot}\rangle$ be an algebraic structure where:

$(x \vee y)$ is the supremum of $\{x, y\}$ with respect to $\langle\{t, f, n, b\}, \leq\rangle$
$(x \wedge y)$ is the infimum of $\{x, y\}$ with respect to $\langle\{t, f, n, b\}, \leq\rangle$

$$\blacklozenge\, x = \begin{cases} t & \text{if } x = n \\ x & \text{otherwise} \end{cases} \qquad \lozenge\, x = \begin{cases} t & \text{if } x = b \\ x & \text{otherwise} \end{cases}$$

$$\blacksquare\, x = \begin{cases} f & \text{if } x = n \\ x & \text{otherwise} \end{cases} \qquad \square\, x = \begin{cases} f & \text{if } x = b \\ x & \text{otherwise} \end{cases}$$

$$\vartriangle x = \lozenge \blacksquare\, x \qquad \triangledown x = \blacklozenge \square\, x$$

$$\overline{x} = \begin{cases} f & \text{if } x = t \\ t & \text{if } x = f \\ x & \text{otherwise} \end{cases}$$

Using this structure, four particular many-valued assignments are introduced.

3.2.1. A many-valued assignment for K_3 and LP

The interpretation of the logical symbols given by the many-valued assignment that maps each model to the function $\mathcal{M}_1$ is none other than that of Dunn-Belnap's [see 13, 5] four-valued logic L_4. From a three-valued point of view, it can be regarded as a joint extension of Kleene's [19] strong three-valued logic K_3 and Priest's [24] logic of paradox LP.

$\mathcal{M}_1\colon \mathcal{F} \rightarrow \{t, f, n, b\}$ such that:

$$\begin{aligned}
\mathcal{M}_1[Rt_1...t_n] &= v_{\mathcal{M}}[Rt_1...t_n] \\
\mathcal{M}_1[\sim A] &= \overline{\mathcal{M}_1[A]} \\
\mathcal{M}_1[(A \cap B)] &= (\mathcal{M}_1[A] \wedge \mathcal{M}_1[B]) \\
\mathcal{M}_1[(A \cup B)] &= (\mathcal{M}_1[A] \vee \mathcal{M}_1[B]) \\
\mathcal{M}_1[(A \supset B)] &= (\overline{\mathcal{M}_1[A]} \vee \mathcal{M}_1[B])
\end{aligned}$$

$\sim$	
t	f
f	t
n	n
b	b

$\cap$	t	f	n	b
t	t	f	n	b
f	f	f	f	f
n	n	f	n	f
b	b	f	f	b

$\cup$	t	f	n	b
t	t	t	t	t
f	t	f	n	b
n	t	n	n	t
b	t	b	t	b

$\supset$	t	f	n	b
t	t	f	n	b
f	t	t	t	t
n	t	n	n	t
b	t	b	t	b

3.2.2. A many-valued assignment for $Ł_3$ and RM_3

The many-valued assignment that maps each model to the function $\mathcal{M}_2$ corresponds to the matrices of the four-valued logic M_4 proposed by Brady [7] in the context of relevance logic. The function $\mathcal{M}_2$ differs from $\mathcal{M}_1$ only in the interpretation of the logical symbol of implication. This interpretation corresponds to the implication of Łukasiewicz's [21] three-valued logic $Ł_3$ when the domain of the function is restricted to the set $\{t, f, n\}$ and to the implication of Dunn's [see 1] R-mingle three-valued logic RM_3 when it is restricted to the set $\{t, f, b\}$.

$\mathcal{M}_2\colon \mathcal{F} \rightarrow \{t, f, n, b\}$ such that:

$$\begin{aligned}
\mathcal{M}_2[Rt_1...t_n] &= v_{\mathcal{M}}[Rt_1...t_n] \\
\mathcal{M}_2[\sim A] &= \overline{\mathcal{M}_2[A]} \\
\mathcal{M}_2[(A \cap B)] &= (\mathcal{M}_2[A] \wedge \mathcal{M}_2[B]) \\
\mathcal{M}_2[(A \cup B)] &= (\mathcal{M}_2[A] \vee \mathcal{M}_2[B]) \\
\mathcal{M}_2[(A \supset B)] &= ((\overline{\triangle\, \mathcal{M}_2[A]} \vee \mathcal{M}_2[B]) \wedge (\overline{\mathcal{M}_2[A]} \vee \triangledown\, \mathcal{M}_2[B]))
\end{aligned}$$

$\sim$	
t	f
f	t
n	n
b	b

$\cap$	t	f	n	b
t	t	f	n	b
f	f	f	f	f
n	n	f	n	f
b	b	f	f	b

$\cup$	t	f	n	b
t	t	t	t	t
f	t	f	n	b
n	t	n	n	t
b	t	b	t	b

$\supset$	t	f	n	b
t	t	f	n	f
f	t	t	t	t
n	t	n	t	n
b	t	f	n	b

3.2.3. A many-valued assignment for G_3 and G_3^*

The many-valued assignment that maps each model to the function $\mathcal{M}_3$ results from the combination of the definition of the logical connectives involved in Gödel's [18] three-valued logic G_3 and its dual version proposed by Brunner and Carnielli [8], here referred to as Gödel's dual three-valued logic G_3^*. Its main feature concerns the interpretation of the negation symbol, which differs both from $\mathcal{M}_1$ and $\mathcal{M}_2$.

$\mathcal{M}_3\colon \mathcal{F} \to \{t, f, n, b\}$ such that:

$$\mathcal{M}_3[Rt_1...t_n] = v_{\mathcal{M}}[Rt_1...t_n]$$
$$\mathcal{M}_3[\sim A] = \overline{\triangledown\, \mathcal{M}_3[A]}$$
$$\mathcal{M}_3[(A \cap B)] = (\mathcal{M}_3[A] \wedge \mathcal{M}_3[B])$$
$$\mathcal{M}_3[(A \cup B)] = (\mathcal{M}_3[A] \vee \mathcal{M}_3[B])$$
$$\mathcal{M}_3[(A \supset B)] = ((\overline{\triangledown\, \mathcal{M}_3[A]} \vee \triangledown\, \mathcal{M}_3[B]) \wedge (\overline{\blacksquare\, \mathcal{M}_3[A]} \vee \lozenge\, \mathcal{M}_3[B]))$$

$\sim$	
t	f
f	t
n	f
b	t

$\cap$	t	f	n	b
t	t	f	n	b
f	f	f	f	f
n	n	f	n	f
b	b	f	f	b

$\cup$	t	f	n	b
t	t	t	t	t
f	t	f	n	b
n	t	n	n	t
b	t	b	t	b

$\supset$	t	f	n	b
t	t	f	n	f
f	t	t	t	t
n	t	f	t	f
b	t	b	t	t

3.2.4. A many-valued assignment for I^1 and P^1

The interpretation of the logical symbols as defined by the many-valued assignment that maps each model to the function $\mathcal{M}_4$ is characterized by the fact that the image of the functions corresponding to these symbols is the set of classical truth-values. When the domain of these functions is restricted to the set $\{t, f, n\}$, these correspond to the definition of the logical connectives introduced by Sette and Carnielli [26] for their maximal paracomplete three-valued logic I^1. On the other hand, when these functions are restricted to the set $\{t, f, b\}$, these correspond to the interpretation of the logical symbols of the maximal paraconsistent three-valued logic P^1 proposed by Sette [25].

$\mathcal{M}_4\colon \mathcal{F} \to \{t, f, n, b\}$ such that:

$$\mathcal{M}_4[Rt_1...t_n] = v_{\mathcal{M}}[Rt_1...t_n]$$
$$\mathcal{M}_4[\sim A] = \overline{\triangledown\, \mathcal{M}_4[A]}$$
$$\mathcal{M}_4[(A \cap B)] = (\vartriangle \mathcal{M}_4[A] \wedge \vartriangle \mathcal{M}_4[B])$$

$$\mathcal{M}_4[(A \cup B)] = (\vartriangle \mathcal{M}_4[A] \vee \vartriangle \mathcal{M}_4[B])$$
$$\mathcal{M}_4[(A \supset B)] = (\overline{\vartriangle \mathcal{M}_4[A]} \vee \vartriangle \mathcal{M}_4[B])$$

$\sim$	
t	f
f	t
n	f
b	t

$\cap$	t	f	n	b
t	t	f	f	t
f	f	f	f	f
n	f	f	f	f
b	t	f	f	t

$\cup$	t	f	n	b
t	t	t	t	t
f	t	f	f	t
n	t	f	f	t
b	t	t	t	t

$\supset$	t	f	n	b
t	t	f	f	t
f	t	t	t	t
n	t	t	t	t
b	t	f	f	t

3.3. Translation and many-valued assignment

Several surface semantics can be developed on the basis of the deep semantics. While a surface language is not functionally complete with regard to every many-valued assignment, Lemma 1 states that any deep language is functionally complete with regard to the deep semantics. This result follows directly from the fact that any language containing the two logical connectives $-$ and $\ominus$ is functionally complete with regard to the deep semantics. As a consequence of this characteristic, Theorem 1 establishes that any surface semantics can be embedded in the deep semantics through a suitable translation from the surface language into the deep language.

The *many-valued interpretation* of a deep language $\mathcal{L}_\mathcal{D}$ is the function that maps each model $\mathcal{M}$ for $\mathcal{L}$ to the function $\mathcal{I}_\mathcal{M}$ from the set of formulas of $\mathcal{L}_\mathcal{D}$ to the set $\{t, f, n, b\}$ such that for every model $\mathcal{M}$ and for every formula A of $\mathcal{L}_\mathcal{D}$:

$\mathcal{I}_\mathcal{M}[A] = t$ if and only if $\mathcal{M} \vDash^+ A$ and $\mathcal{M} \nvDash^- A$
$\mathcal{I}_\mathcal{M}[A] = f$ if and only if $\mathcal{M} \nvDash^+ A$ and $\mathcal{M} \vDash^- A$
$\mathcal{I}_\mathcal{M}[A] = n$ if and only if $\mathcal{M} \nvDash^+ A$ and $\mathcal{M} \nvDash^- A$
$\mathcal{I}_\mathcal{M}[A] = b$ if and only if $\mathcal{M} \vDash^+ A$ and $\mathcal{M} \vDash^- A$

Based on the definition of many-valued interpretation, truth tables for the logical symbols and the abbreviations of the deep language can be set out. The truth tables for $-$ and $\ominus$ are as follows:

$-$	
t	t
f	f
n	b
b	n

$\ominus$	t	f	n	b
t	b	t	b	t
f	t	n	n	t
n	b	n	f	t
b	t	t	t	t

Lemma 1. *For every n-ary truth-function* $f_\sharp\colon \{t, f, n, b\}^n \to \{t, f, n, b\}$, *there exists a formula* $F_\sharp(P_1, \ldots, P_n)$ *of* $\mathcal{L}_\mathcal{D}$ *such that, for every model* $\mathcal{M}$ *for* $\mathcal{L}_\mathcal{D}$, $f_\sharp(\mathcal{I}_\mathcal{M}[P_1], \ldots, \mathcal{I}_\mathcal{M}[P_n]) = \mathcal{I}_\mathcal{M}[F_\sharp(P_1, \ldots, P_n)]$.

Proof. Although the functional completeness of the set of connectives $\{-, \ominus\}$ can be verified on the basis of previous results, especially those obtained by Muskens [22] and Avron [2], an original and self-contained proof of this result is provided in [11]. ⊣

Theorem 1. *For all many-valued assignments for a surface language* $\mathcal{L}$, *there exists a translation* τ *from* $\mathcal{L}$ *into* $\mathcal{L}_\mathcal{D}$ *such that for every model* $\mathcal{M}$ *and for every formula* A *of* $\mathcal{L}$:

1. $\mathcal{M}[A] = t$ *if and only if* $\mathcal{M} \models^+ \tau[A]$ *and* $\mathcal{M} \not\models^- \tau[A]$
2. $\mathcal{M}[A] = f$ *if and only if* $\mathcal{M} \not\models^+ \tau[A]$ *and* $\mathcal{M} \models^- \tau[A]$
3. $\mathcal{M}[A] = n$ *if and only if* $\mathcal{M} \not\models^+ \tau[A]$ *and* $\mathcal{M} \not\models^- \tau[A]$
4. $\mathcal{M}[A] = b$ *if and only if* $\mathcal{M} \models^+ \tau[A]$ *and* $\mathcal{M} \models^- \tau[A]$

Proof. Let be a many-valued assignment for a surface language $\mathcal{L}$ such that:

$$\begin{aligned}
\mathcal{M}[Rt_1 ... t_n] &= v_\mathcal{M}[Rt_1 ... t_n]\\
\mathcal{M}[\sim A] &= f_\sim(\mathcal{M}[A])\\
\mathcal{M}[(A \cap B)] &= f_\cap(\mathcal{M}[A], \mathcal{M}[B])\\
\mathcal{M}[(A \cup B)] &= f_\cup(\mathcal{M}[A], \mathcal{M}[B])\\
\mathcal{M}[(A \supset B)] &= f_\supset(\mathcal{M}[A], \mathcal{M}[B])
\end{aligned}$$

Using Lemma 1, it can be shown that for every n-ary truth-function $f_\sharp$ defined on the set of truth-values, there is a formula $F_\sharp(P_1, ..., P_n)$ of $\mathcal{L}_\mathcal{D}$ such that for every model $\mathcal{M}$ and for every sequence of formulas $A_1, \ldots, A_n$ of $\mathcal{L}_\mathcal{D}$ $f_\sharp(\mathcal{I}_\mathcal{M}[A_1], ..., \mathcal{I}_\mathcal{M}[A_n]) = \mathcal{I}_\mathcal{M}[F_\sharp(P_1, ..., P_n)[P_1 := A_1, ..., P_n := A_n]]$.

Then, let τ be a translation from $\mathcal{L}$ into $\mathcal{L}_\mathcal{D}$ such that:

$$\begin{aligned}
\tau[Rt_1 ... t_n] &= Rt_1 ... t_n\\
\tau[\sim A] &= F_\sim(P)[P := \tau[A]]\\
\tau[(A \cap B)] &= F_\cap(P_1, P_2)[P_1 := \tau[A], P_2 := \tau[B]]\\
\tau[(A \cup B)] &= F_\cup(P_1, P_2)[P_1 := \tau[A], P_2 := \tau[B]]\\
\tau[(A \supset B)] &= F_\supset(P_1, P_2)[P_1 := \tau[A], P_2 := \tau[B]]
\end{aligned}$$

and such that:

$$f_{\sim}(\mathcal{I}_{\mathcal{M}}[A]) = \mathcal{I}_{\mathcal{M}}[F_{\sim}(P)[P := A]]$$
$$f_{\cap}(\mathcal{I}_{\mathcal{M}}[A_1], \mathcal{I}_{\mathcal{M}}[A_2]) = \mathcal{I}_{\mathcal{M}}[F_{\cap}(P_1, P_2)[P_1 := A_1, P_2 := A_2]]$$
$$f_{\cup}(\mathcal{I}_{\mathcal{M}}[A_1], \mathcal{I}_{\mathcal{M}}[A_2]) = \mathcal{I}_{\mathcal{M}}[F_{\cup}(P_1, P_2)[P_1 := A_1, P_2 := A_2]]$$
$$f_{\supset}(\mathcal{I}_{\mathcal{M}}[A_1], \mathcal{I}_{\mathcal{M}}[A_2]) = \mathcal{I}_{\mathcal{M}}[F_{\supset}(P_1, P_2)[P_1 := A_1, P_2 := A_2]]$$

The proof consists in showing that $\mathcal{M}[A] = \mathcal{I}_{\mathcal{M}}[\tau[A]]$, for every model $\mathcal{M}$ and every formula A of $\mathcal{L}$. The proof proceeds by induction on the complexity of A. The initial step is trivial. As for the induction step, all the connectives are treated in the same way. For example, let us examine the negation. By the definition of the many-valued assignment considered, $\mathcal{M}[\sim A] = f_{\sim}(\mathcal{M}[A])$. In addition, from the induction hypothesis, it follows that $f_{\sim}(\mathcal{M}[A]) = f_{\sim}(\mathcal{I}_{\mathcal{M}}[\tau[A]])$. Also, by the definition of τ, we know that $f_{\sim}(\mathcal{I}_{\mathcal{M}}[\tau[A]]) = \mathcal{I}_{\mathcal{M}}[F_{\sim}(P)[P := \tau[A]]]$ and $\mathcal{I}_{\mathcal{M}}[F_{\sim}(P)[P := \tau[A]]] = \mathcal{I}_{\mathcal{M}}[\tau[\sim A]]$. Finally, we conclude the proof by observing that $\mathcal{M}[\sim A] = \mathcal{I}_{\mathcal{M}}[\tau[\sim A]]$. ⊣

Remark. Theorem 1 also applies to any broader notion of surface language containing additional propositional connectives. For example, the paraconsistent three-valued logic J_3 introduced by D'Ottaviano and da Costa [12] can be easily embedded in the deep semantics through a suitable extension of the translation τ_1. Indeed, the possibility logical connective involved in J_3 can be translated in an obvious way into the deep language by means of the abbreviation $\lozenge A$. Moreover, all theorems in this article can be extended to first-order logic. This can be done by adding a universal quantifier and an existential quantifier both to the surface language and the deep language. A semantic interpretation of these quantifiers as well as a set of rules of inference suitable for this purpose can be found in [10].

By means of the different translations previously proposed, Proposition 1 establishes a correspondence between each of the four many-valued assignments introduced and the general notions of truth and falsehood.

By induction on the complexity of A we obtain:

PROPOSITION 1. *Let $\mathcal{L}$ be a surface language and $\mathcal{L}_{\mathcal{D}}$ be its deep language. Then, for all formulas A of $\mathcal{L}$ and for all i such that $1 \leqslant i \leqslant 4$:*

1. *$\mathcal{M}_i[A] = t$ if and only if $\mathcal{M} \vDash^+ \tau_i[A]$ and $\mathcal{M} \nvDash^- \tau_i[A]$*
2. *$\mathcal{M}_i[A] = f$ if and only if $\mathcal{M} \nvDash^+ \tau_i[A]$ and $\mathcal{M} \vDash^- \tau_i[A]$*
3. *$\mathcal{M}_i[A] = n$ if and only if $\mathcal{M} \nvDash^+ \tau_i[A]$ and $\mathcal{M} \nvDash^- \tau_i[A]$*
4. *$\mathcal{M}_i[A] = b$ if and only if $\mathcal{M} \vDash^+ \tau_i[A]$ and $\mathcal{M} \vDash^- \tau_i[A]$*

3.4. Gappy semantics and glutty semantics

Several semantic approaches can be specified according to the class of models considered. Two of these approaches seem particularly relevant for our purposes, namely the gappy semantics and the glutty semantics. These two types of semantics can be distinguished by defining some properties on the models.

A model $\mathcal{M}$ is *consistent* if $(R^n)^+_{\mathcal{M}} \cap (R^n)^-_{\mathcal{M}} = \emptyset$, for every n-ary relation R on $|\mathcal{M}|$. A model $\mathcal{M}$ is *complete* if $(R^n)^+_{\mathcal{M}} \cup (R^n)^-_{\mathcal{M}} = |\mathcal{M}|^n$, for every n-ary relation R on $|\mathcal{M}|$. In this sense, a model is called *classical* if it is both consistent and complete.

Depending on whether a semantics restricts the class of models to that of consistent or complete models, this semantics will be called *gappy* or *glutty*, respectively. Also, a semantics may be characterized as gappy or glutty regardless of whether we are dealing with a deep semantics or a surface semantics.

The reason why we call these semantics gappy or glutty lies in the fact that they do not obey the 'metalinguistic' Law of Excluded Middle (stating that any sentence of the object-language has at least one of the values true and false) or the 'metalinguistic' Law of Non-Contradiction (stating that any sentence of the object-language has at most one of the values true and false), respectively [see 14]. Furthermore, although any atomic formula of a deep language is always either true or false in a complete model and never both true and false in a consistent model, these properties do not apply to the complex formulas. For this reason, restricting oneself to the class of consistent models does not imply that there is no translation from the surface language to the deep language such that some complex formulas are both true and false. Similarly, restricting oneself to the class of complete models does not mean that there is no translation such that some complex formulas are neither true nor false. In this connection, it is important to note that propositions 2 and 3 do not hold for any translation. Regarding the surface semantics, this feature is reflected in the fact that the truth-functions associated with the logical symbols of a surface language are not necessarily conservative when their domain is restricted to the set $\{t, f, n\}$ or $\{t, f, b\}$ [see 6]. By induction on the complexity of A we obtain:

Proposition 2 (Meta-law of excluded middle). *Let $\mathcal{M}$ be a complete model for a surface language $\mathcal{L}$ and its deep language $\mathcal{L}_{\mathcal{D}}$. Then, for all A of $\mathcal{L}$ and for all i such that $1 \leqslant i \leqslant 4$, $\mathcal{M} \vDash^+ \tau_i[A]$ or $\mathcal{M} \vDash^- \tau_i[A]$.*

PROPOSITION 3 (Meta-law of non-contradiction). *Let $\mathcal{M}$ be a consistent model for a surface language $\mathcal{L}$ and its deep language $\mathcal{L}_{\mathcal{D}}$. Then, for all A of $\mathcal{L}$ and for all i such that $1 \leqslant i \leqslant 4$, $\mathcal{M} \nvDash^{+} \tau_i[A]$ or $\mathcal{M} \nvDash^{-} \tau_i[A]$.*

4. Logical consequence

In order to address the distinction between the partial and paraconsistent notions of logical consequence involved in three-valued logic, we propose a unified proof-theoretic framework based on a generalization of Gentzen's [16] definition of sequent. This notion is closely related to those developed by Girard [17]), Muskens [23], and Bochman [6]. Also be noted that this notion can be regarded as a particular instance of labeled sequent introduced in the context of many-valued logics (see, among others [3, 4]).

A *sequent* is a quadruple $\langle \Pi, \Gamma, \Delta, \Sigma \rangle$, where Π, Γ, Δ, and Σ are finite multisets over the set of formulas of a deep language $\mathcal{L}_{\mathcal{D}}$. The sequent $\langle \Pi, \Gamma, \Delta, \Sigma \rangle$ is denoted $\Pi; \Gamma \Vdash \Delta; \Sigma$ (see [10]).

A multiset is a sequence modulo the ordering. More specifically, a *multiset* M over S is an ordered pair $\langle S, f \rangle$, where S is a set and $f : S \to \mathbb{N}$ is a function that indicates the multiplicity of each element of S. The *underlying set* of a multiset $M = \langle S, f \rangle$ is the set μ such that $\mu = \{s \in S \mid f(s) \neq 0\}$. M is called *finite*, if μ is finite. The sum of the multisets M_1 and M_2 is denoted by M_1, M_2 and the multiset $\langle S, f \rangle$ where $\{s \in S \mid f(s) \neq 0\} = \{A\}$ and $f(A) = 1$ is denoted by A.

Let $\Pi; \Gamma \Vdash \Delta; \Sigma$ be a sequent such that π, γ, δ and σ are the underlying sets of Π, Γ, Δ and Σ, respectively. Then, $\Pi; \Gamma \Vdash \Delta; \Sigma$ is *valid* if for every model $\mathcal{M}$, $\mathcal{M} \nvDash^{-} A$, for all $A \in \pi$, and $\mathcal{M} \vDash^{+} A$, for all $A \in \gamma$, implies $\mathcal{M} \vDash^{+} A$, for some $A \in \delta$, or $\mathcal{M} \nvDash^{-} A$, for some $A \in \sigma$.

This definition of validity can be preserved for the gappy and the glutty semantics. Depending on whether the notion of valid sequent is restricted to the consistent models or to the complete models, a sequent is called *gap-valid* or *glut-valid*, respectively.

4.1. Some well-known many-valued logics

In general, the notion of logical consequence for many-valued logics is defined in terms of designated values selected from a set of truth-values. Roughly speaking, a Gentzen sequent is said to be valid in a many-valued logic L (or L-valid) if some formula in the succedent gets a designated truth-value whenever all the formulas in the antecedent get a designated truth-value.

The many-valued logics mentioned so far can be divided into three categories according to the set of truth-values and the set of designated values. The set of truth-values for the four-valued logics L_4 and M_4 is $\{t, f, n, b\}$ and the set of designated values is $\{t, b\}$. Regarding the three-valued logics K_3, $Ł_3$, G_3, and I^1, the set of truth-values is $\{t, f, n\}$ and the set of designated values is $\{t\}$. As for the three-valued logics LP, RM_3, G_3^*, and P^1, the set of their truth-values is $\{t, f, b\}$ and their designated values are t and b.

However, from the deep semantics standpoint, these logics share a common feature in that their notion of validity can be reformulated in terms of truth-preservation (from antecedent to succedent). In this sense, we see that a sequent is valid in these logics if when all formulas in the antecedent are at least true (that is to say, either 'true and not false' or 'both true and false'), so is some formula in the succedent. Based on these observations, propositions 4–6 follow directly from propositions 1–3.

Proposition 4. *Let* $\Gamma \Vdash \Delta$ *be a sequent of the usual kind.*

1. $\Gamma \Vdash \Delta$ *is* L_4*-valid if and only if* $\;;\tau_1[\Gamma] \Vdash \tau_1[\Delta]\,;\;$ *is valid.*
2. $\Gamma \Vdash \Delta$ *is* M_4*-valid if and only if* $\;;\tau_2[\Gamma] \Vdash \tau_2[\Delta]\,;\;$ *is valid.*

Proposition 5. *Let* $\Gamma \Vdash \Delta$ *be a sequent of the usual kind.*

1. $\Gamma \Vdash \Delta$ *is* K_3*-valid if and only if* $\;;\tau_1[\Gamma] \Vdash \tau_1[\Delta]\,;\;$ *is gap-valid.*
2. $\Gamma \Vdash \Delta$ *is* $Ł_3$*-valid if and only if* $\;;\tau_2[\Gamma] \Vdash \tau_2[\Delta]\,;\;$ *is gap-valid.*
3. $\Gamma \Vdash \Delta$ *is* G_3*-valid if and only if* $\;;\tau_3[\Gamma] \Vdash \tau_3[\Delta]\,;\;$ *is gap-valid.*
4. $\Gamma \Vdash \Delta$ *is* I^1*-valid if and only if* $\;;\tau_4[\Gamma] \Vdash \tau_4[\Delta]\,;\;$ *is gap-valid.*

Proposition 6. *Let* $\Gamma \Vdash \Delta$ *be a sequent of the usual kind.*

1. $\Gamma \Vdash \Delta$ *is* LP*-valid if and only if* $\;;\tau_1[\Gamma] \Vdash \tau_1[\Delta]\,;\;$ *is glut-valid.*
2. $\Gamma \Vdash \Delta$ *is* RM_3*-valid if and only if* $\;;\tau_2[\Gamma] \Vdash \tau_2[\Delta]\,;\;$ *is glut-valid.*
3. $\Gamma \Vdash \Delta$ *is* G_3^**-valid if and only if* $\;;\tau_3[\Gamma] \Vdash \tau_3[\Delta]\,;\;$ *is glut-valid.*
4. $\Gamma \Vdash \Delta$ *is* P^1*-valid if and only if* $\;;\tau_4[\Gamma] \Vdash \tau_4[\Delta]\,;\;$ *is glut-valid.*

These propositions can be summarized by means of Table 1.

4.2. Three sequent calculi

To define a sequent calculus and a notion of derivability for a deep language, some rules of inference governing the behavior of the logical connectives are to be set out. The main feature of the rules given below

	τ_1	τ_2	τ_3	τ_4
Validity	L_4	M_4		
Gap-validity	K_3	Ł$_3$	G_3	I^1
Glut-validity	LP	RM_3	G_3^*	P^1

Table 1.

is that the weakening and contraction structural rules are absorbed into the rules of inference [see 27].

$$\frac{\Pi, A; \Gamma \Vdash \Delta; \Sigma}{\Pi; \Gamma, -A \Vdash \Delta; \Sigma}\,-^i_L \qquad \frac{\Pi; \Gamma \Vdash \Delta; A, \Sigma}{\Pi; \Gamma \Vdash -A, \Delta; \Sigma}\,-^i_R$$

$$\frac{\Pi; \Gamma, A \Vdash \Delta; \Sigma}{\Pi, -A; \Gamma \Vdash \Delta; \Sigma}\,-^e_L \qquad \frac{\Pi; \Gamma \Vdash A, \Delta; \Sigma}{\Pi; \Gamma \Vdash \Delta; -A, \Sigma}\,-^e_R$$

$$\frac{\Pi; \Gamma, A \Vdash \Delta; \Sigma \qquad \Pi; \Gamma, B \Vdash \Delta; \Sigma}{\Pi; \Gamma, (A \ominus B) \Vdash \Delta; \Sigma}\,\ominus^i_L \qquad \frac{\Pi; \Gamma \Vdash A, B, \Delta; \Sigma}{\Pi; \Gamma \Vdash (A \ominus B), \Delta; \Sigma}\,\ominus^i_R$$

$$\frac{\Pi; \Gamma \Vdash \Delta; A, \Sigma \qquad \Pi; \Gamma \Vdash \Delta; B, \Sigma}{\Pi, (A \ominus B); \Gamma \Vdash \Delta; \Sigma}\,\ominus^e_L \qquad \frac{\Pi, A, B; \Gamma \Vdash \Delta; \Sigma}{\Pi; \Gamma \Vdash \Delta; (A \ominus B), \Sigma}\,\ominus^e_R$$

The notion of *derivation* as well as those of *initial sequent* and *end-sequent* are defined inductively in the usual way. Roughly speaking, a derivation is a finite rooted tree in which the nodes are sequents. The root of the tree (at the bottom) is called the endsequent and the leaves of the tree (at the top) are called initial sequents. The *length* of a derivation is the number of sequents in that derivation.

Starting with the single set of rules of inference set out above, three notions of derivability are distinguished so that they differ only in the definition of axiomatic sequent. A sequent is *derivable*, *gap-derivable*, or *glut-derivable* if there exists a derivation in which it is the endsequent and all initial sequents are respectively axiomatic, gap-axiomatic, or glut-axiomatic.

A sequent $\Pi; \Gamma \Vdash \Delta; \Sigma$ is *axiomatic* if there exists an atomic formula P such that $P \in \gamma \cap \delta$ or $P \in \pi \cap \sigma$.

A sequent $\Pi; \Gamma \Vdash \Delta; \Sigma$ is *gap-axiomatic* if it is axiomatic or there is an atomic formula P such that $P \in \gamma \cap \sigma$.

A sequent $\Pi; \Gamma \Vdash \Delta; \Sigma$ is *glut-axiomatic* if it is axiomatic or there is an atomic formula P such that $P \in \pi \cap \delta$.

A set of rules involving the previously mentioned abbreviations are provided below. These rules of inference are admissible in the three sequent calculi. In other words, if all the premisses of one of these rules are derivable, gap-derivable, or glut-derivable, then so is the conclusion. These rules essentially play a practical role. They facilitate the use of the sequent calculi by greatly reducing the number of instances of rules in a derivation.

$$\frac{\Pi;\Gamma\Vdash\Delta;A,\Sigma}{\Pi;\Gamma,\neg A\Vdash\Delta;\Sigma}\,\neg^i_L \qquad \frac{\Pi,A;\Gamma\Vdash\Delta;\Sigma}{\Pi;\Gamma\Vdash\neg A,\Delta;\Sigma}\,\neg^i_R$$

$$\frac{\Pi;\Gamma\Vdash A,\Delta;\Sigma}{\Pi,\neg A;\Gamma\Vdash\Delta;\Sigma}\,\neg^e_L \qquad \frac{\Pi;\Gamma,A\Vdash\Delta;\Sigma}{\Pi;\Gamma\Vdash\Delta;\neg A,\Sigma}\,\neg^e_R$$

$$\frac{\Pi;\Gamma,A\Vdash\Delta;\Sigma \qquad \Pi;\Gamma,B\Vdash\Delta;\Sigma}{\Pi;\Gamma,(A\vee B)\Vdash\Delta;\Sigma}\,\vee^i_L \qquad \frac{\Pi;\Gamma\Vdash A,B,\Delta;\Sigma}{\Pi;\Gamma\Vdash(A\vee B),\Delta;\Sigma}\,\vee^i_R$$

$$\frac{\Pi,A;\Gamma\Vdash\Delta;\Sigma \qquad \Pi,B;\Gamma\Vdash\Delta;\Sigma}{\Pi,(A\vee B);\Gamma\Vdash\Delta;\Sigma}\,\vee^e_L \qquad \frac{\Pi;\Gamma\Vdash\Delta;A,B,\Sigma}{\Pi;\Gamma\Vdash\Delta;(A\vee B),\Sigma}\,\vee^e_R$$

$$\frac{\Pi;\Gamma,A,B\Vdash\Delta;\Sigma}{\Pi;\Gamma,(A\wedge B)\Vdash\Delta;\Sigma}\,\wedge^i_L \qquad \frac{\Pi;\Gamma\Vdash A,\Delta;\Sigma \qquad \Pi;\Gamma\Vdash B,\Delta;\Sigma}{\Pi;\Gamma\Vdash(A\wedge B),\Delta;\Sigma}\,\wedge^i_R$$

$$\frac{\Pi,A,B;\Gamma\Vdash\Delta;\Sigma}{\Pi,(A\wedge B);\Gamma\Vdash\Delta;\Sigma}\,\wedge^e_L \qquad \frac{\Pi;\Gamma\Vdash\Delta;A,\Sigma \qquad \Pi;\Gamma\Vdash\Delta;B,\Sigma}{\Pi;\Gamma\Vdash\Delta;(A\wedge B),\Sigma}\,\wedge^e_R$$

$$\frac{\Pi;\Gamma\Vdash\Delta;A,\Sigma \qquad \Pi;\Gamma,B\Vdash\Delta;\Sigma}{\Pi;\Gamma,(A\to B)\Vdash\Delta;\Sigma}\,\to^i_L \qquad \frac{\Pi,A;\Gamma\Vdash B,\Delta;\Sigma}{\Pi;\Gamma\Vdash(A\to B),\Delta;\Sigma}\,\to^i_R$$

$$\frac{\Pi;\Gamma\Vdash A,\Delta;\Sigma \qquad \Pi,B;\Gamma\Vdash\Delta;\Sigma}{\Pi,(A\to B);\Gamma\Vdash\Delta;\Sigma}\,\to^e_L \qquad \frac{\Pi;\Gamma,A\Vdash\Delta;B,\Sigma}{\Pi;\Gamma\Vdash\Delta;(A\to B),\Sigma}\,\to^e_R$$

$$\frac{\Pi;\Gamma,A\Vdash\Delta;\Sigma \qquad \Pi,A;\Gamma\Vdash\Delta;\Sigma}{\Pi;\Gamma,\blacklozenge A\Vdash\Delta;\Sigma}\,\blacklozenge^i_L \qquad \frac{\Pi;\Gamma\Vdash A,\Delta;A,\Sigma}{\Pi;\Gamma\Vdash\blacklozenge A,\Delta;\Sigma}\,\blacklozenge^i_R$$

$$\frac{\Pi,A;\Gamma\Vdash\Delta;\Sigma}{\Pi,\blacklozenge A;\Gamma\Vdash\Delta;\Sigma}\,\blacklozenge^e_L \qquad \frac{\Pi;\Gamma\Vdash\Delta;A,\Sigma}{\Pi;\Gamma\Vdash\Delta;\blacklozenge A,\Sigma}\,\blacklozenge^e_R$$

$$\frac{\Pi;\Gamma,A\Vdash\Delta;\Sigma}{\Pi;\Gamma,\lozenge A\Vdash\Delta;\Sigma}\,\lozenge^i_L \qquad \frac{\Pi;\Gamma\Vdash A,\Delta;\Sigma}{\Pi;\Gamma\Vdash\lozenge A,\Delta;\Sigma}\,\lozenge^i_R$$

$$\frac{\Pi;\Gamma,A\Vdash\Delta;\Sigma \qquad \Pi,A;\Gamma\Vdash\Delta;\Sigma}{\Pi,\lozenge A;\Gamma\Vdash\Delta;\Sigma}\,\lozenge^e_L \qquad \frac{\Pi;\Gamma\Vdash A,\Delta;A,\Sigma}{\Pi;\Gamma\Vdash\Delta;\lozenge A,\Sigma}\,\lozenge^e_R$$

$$\frac{\Pi;\Gamma,A\Vdash\Delta;\Sigma}{\Pi;\Gamma,\blacksquare A\Vdash\Delta;\Sigma}\,\blacksquare^i_L \qquad \frac{\Pi;\Gamma\Vdash A,\Delta;\Sigma}{\Pi;\Gamma\Vdash\blacksquare A,\Delta;\Sigma}\,\blacksquare^i_R$$

$$\frac{\Pi,A;\Gamma,A\Vdash\Delta;\Sigma}{\Pi,\blacksquare A;\Gamma\Vdash\Delta;\Sigma}\,\blacksquare^e_L \qquad \frac{\Pi;\Gamma\Vdash A,\Delta;\Sigma \qquad \Pi;\Gamma\Vdash\Delta;A,\Sigma}{\Pi;\Gamma\Vdash\Delta;\blacksquare A,\Sigma}\,\blacksquare^e_R$$

$$\frac{\Pi,A;\Gamma,A\Vdash\Delta;\Sigma}{\Pi;\Gamma,\square A\Vdash\Delta;\Sigma}\,\square^i_L \qquad \frac{\Pi;\Gamma\Vdash A,\Delta;\Sigma \qquad \Pi;\Gamma\Vdash\Delta;A,\Sigma}{\Pi;\Gamma\Vdash\square A,\Delta;\Sigma}\,\square^i_R$$

$$\frac{\Pi, A; \Gamma \Vdash \Delta; \Sigma}{\Pi, \Box A; \Gamma \Vdash \Delta; \Sigma} \Box^e_L \qquad \frac{\Pi; \Gamma \Vdash \Delta; A, \Sigma}{\Pi; \Gamma \Vdash \Delta; \Box A, \Sigma} \Box^e_R$$

4.3. Soundness and completeness

This section is devoted to showing that the general sequent calculus as well as the gappy and glutty sequent calculi are sound and complete with regard to the deep semantics. These properties are contained in theorems 2 and 3.

Lemma 2. *The conclusion of a rule instance is valid, gap-valid, or glut-valid iff every premiss of that instance is valid, gap-valid, or glut-valid, respectively.*

Proof. These properties should be proved for each rule of inference. For example, let us show that the $\ominus^e_L$ rule preserves validity both from premisses to conclusion and from conclusion to premisses. The proof is entirely similar when we only consider the class of consistent or complete models.

Assume that $\Pi; \Gamma \Vdash \Delta; A, \Sigma$ and $\Pi; \Gamma \Vdash \Delta; B, \Sigma$ are valid. Let $\mathcal{M}$ be a model such that $\mathcal{M} \not\models^- C$, for all $C \in \pi$, $\mathcal{M} \models^+ C$, for all $C \in \gamma$, and $\mathcal{M} \not\models^- (A \ominus B)$. From the semantic definition of $\ominus$, it follows that $\mathcal{M} \models^- A$ or $\mathcal{M} \models^- B$. By the validity of premisses, we conclude that $\mathcal{M} \models^+ C$, for some $C \in \delta$, or $\mathcal{M} \not\models^- C$, for some $C \in \sigma$.

Assume that $\Pi, (A \ominus B); \Gamma \Vdash \Delta; \Sigma$ is valid. Let $\mathcal{M}$ be a model such that $\mathcal{M} \not\models^- C$, for all $C \in \pi$ and $\mathcal{M} \models^+ C$, for all $C \in \gamma$. If there is some $C \in \delta$ such that $\mathcal{M} \models^+ C$ or some $C \in \sigma$ such that $\mathcal{M} \not\models^- C$, the two premisses of the rule are valid. On the other hand, if there is no such formula, it follows from the validity of the conclusion that $\mathcal{M} \models^- (A \ominus B)$. By the semantic definition of $\ominus$, this means that $\mathcal{M} \not\models^- A$ and $\mathcal{M} \not\models^- B$. Therefore, we conclude that, in this case also, $\Pi; \Gamma \Vdash \Delta; A, \Sigma$ and $\Pi; \Gamma \Vdash \Delta; B, \Sigma$ are valid. ⊣

Theorem 2 (Soundness). *If a sequent is derivable, gap-derivable, or glut-derivable, then it is respectively valid, gap-valid, or glut-valid.*

Proof. The proof proceeds by an induction on the derivation length. The initial step consists in showing that if a sequent is axiomatic, gap-axiomatic, or glut-axiomatic, then it is respectively valid, gap-valid, or glut-valid. The induction step follows immediately from Lemma 2.

As an example, we prove that every glut-axiomatic sequent is glut-valid. Let $\Pi; \Gamma \Vdash \Delta; \Sigma$ be a glut-axiomatic sequent and let $\mathcal{M}$ be a complete model such that $\mathcal{M} \nvDash^- A$, for all $A \in \pi$, and $\mathcal{M} \vDash^+ A$, for all $A \in \gamma$. If there is a formula in $\gamma \cap \delta$ or $\pi \cap \sigma$, then $\Pi; \Gamma \Vdash \Delta; \Sigma$ is obviously valid. If there is an atomic formula P in $\pi \cap \delta$, then $\mathcal{M} \nvDash^- P$. Since $\mathcal{M}$ is complete, it follows that $\mathcal{M} \vDash^+ P$. In other words, some formula $A \in \delta$ is such that $\mathcal{M} \vDash^+ A$. ⊣

THEOREM 3 (Completeness). *If a sequent is valid, gap-valid, or glut-valid, then it is respectively derivable, gap-derivable, or glut-derivable.*

PROOF. Let $\Pi; \Gamma \Vdash \Delta; \Sigma$ be a sequent. Also, let us say that a sequent is atomic if it contains only atomic formulas. The proof consists, firstly, in constructing a derivation in which $\Pi; \Gamma \Vdash \Delta; \Sigma$ is the endsequent and every initial sequent is atomic and, secondly, in showing that if $\Pi; \Gamma \Vdash \Delta; \Sigma$ is valid, then all the initial sequents of that derivation are axiomatic.

The first part of the proof is based on the observation that the number of logical symbols occurring in a sequent is finite and the number of logical symbols contained in each premiss of a rule instance is strictly less than the number of logical symbols contained in the conclusion. From this observation, it follows that a backwards application of the rules of inference from $\Pi; \Gamma \Vdash \Delta; \Sigma$ necessarily leads to a derivation of $\Pi; \Gamma \Vdash \Delta; \Sigma$ in which every initial sequent is atomic.

The second part of the proof is intended to show that if $\Pi; \Gamma \Vdash \Delta; \Sigma$ is valid, then all the initial sequents of such a derivation are axiomatic. By Lemma 2, we know that if the endsequent of a derivation is valid, then every initial sequent of that derivation is also valid. So, it remains to prove that if a sequent is both valid and atomic, then it is axiomatic.

The argument is similar for the gappy and glutty cases. As an example, let us show this last property for the glutty case: if an atomic sequent is not glut-axiomatic, then it is not glut-valid. Let $\Pi'; \Gamma' \Vdash \Delta'; \Sigma'$ be an atomic but not glut-axiomatic sequent. Then, let $\mathcal{M}$ be a model such that:

$\langle t_{1\mathcal{M}}, \ldots, t_{n\mathcal{M}} \rangle \in R^+_{\mathcal{M}}$, if the formula $Rt_1 \ldots t_n$ appears in $\pi' \cup \gamma'$
$\langle t_{1\mathcal{M}}, \ldots, t_{n\mathcal{M}} \rangle \notin R^+_{\mathcal{M}}$, if the formula $Rt_1 \ldots t_n$ does not appear in $\pi' \cup \gamma'$
$\langle t_{1\mathcal{M}}, \ldots, t_{n\mathcal{M}} \rangle \in R^-_{\mathcal{M}}$, if the formula $Rt_1 \ldots t_n$ appears in π'
$\langle t_{1\mathcal{M}}, \ldots, t_{n\mathcal{M}} \rangle \notin R^-_{\mathcal{M}}$, if the formula $Rt_1 \ldots t_n$ does not appear in π'

It suffices to note that $\mathcal{M}$ is complete and $\mathcal{M} \nvDash^- A$ for all $A \in \pi'$, $\mathcal{M} \vDash^+ A$ for all $A \in \gamma'$, $\mathcal{M} \nvDash^+ A$ for all $A \in \delta'$, and $\mathcal{M} \vDash^- A$ for all $A \in \sigma'$. ⊣

By combining theorems 1, 2, and 3, we obtain corollaries 1 and 2.

Corollary 1. *Let $\Gamma \Vdash \Delta$ be a sequent of the usual kind, where Γ and Δ are finite multisets over the set of formulas of a surface language $\mathcal{L}$. Then for all many-valued assignments* MV *for $\mathcal{L}$, there exists a translation τ from $\mathcal{L}$ into $\mathcal{L}_{\mathcal{D}}$ such that $\Gamma \Vdash \Delta$ is truth-preserving with regard to* MV *iff* $\quad ; \tau[\Gamma] \Vdash \tau[\Delta] \, ; \quad$ *is derivable.*

Corollary 2. *Let $\Gamma \Vdash \Delta$ be a sequent of the usual kind, where Γ and Δ are finite multisets over the set of formulas of a surface language $\mathcal{L}$. Then for all many-valued assignments* MV *for $\mathcal{L}$, there exists a translation τ from $\mathcal{L}$ into $\mathcal{L}_{\mathcal{D}}$ such that:*

1. *If* MV *is gappy, then $\Gamma \Vdash \Delta$ is truth-preserving with regard to* MV *if and only if* $\quad ; \tau[\Gamma] \Vdash \tau[\Delta] \, ; \quad$ *is gap-derivable.*
2. *If* MV *is glutty, then $\Gamma \Vdash \Delta$ is truth-preserving with regard to* MV *if and only if* $\quad ; \tau[\Gamma] \Vdash \tau[\Delta] \, ; \quad$ *is glut-derivable.*

On the basis of corollaries 1–2, we conclude that the proposed sequent calculi are suitable for all partial and paraconsistent truth-preserving four-valued logics as well as their gappy and glutty restrictions. Moreover, from the observations made in the proof of Theorem 3, it follows that a uniform proof-search method can be provided for any of these logics by a backwards application of the rules of inference.

Remark. Many other notions of logical consequence can in principle be defined in three-valued and four-valued logics [see 3]. Some of them can be obtained in a natural way from the definition of valid sequent by simply changing the position of the translated formulas within a sequent.

4.4. Cut redundancy

Several formulations of the redundancy of cut are possible in the sequent calculi mentioned above. According to the position of the cut formula, four different forms of the original cut rule are distinguished. The admissibility of these rules can be easily obtained from theorems 2 and 3 by noting that if the premisses of one of these rules are valid, then so is the conclusion. While this proof has the advantage of being very short, it has two main drawbacks: firstly, it fails to provide a better insight into the nature of the sequent calculi involved and, secondly, it makes use of model-theoretic notions that are superfluous and extraneous to these systems.

Instead, we adopt a purely proof-theoretic approach. As part of the study of some formulations of the cut rule, the weakening and contraction structural properties as well as the inversion property of the inference rules are proved for these calculi. These results are contained in propositions 7–9. Although these properties hold for the three sequent calculi, we only give a sketch of the proof for the general sequent calculus.

A sequent is *l-derivable* if it is the endsequent of a derivation such that its length is at most l and all its initial sequents are axiomatic.

PROPOSITION 7 (Weakening). *Let A be a formula of a deep language.*

1. *If $\Pi; \Gamma \Vdash \Delta; \Sigma$ is l-derivable, then $\Pi, A; \Gamma \Vdash \Delta; \Sigma$ is l-derivable.*
2. *If $\Pi; \Gamma \Vdash \Delta; \Sigma$ is l-derivable, then $\Pi; \Gamma, A \Vdash \Delta; \Sigma$ is l-derivable.*
3. *If $\Pi; \Gamma \Vdash \Delta; \Sigma$ is l-derivable, then $\Pi; \Gamma \Vdash A, \Delta; \Sigma$ is l-derivable.*
4. *If $\Pi; \Gamma \Vdash \Delta; \Sigma$ is l-derivable, then $\Pi; \Gamma \Vdash \Delta; A, \Sigma$ is l-derivable.*

PROOF. By induction on the derivation length of $\Pi; \Gamma \Vdash \Delta; \Sigma$. ⊣

PROPOSITION 8 (Inversion). *If the conclusion of a rule instance is l-derivable, then every premiss of that instance is also l-derivable.*

PROOF. The proof consists in showing the inversion property for each of the eight rules of inference by using an induction on the derivation length. As an example, we give the proof for the $\ominus^e_L$ rule. Assume there exists a derivation of length at most l in which $\Pi, (A \ominus B); \Gamma \Vdash \Delta; \Sigma$ is the endsequent and every initial sequent is axiomatic. If $l = 1$, then $\Pi, (A \ominus B); \Gamma \Vdash \Delta; \Sigma$ is axiomatic and so are $\Pi; \Gamma \Vdash \Delta; A, \Sigma$ and $\Pi; \Gamma \Vdash \Delta; B, \Sigma$. If $l > 1$, then $\Pi, (A \ominus B); \Gamma \Vdash \Delta; \Sigma$ is the conclusion of an instance of a rule R. Two cases are possible. If $(A \ominus B)$ is the principal formula of that rule instance, then it follows from the inductive definition of derivation that $\Pi; \Gamma \Vdash \Delta; A, \Sigma$ and $\Pi; \Gamma \Vdash \Delta; B, \Sigma$ are l-derivable. If $(A \ominus B)$ is not the principal formula of that rule instance, we apply the induction hypothesis to each premiss of the rule instance and then use the rule R to obtain derivations of $\Pi; \Gamma \Vdash \Delta; A, \Sigma$ and $\Pi; \Gamma \Vdash \Delta; B, \Sigma$. ⊣

PROPOSITION 9 (Contraction). *Let A be a formula of a deep language.*

1. *If $\Pi, A, A; \Gamma \Vdash \Delta; \Sigma$ is l-derivable, then $\Pi, A; \Gamma \Vdash \Delta; \Sigma$ is l-derivable.*
2. *If $\Pi; \Gamma, A, A \Vdash \Delta; \Sigma$ is l-derivable, then $\Pi; \Gamma, A \Vdash \Delta; \Sigma$ is l-derivable.*
3. *If $\Pi; \Gamma \Vdash A, A, \Delta; \Sigma$ is l-derivable, then $\Pi; \Gamma \Vdash A, \Delta; \Sigma$ is l-derivable.*
4. *If $\Pi; \Gamma \Vdash \Delta; A, A, \Sigma$ is l-derivable, then $\Pi; \Gamma \Vdash \Delta; A, \Sigma$ is l-derivable.*

Proof. The proof of these four assertions proceeds by a simultaneous induction on the derivation length. We only consider the first assertion assuming that the others are treated symmetrically. Assume there exists a derivation of length at most l in which $\Pi, A, A; \Gamma \Vdash \Delta; \Sigma$ is the endsequent and every initial sequent is axiomatic. If $l = 1$, then $\Pi, A, A; \Gamma \Vdash \Delta; \Sigma$ is axiomatic, in which case $\Pi, A; \Gamma \Vdash \Delta; \Sigma$ is also axiomatic. If $l > 1$, then $\Pi, A, A; \Gamma \Vdash \Delta; \Sigma$ is the conclusion of an instance of a rule R. Two cases are possible. If A is the principal formula of that rule, then we distinguish different sub-cases depending on the last rule applied: $-^e_L$ or $\ominus^e_L$. In both sub-cases, the proof consists in applying Proposition 8 and using the induction hypothesis. If A is not the principal formula of that rule instance, we apply the induction hypothesis to each premiss of the rule instance and then extend each new derivation by using the rule R in order to obtain a derivation of $\Pi, A; \Gamma \Vdash \Delta; \Sigma$. $\dashv$

Theorem 4 shows that the cut property is satisfied for the general sequent calculus when the cut formula appears either in the right and left *internal* sides or in the right and left *external* sides of sequents. This property also holds for the gappy and glutty sequent calculi. On the other hand, Theorem 5 states that the gappy and glutty sequent calculi admit one form of cut (restricted to the atomic formulas) in addition to the two that hold for the general sequent calculus. Note that this theorem can be extended to the complex formulas of $\mathcal{L}_\mathcal{D}$ obtained by means of a translation τ_i $(1 \leqslant i \leqslant 4)$ from $\mathcal{L}$ into $\mathcal{L}_\mathcal{D}$.

Theorem 4 (Cut redundancy). *Let A be a formula of a deep language.*

1. *If $\Pi; \Gamma \Vdash A, \Delta; \Sigma$ and $\Pi; \Gamma, A \Vdash \Delta; \Sigma$ are derivable, then $\Pi; \Gamma \Vdash \Delta; \Sigma$ is derivable.*
2. *If $\Pi; \Gamma \Vdash \Delta; A, \Sigma$ and $\Pi, A; \Gamma \Vdash \Delta; \Sigma$ are derivable, then $\Pi; \Gamma \Vdash \Delta; \Sigma$ is derivable.*

Proof. The proof of these two assertions proceeds by a main simultaneous induction on the complexity of A. When A is an atomic formula, the proof uses a sub-induction on the sum of the derivation lengths of the sequents $\Pi; \Gamma \Vdash A, \Delta; \Sigma$ and $\Pi; \Gamma, A \Vdash \Delta; \Sigma$, for the first assertion, or $\Pi; \Gamma \Vdash \Delta; A, \Sigma$ and $\Pi, A; \Gamma \Vdash \Delta; \Sigma$, for the second assertion. We only consider the first assertion. The second is assumed to be treated symmetrically.

Assume that A is an atomic formula. Let be $l = l_1 + l_2$. Then, by induction on l, we show that if $\Pi; \Gamma \Vdash A, \Delta; \Sigma$ is l_1-derivable and

$\Pi;\Gamma,A\Vdash\Delta;\Sigma$ is l_2-derivable, then $\Pi;\Gamma\Vdash\Delta;\Sigma$ is derivable. If $l=2$, then $\Pi;\Gamma\Vdash A,\Delta;\Sigma$ and $\Pi;\Gamma,A\Vdash\Delta;\Sigma$ are axiomatic and so is $\Pi;\Gamma\Vdash\Delta;\Sigma$. If $l>2$, either $l_1>1$ or $l_2>1$. In both cases, the fact that $\Pi;\Gamma\Vdash\Delta;\Sigma$ is derivable can be established by applying Proposition 8 and using the induction hypothesis.

Assume now that A is a formula of the form $-B$. Then, $\Pi;\Gamma\Vdash -B,\Delta;\Sigma$ and $\Pi;\Gamma,-B\Vdash\Delta;\Sigma$ are derivable. By Proposition 8, it follows that $\Pi;\Gamma\Vdash\Delta;B,\Sigma$ and $\Pi,B;\Gamma\Vdash\Delta;\Sigma$ are also derivable. Therefore, by the induction hypothesis on the complexity of A, we conclude that $\Pi;\Gamma\Vdash\Delta;\Sigma$ is derivable.

Finally, assume that A is a formula of the form $(B\ominus C)$. Then, $\Pi;\Gamma\Vdash(B\ominus C),\Delta;\Sigma$ and $\Pi;\Gamma,(B\ominus C)\Vdash\Delta;\Sigma$ are derivable. By Proposition 8, it follows that $\Pi;\Gamma\Vdash B,C,\Delta;\Sigma$ as well as $\Pi;\Gamma,B\Vdash\Delta;\Sigma$ and $\Pi;\Gamma,C\Vdash\Delta;\Sigma$ are derivable. In addition, by Proposition 7, if $\Pi;\Gamma,B\Vdash\Delta;\Sigma$ is derivable, then $\Pi;\Gamma,B\Vdash C,\Delta;\Sigma$ is also derivable. Therefore, by using the induction hypothesis on the complexity of A twice, we conclude that $\Pi;\Gamma\Vdash\Delta;\Sigma$ is derivable. ⊣

Theorem 5 (Cut redundancy). *Let P be an atomic formula of a deep language.*

1. *If $\Pi;\Gamma\Vdash\Delta;P,\Sigma$ and $\Pi;\Gamma,P\Vdash\Delta;\Sigma$ are glut-derivable, then $\Pi;\Gamma\Vdash\Delta;\Sigma$ is glut-derivable.*
2. *If $\Pi;\Gamma\Vdash P,\Delta;\Sigma$ and $\Pi,P;\Gamma\Vdash\Delta;\Sigma$ are gap-derivable, then $\Pi;\Gamma\Vdash\Delta;\Sigma$ is gap-derivable.*

Proof. The proof for the first assertion uses an induction on the sum of the derivation lengths of the sequents $\Pi;\Gamma\Vdash\Delta;P,\Sigma$ and $\Pi;\Gamma,P\Vdash\Delta;\Sigma$. As for the second assertion, it is obtained by a symmetric treatment. Basically, both proofs are similar to that provided for Theorem 4 when A is an atomic formula. ⊣

5. Conclusion

Our aim is to provide a unified framework for partial and paraconsistent three-valued logics. In this connection, the main philosophical idea of the article is that a (surface) many-valued semantics leads to a fragmented understanding of these logics, while a (deep) gappy or glutty semantics involving only truth and falsehood yields a conceptually more fundamental and unified view. This feature is, among others, reflected in the fact that a single notion of model, sequent, and validity are defined regardless of whether we are dealing with a partial three-valued logic, a paraconsistent three-valued logic, or a both partial and paraconsistent four-valued logic.

In addition to providing a unified framework suitable to address the model-theoretic as well as the proof-theoretic issues related to these logics, the main results presented in this article are as follows.

At the model-theoretic level, we showed that two logical symbols (one being unary and the other binary) are enough to define a functionally complete four-valued language. Due to this property, we have also shown that every formula of the surface language can be translated into the deep language while preserving its semantic interpretation. Finally, four translations that capture the notions of validity specific to some well-known three-valued and four-valued logics have been identified.

Regarding the proof-theoretic aspects, we introduced a 'general' sequent calculus that can be easily modified to obtain either a 'gappy' sequent calculus or a 'glutty' sequent calculus, by simply changing the definition of axiomatic sequent. These three calculi (sharing the same rules of inference) feature several interesting properties.

First, the weakening and contraction structural properties as well as the inversion property of the rules of inference are satisfied by these three calculi. Furthermore, they admit the two most natural formulations of the original cut rule. In particular, the gappy and glutty sequent calculi each admit an additional hybrid version of the cut rule where the cut formula appears in the internal side of one of the premisses and in the external side of the other.

Then, we also pointed out that the three sequent calculi allow a uniform proof-search procedure which is ultimately based on the inversion property of the rules of inference. This method consists in constructing a derivation from the bottom up, i.e. by means of a backwards application of the rules of inference from the sequent supposed to be derivable to sequents containing no logical symbol. In this way, if all the initial sequents of the derivation are axiomatic, then the endsequent is derivable. But it is also true that if some initial sequent of such a derivation is not axiomatic, then the endsequent is not derivable.

Finally, we showed that the three sequent calculi are sound and complete with respect to the deep semantics as well as its gappy and glutty restrictions. In addition, the soundness and the completeness proof provided in this article are general enough to apply to any partial or paraconsistent truth-preserving logic modulo a suitable translation from the surface language into the deep language.

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