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# HYPERSEQUENT CALCULI FOR S5: The methods of cut elimination

**Abstract.** **S5** is one of the most important modal logic with nice syntactic, semantic and algebraic properties. In spite of that, a successful (i.e. cut-free) formalization of **S5** on the ground of standard sequent calculus (SC) was problematic and led to the invention of numerous nonstandard, generalised forms of SC. One of the most interesting framework which was very often used for this aim is that of hypersequent calculi (HC). The paper is a survey of HC for **S5** proposed by Pottinger, Avron, Restall, Poggiolesi, Lahav and Kurokawa. We are particularly interested in examining different methods which were used for proving the eliminability/admissibility of cut in these systems and present our own variant of a system which admits relatively simple proof of cut elimination.

**Keywords**: cut elimination; modal logic; hypersequent calculi; proof theory

## 1. Introduction

The problem of providing a cut-free sequent calculus for **S5** has a long history. On the ground of standard sequent calculus it was dealt with by Ohnishi and Matsumoto [24, 25], Sato [30], Mints [21], Fitting [12], Takano [33] and Braüner [6]. There are also numerous nonstandard sequent calculi offering some solution to the problem. One can mention here for example: Kanger's indexed sequent calculus [17] Belnap's display calculus (see an exposition in Wansing [34] or [10]), Negri's labelled sequent calculus [22], Indrzejczak's double sequent calculus [14] or Stouppa's nested sequent calculus [31, 32].

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It seems however that the best solutions were offered on the ground of hypersequent calculi (HC), a generalised form of ordinary sequent calculi invented independently by Pottinger [28] and Avron [1] who developed significantly their theory in 90s. The main idea is to define rules on hypersequents which are (multi)sets of ordinary sequents. Despite of the simplicity of additional machinery this kind of a proof system proved to be very useful in providing cut-free formalizations for many nonclassical logics including several many-valued, relevant, and paraconsistent logics (see in particular: Avron [2, 3]; Baaz, Ciabattoni, and Fermüller [8]; Metcalfe, Olivetti, and Gabbay [20]; Ciabattoni, Ramanayake, and Wansing [10]).

Although originally HC was introduced for some modal logics (Pottinger [28]), and despite the popularity of HC in proof theory for nonclassical logics, it was not frequently used in the field of modal logics. In addition to the variety of HC for **S5** which will be dealt in detail below, one can find only one general treatment in Lahav [19] and the case studies of some modal logics of relational frames with linear accessibility relation (Indrzejczak [15, 16]), and of some other extensions of **S4** (Kurokawa [18]).

However **S5** was dealt with in this framework many times with success; at least six different systems were provided, due to Pottinger [28], Avron [3], Restall [29], Poggiolesi [26], Lahav [19], and Kurokawa [18]. All of them are cut-free but the result is proven by means of different methods and in some cases only sketched or even just mentioned without proof. In particular, for Pottinger's system no proof of cut elimination theorem was presented so far and we will explain the reason for that.

The aim of the paper is to compare proposed solutions and to focus on the proofs of cut elimination for these systems. In particular, we present in detail the proof of cut elimination for Avron's system and discuss some of its peculiarities, namely the application of two different forms of cut (or rather mix) rule[1] Then we show in what way this difficulty (i.e., necessity of using two rules) may be overcome in other systems. Finally we show that a slight modification of Pottinger's rules may lead to a system which admits relatively simple proof of cut elimination theorem. Thus we provide in fact one more HC for **S5** which may be of some interest. It

[1] In what follows we will use terms 'cut', 'mix' in generic sense for covering any version of these rules; specific cases with suitable individual names will be presented in Section 5.

seems that the case study we offer below may be helpfull in finding proofs of cut elimination for other logics formalised via hypersequent calculi.

## 2. S5 – Basic Facts

Let us recall the basic facts concerning modal logic **S5** in the standard characterization, i.e., as an axiomatic system adequate with respect to suitable classes of relational (Kripke) frames. We will use standard monomodal language with countable set PV of propositional variables, $\Box$ – unary modal necessity operator, and ordinary Boolean constants. One can axiomatize **S5** by adding to all (modal) instances of some system of classical propositional logic the following schemata:

$$\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta) \tag{K}$$

$$\Box\alpha \to \alpha \tag{T}$$

$$\Box\alpha \to \Box\Box\alpha \tag{4}$$

$$\neg\Box\alpha \to \Box\neg\Box\alpha \tag{5}$$

and closing under modus ponens and Gödel's rule:

$$\frac{\alpha \quad \alpha \to \beta}{\beta} \tag{MP}$$

$$\frac{\alpha}{\Box\alpha} \tag{GR}$$

Instead of (5) one can use:

$$\neg\alpha \to \Box\neg\Box\alpha \tag{B}$$

The simplest semantical characterization of **S5** may be obtained by means of Kripke frames without accessibility relation. A *model* for **S5** is thus any pair $\mathfrak{M} = \langle W, V\rangle$, where $W$ ia a nonempty set and $V\colon \mathrm{PV} \longrightarrow \mathcal{P}(W)$. Satisfaction in a world $w$ of a model $\mathfrak{M}$ is inductively defined in the following way:

- $w \vDash_{\mathfrak{M}} \alpha$ iff $w \in V(\alpha)$, for any $\alpha \in \mathrm{PV}$,
- $w \vDash_{\mathfrak{M}} \neg\alpha$ iff $w \nvDash_{\mathfrak{M}} \alpha$,
- $w \vDash_{\mathfrak{M}} \alpha \wedge \beta$ iff $w \vDash_{\mathfrak{M}} \alpha$ and $w \vDash_{\mathfrak{M}} \beta$,
- $w \vDash_{\mathfrak{M}} \alpha \vee \beta$ iff $w \vDash_{\mathfrak{M}} \alpha$ or $w \vDash_{\mathfrak{M}} \beta$,
- $w \vDash_{\mathfrak{M}} \alpha \to \beta$ iff $w \nvDash_{\mathfrak{M}} \alpha$ or $w \vDash_{\mathfrak{M}} \beta$,
- $w \vDash_{\mathfrak{M}} \Box\alpha$ iff $v \vDash_{\mathfrak{M}} \alpha$, for any $v \in W$.

A formula is **S5**-valid iff it is true in every world of every model.

## 3. Basic Hypersequent Calculus

In this paper we define hypersequents as finite multisets of ordinary Gentzen's sequents. We will use the following notation:

- $\Gamma$ and $\Delta$ – for finite multisets of formulae;
- $\Gamma \Rightarrow \Delta$ and $s$ – for sequents;
- $G$ and $H$ – for hypersequents;
- $G \mid s$ or $s \mid G$ (resp. $G \mid \Gamma \Rightarrow \Delta$ or $\Gamma \Rightarrow \Delta \mid G$) – for hypersequents with displayed sequent $s$ (resp. $\Gamma \Rightarrow \Delta$);
- $\Box\Gamma$ – for finite multisets containing all elements of $\Gamma$ with added $\Box$;
- $\Gamma^{\Box}$ – for finite multisets of these elements of $\Gamma$ which are already preceded with $\Box$.

For example, for $\Gamma = \{\Box p, q, \Box(r \wedge s)\}$ we have $\Gamma^{\Box} = \{\Box p, \Box(r \wedge s)\}$ and $\Box\Gamma = \{\Box\Box p, \Box q, \Box\Box(r \wedge s)\}$.

The calculus HC for **CPL** (Classical Propositional Logic) consists of axioms of the form $\alpha \Rightarrow \alpha$ and the following rules:

*Structural rules*:

$$(\Rightarrow\mathrm{W})\ \frac{G \mid \Gamma \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta, \alpha} \qquad (\mathrm{W}\Rightarrow)\ \frac{G \mid \Gamma \Rightarrow \Delta}{G \mid \Gamma, \alpha \Rightarrow \Delta}$$

$$(\Rightarrow\mathrm{C})\ \frac{G \mid \Gamma \Rightarrow \Delta, \alpha, \alpha}{G \mid \Gamma \Rightarrow \Delta, \alpha} \qquad (\mathrm{C}\Rightarrow)\ \frac{G \mid \Gamma, \alpha, \alpha \Rightarrow \Delta}{G \mid \Gamma, \alpha \Rightarrow \Delta}$$

$$(\mathrm{EC})\ \frac{G \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta} \qquad (\mathrm{EW})\ \frac{G}{G \mid \Gamma \Rightarrow \Delta}$$

The first four are internal weakening and contraction rules, whereas the last two are external ones. In what follows we will use (IC) and (IW) as shortcuts for applications of internal rules.

*Logical rules*:

$$(\Rightarrow\wedge)\ \frac{G \mid \Gamma \Rightarrow \Delta, \alpha \qquad G \mid \Gamma \Rightarrow \Delta, \beta}{G \mid \Gamma \Rightarrow \Delta, \alpha \wedge \beta}$$

$$(\wedge\Rightarrow)\ \frac{G \mid \Gamma, \alpha \Rightarrow \Delta}{G \mid \Gamma, \alpha \wedge \beta \Rightarrow \Delta} \qquad \frac{G \mid \Gamma, \beta \Rightarrow \Delta}{G \mid \Gamma, \alpha \wedge \beta \Rightarrow \Delta}$$

$$(\Rightarrow\vee)\ \frac{G \mid \Gamma \Rightarrow \Delta, \beta}{G \mid \Gamma \Rightarrow \Delta, \alpha \vee \beta} \qquad \frac{G \mid \Gamma \Rightarrow \Delta, \alpha}{G \mid \Gamma \Rightarrow \Delta, \alpha \vee \beta}$$

$$(\vee\Rightarrow)\ \frac{G \mid \Gamma, \alpha \Rightarrow \Delta \qquad G \mid \Gamma, \beta \Rightarrow \Delta}{G \mid \Gamma, \alpha \vee \beta \Rightarrow \Delta}$$

$$(\Rightarrow\to)\ \frac{G \mid \Gamma, \alpha \Rightarrow \Delta, \beta}{G \mid \Gamma \Rightarrow \Delta, \alpha \to \beta}$$

$$(\to\Rightarrow)\ \frac{G \mid \Gamma \Rightarrow \Delta, \alpha \qquad G \mid \Gamma, \beta \Rightarrow \Delta}{G \mid \Gamma, \alpha \to \beta \Rightarrow \Delta}$$

$$(\Rightarrow\neg)\ \frac{G \mid \Gamma, \alpha \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta, \neg\alpha} \qquad (\neg\Rightarrow)\ \frac{G \mid \Gamma \Rightarrow \Delta, \alpha}{G \mid \Gamma, \neg\alpha \Rightarrow \Delta}$$

This set of rules is taken from Avron [3]. In particular, all logical rules are for $\vee$, $\wedge$, and $\to$ are in additive form, except $(\Rightarrow\to)$ which is multiplicative. It is of no special importance but it should be noted that some of the calculi presented below use other variants of rules. In particular, one-premiss rules for $\vee$ and $\wedge$ in multiplicative form (i.e., with both components present in the premiss) in order to provide invertibility of all Boolean rules. We will mention these changes when necessary. Modal rules of several systems will be introduced systematically in Section 4.

A derivation $d$ of a hypersequent $G$ in HC is defined in the usual way as a tree of hypersequents with $G$ as the root and axioms as leafs.

We extend semantical notions to hypersequents in the following way for any model $\mathfrak{M} = \langle W_{\mathfrak{M}}, V_{\mathfrak{M}} \rangle$:

- $w \vDash_{\mathfrak{M}} \Gamma \Rightarrow \Delta$ iff $w \vDash_{\mathfrak{M}} \bigwedge \Gamma \to \bigvee \Delta$, where $\bigwedge \Gamma$ (resp. $\bigvee \Gamma$) stands for the conjunction (resp. disjunction) of all elements of $\Gamma$;
- $\mathfrak{M} \models s$ iff $w \vDash_{\mathfrak{M}} s$, for any $w \in W_{\mathfrak{M}}$;
- $\mathfrak{M} \models G$ iff there is $s \in G$ such that $\mathfrak{M} \models s$.

We say that a hypersequent $G$ *is valid* (in short: $\models G$) iff $\mathfrak{M} \models G$, for any model $\mathfrak{M}$. Note that — as a consequence — we obtain: $\mathfrak{M} \not\models s$ iff there is $w \in W_{\mathfrak{M}}$ such that $w \nvDash_{\mathfrak{M}} s$; $\mathfrak{M} \not\models G$ iff $\mathfrak{M} \not\models s$, for every $s \in G$; and at the end $\not\models G$ iff there is a model $\mathfrak{M}$ such that $\mathfrak{M} \not\models G$, i.e., iff there is $\mathfrak{M}$ such that for every $s \in G$ there is $w \in W_{\mathfrak{M}}$ such that $w \nvDash_{\mathfrak{M}} s$.

Lemma 1 (Soundness Lemma). *All rules of* HC *are validity-preserving in* **CPL**.

We need also a few technical concepts:

Definition 1. The height of a derivation $d$ (in short: $\mathrm{h}(d)$) for our purposes is defined as the number of hypersequents in the longest branch of a tree-derivation minus one; formally:

1. For $d = \alpha \Rightarrow \alpha$, $\mathrm{h}(d) = 0$;
2. if $d$ ends with $\frac{s'}{s}$ and $d'$ is a derivation of $s'$, then $\mathrm{h}(d) = \mathrm{h}(d') + 1$;
3. if $d$ ends with $\frac{s' \quad s”}{s}$ and $d'$, $d”$ are derivations of $s'$ and $s”$ respectively, then $\mathrm{h}(d) = \max(\mathrm{h}(d'), \mathrm{h}(d”)) + 1$.

Definition 2 (Admissibility: height-preserving admissibility). A rule $\mathcal{R}$ is (*height-preserving*) *admissible* in a calculus HC iff $\mathcal{R}$ satisfies the following condition:

- if in HC there exist derivations of the premisses of $\mathcal{R}$, then there is a derivation of the conclusion of $\mathcal{R}$ that contains no application of $\mathcal{R}$ (with the height at most $n$, where $n$ is the maximal height of derivations of premisses).

Definition 3 (Invertibility: height-preserving invertibility). A rule $\mathcal{R}$ is *invertible* iff derivability of its conclusion entails dedrivability of its premiss(es) (with the height at most $n$, where $n$ is the height of a derivation of the conclusion).

Note that in the set of logical rules displayed above, only the rules $(\Rightarrow\wedge)$ and $(\vee\Rightarrow)$ are not invertible.

Definition 4. The complexity of the formula is defined as the number of logical connectives contained in it.

## 4. Hypersequent Calculi for S5

The number of different hypersequential calculi proposed for **S5** may seem quite surprising but we will observe a lot of similarities between them as well. Several HC proposed for nonclassical logics, and in particular for **S5**, may be roughly divided into two groups: those that are more proof-theoretically oriented and those which are more semantically oriented on actual search of either a proof or a falsifying model.

### 4.1. Pottinger

The first system was stated very briefly in the half-page long abstract of Pottinger [28] and, as far as we know, was not presented in full version. The abstract contains rules of HC for modal logics **T**, **S4**, and **S5**. His system is based on hypersequents being finite sequences of Gentzen's sequents (i.e., built from finite sequences of formulae) rather than finite multisets, and his rules are so constructed as to absorb the effect of structural rules of weakening, contraction and permutation. In particular, axioms have general form $\alpha, \Gamma \Rightarrow \Delta, \alpha$ and $(\wedge\Rightarrow)$ is in multiplicative form, i.e., with all components of the conjunction in the antecedent of the premiss (disjunction is not considered). These changes do not affect the structure of specific modal rules and we can present them in the way suitable for our format. Two modal rules for introduction of $\Box$ are of the form:

$$(\Box\Rightarrow^{\mathrm{P}})\quad \frac{\alpha, \Box\alpha, \Gamma \Rightarrow \Delta \mid \Box\alpha, \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Box\alpha, \Gamma_n \Rightarrow \Delta_n}{\Box\alpha, \Gamma \Rightarrow \Delta \mid \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n}$$

$$(\Rightarrow\Box^{\mathrm{P}})\quad \frac{\Gamma \Rightarrow \Delta \mid \Gamma^{\Box} \Rightarrow \alpha \mid G}{\Gamma \Rightarrow \Delta, \Box\alpha \mid G}$$

Both rules are rather semantically oriented on actual search of either a proof or a falsifying model and in effect are quite redundant.

For example a proof of (K) looks like this, where $s$ is the sequent $\Box(\alpha \to \beta), \Box(\alpha \to \beta), \Box\alpha, \Box\alpha \Rightarrow$:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{s \mid \Box(\alpha \to \beta), \Box\alpha, \alpha \Rightarrow \beta, \alpha \qquad s \mid \Box(\alpha \to \beta), \Box\alpha, \alpha, \beta \Rightarrow \beta}{\Box(\alpha \to \beta), \Box(\alpha \to \beta), \Box\alpha, \Box\alpha \Rightarrow \mid \Box(\alpha \to \beta), \Box\alpha, \alpha \to \beta, \alpha \Rightarrow \beta}\,(\to\Rightarrow)}{\Box(\alpha \to \beta), \Box\alpha, \Box\alpha \Rightarrow \mid \Box(\alpha \to \beta), \Box\alpha, \alpha \Rightarrow \beta}\,(\Box\Rightarrow^{\mathrm{P}})}{\Box(\alpha \to \beta), \Box\alpha \Rightarrow \mid \Box(\alpha \to \beta), \Box\alpha \Rightarrow \beta}\,(\Box\Rightarrow^{\mathrm{P}})}{\Box(\alpha \to \beta), \Box\alpha \Rightarrow \Box\beta}\,(\Rightarrow\Box^{\mathrm{P}})}{\Box(\alpha \to \beta) \Rightarrow \Box\alpha \to \Box\beta}\,(\Rightarrow\to)}{\Rightarrow \Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)}\,(\Rightarrow\to)
$$

The proof of (5) looks like that:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Box\alpha \Rightarrow \Box\alpha \mid \Box\alpha, \alpha \Rightarrow}{\Rightarrow \Box\alpha \mid \Box\alpha \Rightarrow}\,(\Box\Rightarrow^{\mathrm{P}})}{\Rightarrow \Box\alpha \mid \ \Rightarrow \neg\Box\alpha}\,(\Rightarrow\neg)}{\Rightarrow \Box\alpha, \Box\neg\Box\alpha}\,(\Rightarrow\Box^{\mathrm{P}})}{\neg\Box\alpha \Rightarrow \Box\neg\Box\alpha}\,(\neg\Rightarrow)}{\Rightarrow \neg\Box\alpha \to \Box\neg\Box\alpha}\,(\Rightarrow\to)
$$

The abstract of Pottinger does not contain any proof; the system is only claimed to be adequate in cut-free form but it is not known if it was proved syntactically or semantically. It is easy to prove soundness of both rules. We demonstrate the proof for $(\Box\Rightarrow^{\mathrm{P}})$ as an example.

Assume that $\models \alpha, \Box\alpha, \Gamma \Rightarrow \Delta \mid \Box\alpha, \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Box\alpha, \Gamma_n \Rightarrow \Delta_n$ and $\not\models \Box\alpha, \Gamma \Rightarrow \Delta \mid \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$. So for some model $\mathfrak{M} = \langle W_{\mathfrak{M}}, V_{\mathfrak{M}} \rangle$ and $w_0, w_1, \ldots, w_n \in W$ we have $w_0 \nvDash_{\mathfrak{M}} \Box\alpha, \Gamma \Rightarrow \Delta$, $w_1 \nvDash_{\mathfrak{M}} \Gamma_1 \Rightarrow \Delta_1$, $\ldots$, $w_n \nvDash_{\mathfrak{M}} \Gamma_n \Rightarrow \Delta_n$. Hence $w_0 \vDash_{\mathfrak{M}} \Box\alpha$, which implies that $w \vDash_{\mathfrak{M}} \Box\alpha$ and $w \vDash_{\mathfrak{M}} \alpha$, for any $w \in W_{\mathfrak{M}}$. But this falsifies all sequents in the premiss and contradicts our assumption.

As for completeness it is easier to prove it semantically by Hintikka-style argument showing how to construct a falsifying model for failed proof construction. The structure of modal rules make them not very suitable for constructing syntactic proof of cut elimination which we will demonstrate in Section 5.2.

### 4.2. Avron

The second HC for **S5** was constructed by Avron [3] and — in contrast to Pottinger's system — it is more proof-theoretically oriented. In fact, Avron's general policy in developing unified hypersequential environment for several nonclassical logics was rather proof-theoretical[2] and this applies also to his HC proposed for **S5**. The system is based on Ohnishi and Matsumoto standard sequent system for **S4** (see, an exposition in Zeman [35] or Wansing [34]) and modal introduction rules are just taken from their system but in the hypersequent shape, i.e.:

$$(\Box{\Rightarrow}^{A})\ \frac{\alpha, \Gamma \Rightarrow \Delta \mid G}{\Box\alpha, \Gamma \Rightarrow \Delta \mid G} \qquad\qquad ({\Rightarrow}\Box^{A})\ \frac{\Box\Gamma \Rightarrow \alpha \mid G}{\Box\Gamma \Rightarrow \Box\alpha \mid G}$$

Moreover, Avron introduced a special rule (MS) of modal splitting which has combined character; it is partly structural but with displayed multisets of modal formulae:[3]

$$(\mathrm{MS})\ \frac{\Box\Gamma, \Pi \Rightarrow \Box\Delta, \Sigma \mid G}{\Box\Gamma \Rightarrow \Box\Delta \mid \Pi \Rightarrow \Sigma \mid G}$$

In fact, this special rule defined for obtaining cut-free HC for **S5** is so strong that $({\Rightarrow}\Box^{A})$ may be modified in many ways. One can strengthen

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[2] See e.g. [3] for an exposition of desiderata for good, semantically-independent proof system.

[3] It may be noticed that (MS) is based on the similar idea as the special rule of Mints [21] but whereas the latter is defined for standard sequent calculus and destroys subformula property, Avron's rule, due to extra machinery of HC, allows for saving subformula property.

it and use instead a hypersequent version of suitable rule for **S5**, i.e., with $\Box\Delta$ added in the succedent of the premise and conclusion, or weaken it by replacing with a rule which is sound for **K** (i.e., with $\Gamma$ in the antecedent of the premise) or even to hypersequential version of Gödel's rule. All these possible substitutes are listed below:

$$(\Rightarrow\Box^{S5})\ \dfrac{\Box\Gamma \Rightarrow \Box\Delta, \alpha \mid G}{\Box\Gamma \Rightarrow \Box\Delta, \Box\alpha \mid G} \qquad (\Rightarrow\Box^{K})\ \dfrac{\Gamma \Rightarrow \alpha \mid G}{\Box\Gamma \Rightarrow \Box\alpha \mid G}$$

$$(\Rightarrow\Box^{G})\ \dfrac{\Rightarrow \alpha \mid G}{\Rightarrow \Box\alpha \mid G}$$

Such a system strongly depends on the application of structural rules of contraction both in internal (standard) and external version. The proof of (5) is the following:

$$\text{(}\Rightarrow\rightarrow\text{)}\ \dfrac{\text{(EC)}\ \dfrac{\text{(IW)}\times 2\ \dfrac{(\Rightarrow\Box^{A})\ \dfrac{(\Rightarrow\neg)\ \dfrac{\text{(MS)}\ \dfrac{(\neg\Rightarrow)\ \dfrac{\Box\alpha \Rightarrow \Box\alpha}{\Box\alpha, \neg\Box\alpha \Rightarrow}}{\Box\alpha \Rightarrow \ \mid \neg\Box\alpha \Rightarrow}}{\Rightarrow \neg\Box\alpha \mid \neg\Box\alpha \Rightarrow}}{\Rightarrow \Box\neg\Box\alpha \mid \neg\Box\alpha \Rightarrow}}{\neg\Box\alpha \Rightarrow \Box\neg\Box\alpha \mid \neg\Box\alpha \Rightarrow \Box\neg\Box\alpha}}{\neg\Box\alpha \Rightarrow \Box\neg\Box\alpha}}{\Rightarrow \neg\Box\alpha \rightarrow \Box\neg\Box\alpha}$$

One can also easily show that both rules of Pottinger are derivable in Avron's system:

$$\text{(EC)}\times n\ \dfrac{\text{(IW)}\times n\ \dfrac{\text{(MS)}\times n\ \dfrac{(\text{C}\Rightarrow)\ \dfrac{(\Box\Rightarrow^{A})\ \dfrac{\alpha, \Box\alpha, \Gamma \Rightarrow \Delta \mid \Box\alpha, \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Box\alpha, \Gamma_n \Rightarrow \Delta_n}{\Box\alpha, \Box\alpha, \Gamma \Rightarrow \Delta \mid \Box\alpha, \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Box\alpha, \Gamma_n \Rightarrow \Delta_n}}{\Box\alpha, \Gamma \Rightarrow \Delta \mid \Box\alpha, \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Box\alpha, \Gamma_n \Rightarrow \Delta_n}}{\Box\alpha, \Gamma \Rightarrow \Delta \mid \Box\alpha \Rightarrow \mid \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Box\alpha \Rightarrow \ \mid \Gamma_n \Rightarrow \Delta_n}}{\Box\alpha, \Gamma \Rightarrow \Delta \mid \Box\alpha, \Gamma \Rightarrow \Delta \mid \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Box\alpha, \Gamma \Rightarrow \Delta \mid \Gamma_n \Rightarrow \Delta_n}}{\Box\alpha, \Gamma \Rightarrow \Delta \mid \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n}$$

$$\text{(EC)}\ \dfrac{\text{(IW)}\ \dfrac{(\Rightarrow\Box^{A})\ \dfrac{\Pi \Rightarrow \Delta \mid \Box\Gamma \Rightarrow \alpha \mid G}{\Pi \Rightarrow \Delta \mid \Box\Gamma \Rightarrow \Box\alpha \mid G}}{\Box\Gamma, \Pi \Rightarrow \Box\alpha, \Delta \mid \Box\Gamma, \Pi \Rightarrow \Box\alpha, \Delta \mid G}}{\Box\Gamma, \Pi \Rightarrow \Box\alpha, \Delta \mid G}$$

Soundness of the rules is easy to demonstrate. Let us take (MS) as an example. Assume that $\models \Box\Gamma, \Pi \Rightarrow \Box\Delta, \Sigma \mid G$ and $\not\models \Box\Gamma \Rightarrow \Box\Delta \mid \Pi \Rightarrow \Sigma \mid G$. So there are model $\mathfrak{M} = \langle W_\mathfrak{M}, V_\mathfrak{M}\rangle$ and $w_1, w_2 \in W_\mathfrak{M}$ such that $w_1 \not\models_\mathfrak{M} \Box\Gamma \Rightarrow \Box\Delta$, $w_2 \not\models_\mathfrak{M} \Pi \Rightarrow \Sigma$, and for every $s \in G$ there is $w \in W_\mathfrak{M}$ such that $w \not\models_\mathfrak{M} s$. So $\forall_{\Box\alpha\in\Box\Gamma}\, w_1 \vDash \Box\alpha$ and $\forall_{\Box\beta\in\Box\Delta}\, w_1 \nvDash \Box\beta$, which implies that it holds for each $w \in W_\mathfrak{M}$, in particular for $w_2$. But this falsifies $\Box\Gamma, \Pi \Rightarrow \Box\Delta, \Sigma \mid G$, contrary to the assumption.

Avron provided both a semantic proof of completeness and a constructive proof of cut elimination theorem but in a very sketchy way. In Section 5.1 we provide a detailed presentation of it.

### 4.3. Restall

G. Restall [29] proposed HC for **S5** which is based on his system of proofnets for this logic. It is in a sense the simplest solution to the problem of providing cut-free system for **S5**. Essentially the system applies two rules for introduction of $\Box$:

$$(\Box\Rightarrow^{\mathrm{R}})\ \frac{\alpha, \Gamma \Rightarrow \Delta \mid G}{\Box\alpha \Rightarrow\ \mid \Gamma \Rightarrow \Delta \mid G} \qquad (\Rightarrow\Box^{\mathrm{G}})\ \frac{\Rightarrow \alpha \mid G}{\Rightarrow \Box\alpha \mid G}$$

Soundness of these rules is easy to demonstrate; let us take $(\Box\Rightarrow^{\mathrm{R}})$ as an example.

Assume that $\models \alpha, \Gamma \Rightarrow \Delta \mid G$ and $\not\models \Box\alpha \Rightarrow\ \mid \Gamma \Rightarrow \Delta \mid G$. So there are model $\mathfrak{M} = \langle W_\mathfrak{M}, V_\mathfrak{M}\rangle$ and $w_1, w_2 \in W_\mathfrak{M}$ such that $w_1 \vDash_\mathfrak{M} \Box\alpha$, $w_2 \nvDash_\mathfrak{M} \Gamma \Rightarrow \Delta$, and for every $s \in G$ there is $w \in W_\mathfrak{M}$ such that $w \nvDash_\mathfrak{M} s$. Hence $w_2 \vDash_\mathfrak{M} \alpha$, which falsifies the premiss. But this is contrary to assumption.

Additionaly, Restall's system differs in the selection of structural rules. Both (IC) rules are primitive but instead of (EC) he applies the special rule of *merge* and both external and internal weakening are combined into a pair of rules:

$$(\mathrm{Merge})\ \frac{\Gamma \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi \mid G}{\Gamma, \Sigma \Rightarrow \Delta, \Pi \mid G}$$

$$(\Rightarrow\mathrm{WE})\ \frac{G}{\Rightarrow \alpha \mid G} \qquad (\mathrm{WE}\Rightarrow)\ \frac{G}{\alpha \Rightarrow\ \mid G}$$

It is clear that his special weakening rules allow for derivability of usual (IW) and (EW) rules with the help of (Merge), also (EC) is derivable by means of (Merge) and (IC). On the other hand Restall's weakening rules are just special instances of (EC) and (Merge) is derivable by (IW) and (EC). Hence one can add Restall's modal rules to our basic HC without changes to obtain an adequate system for **S5**. Below we display proofs of (K) and (5) in his system:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha \Rightarrow \alpha \qquad \beta \Rightarrow \beta}{\alpha \to \beta, \alpha \Rightarrow \beta}\,(\to\Rightarrow)}{\Box(\alpha \to \beta) \Rightarrow \ \mid \alpha \Rightarrow \beta}\,(\Box\Rightarrow^{\mathrm{R}})}{\Box(\alpha \to \beta) \Rightarrow \ \mid \Box\alpha \Rightarrow \ \mid \ \Rightarrow \beta}\,(\Box\Rightarrow^{\mathrm{R}})}{\Box(\alpha \to \beta) \Rightarrow \ \mid \Box\alpha \Rightarrow \ \mid \ \Rightarrow \Box\beta}\,(\Rightarrow\Box^{\mathrm{G}})}{\Box(\alpha \to \beta) \Rightarrow \ \mid \Box\alpha \Rightarrow \Box\beta}\,(\mathrm{Merge})}{\Box(\alpha \to \beta) \Rightarrow \ \mid \ \Rightarrow \Box\alpha \to \Box\beta}\,(\Rightarrow\to)}{\Box(\alpha \to \beta) \Rightarrow \Box\alpha \to \Box\beta}\,(\mathrm{Merge})}{\Rightarrow \Box(\alpha \to \beta) \to \Box\alpha \to \Box\beta}\,(\Rightarrow\to)
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha \Rightarrow \alpha}{\Box\alpha \Rightarrow \ \mid \ \Rightarrow \alpha}\,(\Box\Rightarrow^{\mathrm{R}})}{\Box\alpha \Rightarrow \ \mid \ \Rightarrow \Box\alpha}\,(\Rightarrow\Box^{\mathrm{G}})}{\Rightarrow \neg\Box\alpha \mid \ \Rightarrow \Box\alpha}\,(\Rightarrow\neg)}{\Rightarrow \Box\neg\Box\alpha \mid \ \Rightarrow \Box\alpha}\,(\Rightarrow\Box^{\mathrm{G}})}{\Rightarrow \Box\neg\Box\alpha \mid \neg\Box\alpha \Rightarrow}\,(\neg\Rightarrow)}{\neg\Box\alpha \Rightarrow \Box\neg\Box\alpha}\,(\mathrm{Merge})}{\Rightarrow \neg\Box\alpha \to \Box\neg\Box\alpha}\,(\Rightarrow\to)
$$

One can notice that (Merge) is a very handy rule for constructing more compact proofs as it combines applications of (IW) and (EC); the reader should for example prove again (5) or (B) in Avron's system but using Restall's structural rules instead.

$(\Rightarrow\Box^{\mathrm{G}})$ is just Gödel's rule in hypersequential form and we noticed before that it is sufficient also for Avron's system. As for Restall's $(\Box\Rightarrow^{\mathrm{R}})$ it looks like an extremaly simplified version of (MS) combined with Avron's $(\Box\Rightarrow^{\mathrm{A}})$ hence we can look at Restall's system as a drastic simplification of Avron's system keeping only the minimal resources necessary for adequate characterization of **S5**. Moreover, Restall's rules are better behaving from the standpoint of proving cut elimination. We will focus on these matters in Section 5.2.

Restall provided also a variant of his system in the spirit of Kleene's solution for constructive semantic proof of completeness. In case of rules for Boolean constants it comprises in repeating principal formulae in premises; in case of modals we must repeat the whole sequents which yields the following rules:

$$(\Box{\Rightarrow}^{R^*})\ \frac{\Box\alpha, \Pi \Rightarrow \Sigma \mid \alpha, \Gamma \Rightarrow \Delta \mid G}{\Box\alpha, \Pi \Rightarrow \Sigma \mid \Gamma \Rightarrow \Delta \mid G}$$

$$({\Rightarrow}\Box^{R^*})\ \frac{\Gamma \Rightarrow \Delta, \Box\alpha \mid\ \Rightarrow \alpha \mid G}{\Gamma \Rightarrow \Delta, \Box\alpha \mid G}$$

Note that in this form both rules exhibit rather a semantical character in contrast to original ones.

### 4.4. Poggiolesi

Poggiolesi [26] cut-free HS for **S5** is more oriented on actual proof/model-search and her rules are modelled on the behaviour of modal constants in Kripke frames with universal relation of accessibility. Hence instead of structural rules the stress is put on logical rules and the system is fully logical (similarly as Pottinger's system) in having only logical rules as primitive. In fact, she needs structural rules like (Merge) for providing a syntactical proof of cut-elimination; but these are not primitive but admissible, in contrast to Restall's system. The price for that is that she needs two rules for introducing $\Box$ in the antecedent. One of the rules for introducing $\Box$ in the antecedent is just $(\Box{\Rightarrow}^{R^*})$; the remaining two rules are:

$$(\Box{\Rightarrow}^{P^*})\ \frac{\alpha, \Box\alpha, \Gamma \Rightarrow \Delta \mid G}{\Box\alpha, \Gamma \Rightarrow \Delta \mid G} \qquad ({\Rightarrow}\Box^{P^*})\ \frac{\Gamma \Rightarrow \Delta \mid\ \Rightarrow \alpha \mid G}{\Gamma \Rightarrow \Delta, \Box\alpha \mid G}$$

The rule $(\Box{\Rightarrow}^{P^*})$ is a variant of suitable Avron's rule but with $\Box\alpha$ saved in the antecedent of the premise for absorbing the effect of contraction. The rule $({\Rightarrow}\Box^{P^*})$ is like $({\Rightarrow}\Box^{R^*})$ but with no repetition of $\Box\alpha$ in the premiss. It is evident from the shape of rules that semantical process of creating new worlds (via $({\Rightarrow}\Box^{P^*})$) and their saturation with necessary formulae taken from other worlds is separated and admits systematic model construction.[4] There are also some minor changes in the basic HC for **CPL**. As we mentioned there are no primitive structural rules, axioms are of the form $\Gamma, p \Rightarrow p, \Delta \mid G$ and one-premiss rules for $\wedge$ and $\vee$ are multiplicative in order to satisfy invertibility of all logical rules.

[4] Similar idea was applied, e.g. in Indrzejczak [14] where instead of hypersequents a double sequents apparatus was applied to obtain cut-free system for **S5**; see Poggiolesi [27] for discussion of possible translations of several solutions.

The proof of (5) looks like that:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow \mid \alpha \Rightarrow \alpha \mid \Box\alpha \Rightarrow}{\Rightarrow \mid \ \Rightarrow \alpha \mid \Box\alpha \Rightarrow}(\Box{\Rightarrow}^{R^*})}{\Rightarrow \Box\alpha \mid \Box\alpha \Rightarrow}({\Rightarrow}\Box^{P^*})}{\Rightarrow \Box\alpha \mid \ \Rightarrow \neg\Box\alpha}({\Rightarrow}\neg)}{\Rightarrow \Box\alpha, \Box\neg\Box\alpha}({\Rightarrow}\Box^{P^*})}{\neg\Box\alpha \Rightarrow \Box\neg\Box\alpha}(\neg{\Rightarrow})}{\Rightarrow \neg\Box\alpha \to \Box\neg\Box\alpha}({\Rightarrow}{\to})
$$

The comparison with Restall's system, particularly in his second variant is very instructive. At first one can suspect that Restall's rules are insufficient but this is not true. One can derive $(\Box{\Rightarrow}^{P^*})$ in his system in the following way:

$$
\dfrac{\dfrac{\dfrac{\alpha, \Box\alpha, \Gamma \Rightarrow \Delta \mid G}{\Box\alpha \Rightarrow \ \mid \Box\alpha, \Gamma \Rightarrow \Delta \mid G}(\Box{\Rightarrow}^{R})}{\Box\alpha, \Box\alpha, \Gamma \Rightarrow \Delta \mid G}(\text{Merge})}{\Box\alpha, \Gamma \Rightarrow \Delta \mid G}(C{\Rightarrow})
$$

This may lead to the question if Poggiolesi's system is not redundant. No, it is not since it has no structural rules as primitive. Her system in the most direct way encodes semantical features of **S5** in terms of syntactical rules for modals. Moreover, despite of its semantical motivation, it allows for very elegant syntactic proof of admissibility of cut along the lines of Dragalin's proof. We will sketch it in Section 5.2.

One may consider if we can avoid apparent inelegancy of having two rules for introduction of ($\Box$ into antecedent in the system. It seems that using Pottinger's rule $(\Box{\Rightarrow}^{P})$ which combines the features of both Poggiolesi's rules may work. Unfortunately if we make this change the proof of height-preserving admissibility of (Merge) fails. Hence we must have (Merge) (or (EC)) as primitive rule(s), similarly as in Restall's system.

### 4.5. Lahav

Lahav [19] presents a general method for generating hypersequent rules from some frame conditions. His basic system for **K** is defined on sequents built from sets so contraction is implicit but both (IW) and (EW) are primitive. The only modal rule is:

$$
({\Rightarrow}\ \Box^{K}) \quad \dfrac{G \mid \Gamma \Rightarrow \alpha}{G \mid \Box\Gamma \Rightarrow \Box\alpha}
$$

In particular, his solution for **S5** is based on the addition of the following rule encoding the property of universality:

$$
(\mathrm{U})\ \dfrac{\mathit{\Gamma}, \mathit{\Pi} \Rightarrow \mathit{\Delta} \mid G}{\mathit{\Lambda}, \Box\mathit{\Pi} \Rightarrow \mathit{\Sigma} \mid \mathit{\Gamma}, \Box\mathit{\Xi} \Rightarrow \mathit{\Delta} \mid G}
$$

One can prove (5) in the following way:

$$
(\Rightarrow\to)\ \dfrac{(\mathrm{EC})\ \dfrac{(\mathrm{IW})\ \dfrac{(\neg\Rightarrow)\ \dfrac{(\Rightarrow\Box^{\mathrm{K}})\ \dfrac{(\Rightarrow\Box^{\mathrm{K}})\ \dfrac{(\Rightarrow\neg)\ \dfrac{(\mathrm{U})\ \dfrac{\alpha \Rightarrow \alpha}{\Box\alpha \Rightarrow\ \mid\ \Rightarrow \alpha}}{\Rightarrow \neg\Box\alpha \mid \Rightarrow \alpha}}{\Rightarrow \neg\Box\alpha \mid \Rightarrow \Box\alpha}}{\Rightarrow \Box\neg\Box\alpha \mid \Rightarrow \Box\alpha}}{\Rightarrow \Box\neg\Box\alpha \mid \neg\Box\alpha \Rightarrow}}{\neg\Box\alpha \Rightarrow \Box\neg\Box\alpha \mid \neg\Box\alpha \Rightarrow \Box\neg\Box\alpha}}{\neg\Box\alpha \Rightarrow \Box\neg\Box\alpha}}{\Rightarrow \neg\Box\alpha \to \Box\neg\Box\alpha}
$$

There is no syntactic proof of cut admisibility; all adequacy proofs are semantical. Closer inspection shows that Lahav's specific rule for **S5** may be seen as a (weaker) variant of (MS) with additional deletion of $\Box$ in elements of $\mathit{\Pi}$ in the premiss. In fact (U) may be easily derived in Avron's system:

$$
(\mathrm{IW})\ \dfrac{(\mathrm{MS})\ \dfrac{(\Box\Rightarrow)\ \dfrac{\mathit{\Gamma}, \mathit{\Pi} \Rightarrow \mathit{\Delta} \mid G}{\mathit{\Gamma}, \Box\mathit{\Pi} \Rightarrow \mathit{\Delta} \mid G}}{\Box\mathit{\Pi} \Rightarrow\ \mid \mathit{\Gamma} \Rightarrow \mathit{\Delta} \mid G}}{\mathit{\Lambda}, \Box\mathit{\Pi} \Rightarrow \mathit{\Sigma} \mid \mathit{\Gamma}, \Box\Xi \Rightarrow \mathit{\Delta} \mid G}
$$

To derive Avron's rules in Lahav's system we need to use cut, which is interpreted by the author as showing that his system is in a sense stronger as it implies the admissibility of cut in Avron's calculus. Note that the qualification of this rule as a weaker version of (MS) is not connected with the lack of $\Box$ in front of elements of $\mathit{\Pi}$ in the premiss but with the fact that only boxed formulae from the antecedent (of one of the sequent in the conclusion) are put in the antecedent of the premiss (without boxes however). In the next subsection we will explain what advantages follow from such modification.

### 4.6. Kurokawa

Recently a paper [18] of Kurokawa shows HC for some extensions of **S4** including **S5**. His basic system is then exactly like Avron's calculus but instead of (MS) he is using its weaker version:

$$(\mathrm{MS}^{\mathrm{K}})\ \ \frac{\Box\mathit{\Gamma}, \mathit{\Pi} \Rightarrow \mathit{\Sigma} \mid G}{\Box\mathit{\Gamma} \Rightarrow \ \mid \mathit{\Pi} \Rightarrow \mathit{\Sigma} \mid G}$$

In contrast to Lahav's rule (U) we can observe immediately in what sense it is a weaker version of (MS). One should check again the proof of (5) in Avron's system to see that the omission of boxed formulae in the succedent of the premiss and (one of the) conclusion sequent makes no harm. Below we put a proof of (B) to show that such a solution works:

$$\begin{array}{rc}
 & \alpha \Rightarrow \alpha \\
(\Box\Rightarrow) & \overline{\Box\alpha \Rightarrow \alpha} \\
(\neg\Rightarrow) & \overline{\Box\alpha, \neg\alpha \Rightarrow} \\
(\mathrm{MS}^{\mathrm{K}}) & \overline{\Box\alpha \Rightarrow \ \mid \neg\alpha \Rightarrow} \\
(\Rightarrow\neg) & \overline{\Rightarrow \neg\Box\alpha \mid \neg\alpha \Rightarrow} \\
(\Rightarrow\Box^{\mathrm{A}}) & \overline{\Rightarrow \Box\neg\Box\alpha \mid \neg\alpha \Rightarrow} \\
IW \times 2 & \overline{\neg\alpha \Rightarrow \Box\neg\Box\alpha \mid \neg\alpha \Rightarrow \Box\neg\Box\alpha} \\
(\mathrm{EC}) & \overline{\neg\alpha \Rightarrow \Box\neg\Box\alpha} \\
(\Rightarrow\to) & \overline{\Rightarrow \neg\alpha \to \Box\neg\Box\alpha}
\end{array}$$

Kurokawa proves eliminability of cut for his system using a general method of cut elimination for HC proposed first in Metcalfe, Olivetti, and Gabbay [20] for fuzzy logics and applied in Ciabattoni, Metcalfe, and Montagna [9] and in Indrzejczak [16] to some modal logics. In the next section we will explain in what way the modification of (MS) makes HC for **S5** better behaving with respect to syntactic proofs of cut elimination.

Summing up our observations one may notice that there are two kinds of HC systems for **S5**. An approach of Pottinger, characteristic also for Restall's and Poggiolesi's system consists in providing special modal rules introducing $\Box$ just for **S5** on the basis of HC for **CPL**. On the other hand, the approach of Avron, characteristic also for Kurokawa and Lahav, builds the system for **S5** by means of a special quasi-structural rule added to HC system which is already equipped with for modal rules introducing $\Box$ adequate for **S4** or even **K**. In general, Pottinger's approach is semantically oriented and Avron's approach is syntactically oriented, but one should note that Restal's system has also syntactical character, whereas Lahav's characteristic rule is obtained by means of translation from semantical condition.

## 5. Cut Elimination

A lot of different methods for proving cut elimination/admissibility in the framework of HC was offered so far. Some syntactical proofs of cut admissibility (elimination) for HC are performed by means of suitable technique for tracing the cut-formula through a proof (see e.g. the "history technique" of Avron [1] or the "decoration technique" of Baaz and Ciabattoni [7]); these were not applied to HC for **S5** and we will not discuss them below.

Another strategy is based on the application of a special multicut version suitable for hypersequents (see e.g. Avron [2]). We will present in detail the version from Avron [3] in Section 5.1.

An original proof of cut elimination for hypersequent formalization of **S5** was provided by Poggiolesi [27] which is based on Dragalin's method (as presented e.g. in Negri and von Plato [22]) and avoids both solutions. It requires (mostly) invertible rules which makes possible to avoid contraction as a primitive rule.

Very general and elegant method of proving cut elimination in the presence of contraction for numerous fuzzy logics was presented in Metcalfe, Olivetti, and Gabbay [20]. The method is in some sense a half-way between original Gentzen's proof of cut elimination and proofs based on global transformations of derivations like in Curry [11]. In contrast to Gentzen's proof we eliminate cut rule, not multicut (or Mix) being its generalization absorbing contraction. But the proof is strongly based on the predefined notion of "substitutivity" of rules where the result of multiple applications of cut is absorbed. The adaptation of this method of proving cut elimination to extensions of t-norm logic **MTL** and related fuzzy logics with truth stresser modalities was provided by Ciabattoni, Metcalfe and Montagna [9]. This proof is particularly important since it deals with rules which are not "substitutive" in the sense of [20] and modal rules of Kurokawa are also of this kind which makes possible an application of this strategy to his system for **S5**.

In order for easier comparison of different strategies let us distinguish between different forms of cut encountered in the framework of HC. The most direct adaptation of a standard (multiplicative) cut is (H-Cut):

$$\text{(H-Cut)} \quad \frac{G \mid \Gamma \Rightarrow \Delta, \alpha \qquad H \mid \alpha, \Sigma \Rightarrow \Pi}{G \mid H \mid \Gamma, \Sigma \Rightarrow \Delta, \Pi}$$

As is well known in order to deal with the application of contraction, Gentzen introduced instead a rule Mix which allows for cutting all occurrences of suitable formula in one step. Its hypersequential counterpart is (H-Mix):

$$\text{(H-Mix)} \quad \frac{G \mid \Gamma \Rightarrow \Delta, [\alpha]^{\lambda} \qquad H \mid [\alpha]^{\mu}, \Sigma \Rightarrow \Pi}{G \mid H \mid \Gamma, \Sigma \Rightarrow \Delta, \Pi}$$

where $\lambda$ and $\mu$ are used to denote the number of occurrences of $\varphi$. We tacitly assume that there is no other occurrences of $\varphi$ in $\Delta$ and $\Sigma$.

Similarly as in the standard sequent calculus both rules are equivalent in the presence of weakening and contraction, we thus have:

LEMMA 2. *$G$ is provable in* HC *with* (H-Cut) *iff $G$ is provable in* HC *with* (H-Mix).

One can also consider a hypersequent version of Multicut rule which allows for cutting more then one occurrence of some formula but, in contrast to Mix, not necessarily all. Clearly this version is also equivalent to the version with (H-Cut).

In HC we can have also rules like (EC) or (Merge) as primitive which introduces additional complications. In order to deal with them Avron introduced yet more general version of (H-Mix), which we call here (SH-Mix) ('S' for strong):

(SH-Mix)

$$\frac{G \mid \Gamma_1 \Rightarrow \Delta_1, [\alpha]^{\lambda_1} \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n, [\alpha]^{\lambda_n} \quad H \mid [\alpha]^{\mu_1}, \Sigma_1 \Rightarrow \Pi_1 \mid \ldots \mid [\alpha]^{\mu_k}, \Sigma_k \Rightarrow \Pi_k}{G \mid H \mid \Gamma_1, \ldots, \Gamma_n, \Sigma_1, \ldots, \Sigma_k \Rightarrow \Delta_1, \ldots, \Delta_n, \Pi_1, \ldots, \Pi_k}$$

In this way we can cut not only multiple occurrences of a formula in one sequent of a premise but in many sequents in one step. If we do not require deletion of $[\alpha]$ in all sequents we can call it (SH-Multicut).

Note that in case of these rules the situation is a bit different with respect to their strength; we have:

LEMMA 3. *If $G$ is provable in* HC *with* (SH-Mix)*, then $G$ is provable in* HC *with* (H-Cut).

PROOF. In order to simulate an application of (SH-Mix) with (H-Cut) it is enough to apply successively (IW), (EC), and (IC) to each premise. This way from the left premiss $G \mid \Gamma_1 \Rightarrow \Delta_1, [\alpha]^{\lambda_1} \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n, [\alpha]^{\lambda_n}$

we obtain $G \mid \Gamma_1, \dots, \Gamma_n \Rightarrow \Delta_1, \dots \Delta_n, \alpha$, and similarly for the right premiss. From these hypersequents the result of the application of (SH-Mix) follows by (H-Cut). $\dashv$

However, not every application of (H-Cut) may be simulated by (SH-Mix). In order to have equal strength of rules we must rather use (SH-Multicut) since in this case (H-Cut) is just a special case of it and we obtain:

Lemma 4. *$G$ is provable in* HC *with* (H-Cut) *iff $G$ is provable in HC with* (SH-Multicut)*.*

Let us notice that from the proof of cut elimination in Kurokawa's system yet another rule (or rather a pair of rules) may be extracted which we call (WH-Mix) ('W' for weak):

(WH-Mix)

$$\frac{G \mid \Gamma \Rightarrow \Delta, \alpha \qquad H \mid [\alpha]^{\mu_1}, \Sigma_1 \Rightarrow \Pi_1 \mid \dots \mid [\alpha]^{\mu_k}, \Sigma_k \Rightarrow \Pi_k}{G \mid H \mid \Gamma^{\mu_1}, \Sigma_1 \Rightarrow \Pi_1, \Delta^{\mu_1} \mid \dots \mid \Gamma^{\mu_k}, \Sigma_k \Rightarrow \Pi_k, \Delta^{\mu_k}}$$

$$\frac{G \mid \Gamma_1 \Rightarrow \Delta_1, [\alpha]^{\lambda_1} \mid \dots \mid \Gamma_n \Rightarrow \Delta_n, [\alpha]^{\lambda_n} \qquad H \mid \alpha, \Sigma \Rightarrow \Pi}{G \mid H \mid \Sigma^{\lambda_1}, \Gamma_1 \Rightarrow \Delta_1, \Pi^{\lambda_1} \mid \dots \mid \Sigma^{\lambda_n}, \Gamma_n \Rightarrow \Delta_n, \Pi^{\lambda_n}}$$

and again in the version which does not require deletion of $[\alpha]$ in all sequents of one of the premisses we can call it (WH-Multicut). The relations between these rules and (H-Cut) are the same as in case of (SH-Mix) and (SH-Multicut).

Because of the defficiences of different versions of Mix stated above, in what follows we prefer to use corresponding Multicut rules which are strictly equivalent to ordinary cut. We start with a detailed presentation of Avron's proof.

### 5.1. Avron's proof

The most interesting point with Avron's rules is that despite the generalization of (H-Mix) to (SH-Mix) to deal with (EC) he was still unable to prove syntactic proof of cut elimination. Let us look at the following figure:

$$(\text{SH}-\text{Multicut})\ \frac{(\text{MS})\ \dfrac{G \mid \Gamma, \Box\Pi \Rightarrow \Delta, \Box\Sigma, \Box\alpha}{G \mid \Gamma \Rightarrow \Delta \mid \Box\Pi \Rightarrow \Box\Sigma, \Box\alpha} \qquad H \mid \Box\alpha, \Lambda \Rightarrow \Theta}{G \mid H \mid \Gamma \Rightarrow \Delta \mid \Box\Pi, \Lambda \Rightarrow \Box\Sigma, \Theta}$$

where we assume that $\Lambda$ and $\Theta$ consist of nonmodal formulae and that $\Box\alpha$ does not belong to $\Delta$. If we want to reduce the height of (SH-Multicut) we obtain:

$$\text{(SH-Multicut)}\ \dfrac{G \mid \Gamma, \Box\Pi \Rightarrow \Delta, \Box\Sigma, \Box\alpha \qquad H \mid \Box\alpha, \Lambda \Rightarrow \Theta}{\text{(MS)}\ \dfrac{G \mid H \mid \Gamma, \Box\Pi, \Lambda \Rightarrow \Delta, \Box\Sigma, \Theta}{G \mid H \mid \Gamma, \Lambda \Rightarrow \Delta, \Theta \mid \Box\Pi \Rightarrow \Box\Sigma}}$$

From the last hypersequent we have no way to obtain the last sequent of the original proof. In order to deal with the problem Avron restricted the application of (SH-Mix) (or rather (SH-Multicut)) to nonmodal formulae and introduced one more special form of mix for cutting boxed formulae which we call (BSH-Mix) (resp. (BSH-Multicut)) ('B' for boxed):

(BSH-Multicut)

$$\dfrac{G \mid \Gamma_1 \Rightarrow \Delta_1, [\Box\alpha]^{\lambda_1} \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n, [\Box\alpha]^{\lambda_n} \quad H \mid [\Box\alpha]^{\mu_1}, \Sigma_1 \Rightarrow \Pi_1 \mid \ldots \mid [\Box\alpha]^{\mu_k}, \Sigma_k \Rightarrow \Pi_k}{G \mid H \mid \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n \mid \Sigma_1 \Rightarrow \Pi_1 \mid \ldots \mid \Sigma_k \Rightarrow \Pi_k}$$

As we can observe the additional rule follows rather the schema of (WH-Mix) or (WH-Multicut) in keeping sequents from premisses still isolated. However one can easilly check that this rule is sound only for **S5** hence it cannot be used in general as an admissible form of cut for HC.

Strictly speaking, Avron did not introduce (BSH-Mix) (or even (SH-Mix)) as special rules but rather demonstrate the admissibility of (H-Cut) by means of more general theorem where both forms of Mix are involved in the induction hypothesis. However it seems to be most transparent to define a special HC calculus with both rules explicitly formulated. We follow in this respect the form of presentation of HC for Gödel logics in Baaz, Ciabattoni and Fermüller [8].

Let us call HC2S5 a system of Avron with both forms of Multicut, i.e. (SH-Multicut) and (BSH-Multicut), whereas the system of Avron with nonrestricted applications of (SH-Multicut) is called just HCS5. For brevity we will use names (SHM) and (BSHM) in proof figures. We will show:

Theorem 1. *$G$ is provable in* HCS5 *iff $G$ is provable in* HC2S5.

Proof. From left to right it is enough to show that any application of (SH-Multicut) with modal formula as cut-formula may be simulated by (BSH-Multicut). It works like that, where $*$ is (BSH-Multicut):

$$
\dfrac{G \mid \Gamma_1 \Rightarrow \Delta_1, [\Box\alpha]^{\lambda_1} \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n, [\Box\alpha]^{\lambda_n} \qquad H \mid [\Box\alpha]^{\mu_1}, \Sigma_1 \Rightarrow \Pi_1 \mid \ldots \mid [\Box\alpha]^{\mu_k}, \Sigma_k \Rightarrow \Pi_k}{\text{IW, EC}\ \dfrac{G \mid H \mid \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n \mid \Sigma_1 \Rightarrow \Pi_1 \mid \ldots \mid \Sigma_k \Rightarrow \Pi_k}{G \mid H \mid \Gamma_1, \ldots, \Gamma_n, \Sigma_1, \ldots, \Sigma_k \Rightarrow \Delta_1, \ldots, \Delta_n, \Pi_1, \ldots, \Pi_k}} *
$$

From right to left it is sufficient to show that every application of (BSH-Multicut) may be simulated by (SH-Multicut). It looks like that:

$$
\begin{array}{l}
(\text{MS})\ \dfrac{G \mid \Gamma_1 \Rightarrow \Delta_1, [\Box\alpha]^{\lambda_1} \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n, [\Box\alpha]^{\lambda_n}}{G \mid \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n \mid\ \Rightarrow [\Box\alpha]^{\lambda_1} \mid \ldots \mid\ \Rightarrow [\Box\alpha]^{\lambda_n}} \qquad \dfrac{(\dagger)}{(\ddagger)} \\
(\text{SHM})\ \overline{G \mid H \mid \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n \mid\ \Rightarrow\ \mid \Sigma_1 \Rightarrow \Pi_1 \mid \ldots \mid \Sigma_k \Rightarrow \Pi_k} \\
(\text{IW})\ \overline{G \mid H \mid \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n \mid \Gamma_n \Rightarrow \Delta_n \mid \Sigma_1 \Rightarrow \Pi_1 \mid \ldots \mid \Sigma_k \Rightarrow \Pi_k} \\
(\text{EC})\ \overline{G \mid H \mid \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n \mid \Sigma_1 \Rightarrow \Pi_1 \mid \ldots \mid \Sigma_k \Rightarrow \Pi_k}
\end{array}
$$

where $\frac{(\dagger)}{(\ddagger)}$ is:

$$
(\text{MS}) \times n\ \dfrac{H \mid [\Box\alpha]^{\mu_1}, \Sigma_1 \Rightarrow \Pi_1 \mid \ldots \mid [\Box\alpha]^{\mu_k}, \Sigma_k \Rightarrow \Pi_k}{H \mid [\Box\alpha]^{\mu_1} \Rightarrow\ \mid \ldots \mid [\Box\alpha]^{\mu_k} \Rightarrow\ \mid \Sigma_1 \Rightarrow \Pi_1 \mid \ldots \mid \Sigma_k \Rightarrow \Pi_k}
$$

$\dashv$

In proofs of cut elimination dealing with Mix usually a subsidiary induction is on the rank of a Mix, a parameter introduced by Gentzen in his original proof. Avron is not very precise in his original presentation but he claims that the proof is by induction on the height. In fact it was shown by Girard [13] how to provide such a proof for ordinary sequent calculus with Mix hence below we will adapt this method to HC. It has the advantage that rank is a more complicated measure already for ordinary SC and its adaptation to hypersequent calculi encounters further difficulties whereas such a measure like height is generally simple. Hence we provide the proof of admissibility of cut by induction on the complexity of cut-formula and the height of the premisses of applied Multicut.

Theorem 2. Cut *admissibility holds for* HC2S5, *and hence* HCS5*.*

The proof is by induction on the complexity of cut-formula $\alpha$ and the sum of heights of proofs of both premisses of (SH-Multicut) (or (BSH-Multicut)). We divide it into three main parts:

1. at least one premiss is an axiom;
2. in at least one premiss all occurrences of cut-formula are parametric;
3. in both premisses cut-formula is principal.

*Ad* 1. If at least one premiss is an axiom, we have two situations:
(a) $\alpha$ is nonmodal:

$$\text{(SHM)}\ \frac{\alpha \Rightarrow \alpha \qquad [\alpha]^{\lambda_1}, \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid [\alpha]^{\lambda_n}, \Gamma_n \Rightarrow \Delta_n \mid H}{\alpha, \Gamma_1 ..., \Gamma_n, \Rightarrow \Delta_1, ..., \Delta_n \mid H}$$

and the conclusion follows from the right premiss by (IW), (EC), (IC).
(b) When $\alpha = \Box\beta$:

$$\text{(BSHM)}\ \frac{\Box\beta \Rightarrow \Box\beta \qquad [\Box\beta]^{\lambda_1}, \Gamma_1 \Rightarrow \Delta_1 \mid ... \mid [\Box\beta]^{\lambda_n}, \Gamma_n \Rightarrow \Delta_n \mid H}{\Box\beta \Rightarrow \mid \Gamma_1 \Rightarrow \Delta_1 \mid ... \mid \Gamma_n \Rightarrow \Delta_n \mid H}$$

and the conclusion follows from the right premiss by $n$ applications of (MS) followed by (EC) and (IC). Cases of axiomatic right premisses are dual.

*Ad* 2. If in at least one premiss all occurrences of $\alpha$ are parametric, then we apply reduction on the height of this premiss. Below we consider some cases as examples.

(a) Let the last rule be (EC) in the left premiss. Assume that $\alpha$ is nonmodal:

$$\text{(SHM)}\ \frac{\text{(EC)}\ \dfrac{G \mid \Gamma_1 \Rightarrow \Delta_1, [\alpha]^{\lambda_1} \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n, [\alpha]^{\lambda_n} \mid \Gamma_n \Rightarrow \Delta_n, [\alpha]^{\lambda_n}}{G \mid \Gamma_1 \Rightarrow \Delta_1, [\alpha]^{\lambda_1} \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n, [\alpha]^{\lambda_n}} \qquad (\star)}{G \mid H \mid \Gamma_1, \ldots, \Gamma_n, \Sigma_1, \ldots, \Sigma_k \Rightarrow \Delta_1, \ldots, \Delta_n, \Pi_1, \ldots, \Pi_k}$$

where

$$(\star) \text{ is } H \mid [\alpha]^{\mu_1}, \Sigma_1 \Rightarrow \Pi_1 \mid \ldots \mid [\alpha]^{\mu_k}, \Sigma_k \Rightarrow \Pi_k.$$

We reduce the height on the left obtaining:

$$\text{(IC)}\ \frac{\text{(SHM)}\ \dfrac{G \mid \Gamma_1 \Rightarrow \Delta_1, [\alpha]^{\lambda_1} \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n, [\alpha]^{\lambda_n} \mid \Gamma_n \Rightarrow \Delta_n, [\alpha]^{\lambda_n} \qquad (\star)}{G \mid H \mid \Gamma_1, \ldots, \Gamma_n, \Gamma_n, \Sigma_1, \ldots, \Sigma_k \Rightarrow \Delta_1, \ldots, \Delta_n, \Delta_n, \Pi_1, \ldots, \Pi_k}}{G \mid H \mid \Gamma_1, \ldots, \Gamma_n, \Sigma_1, \ldots, \Sigma_k \Rightarrow \Delta_1, \ldots, \Delta_n, \Pi_1, \ldots, \Pi_k}$$

If $\alpha$ is modal, then we apply (BSHM) and after transformation we must use (EC) instead.

(b) Let us consider (MS) with $\alpha$ not modal:

$$\text{(SHM)}\ \frac{\text{(MS)}\ \dfrac{G \mid \Box\Gamma, \Gamma_1 \Rightarrow \Box\Delta, \Delta_1, [\alpha]^{\lambda_1} \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n, [\alpha]^{\lambda_n}}{G \mid \Box\Gamma \Rightarrow \Box\Delta \mid \Gamma_1 \Rightarrow \Delta_1, [\alpha]^{\lambda_1} \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n, [\alpha]^{\lambda_n}} \qquad (\star)}{G \mid H \mid \Box\Gamma \Rightarrow \Box\Delta \mid \Gamma_1, ..., \Gamma_n, \Sigma_1, ..., \Sigma_k \Rightarrow \Delta_1, ..., \Delta_n, \Pi_1, ..., \Pi_k}$$

which after transformation looks like that:

$$\dfrac{\text{(MS)}\ \dfrac{\text{(SHM)}\ \dfrac{G \mid \Box\Gamma, \Gamma_1 \Rightarrow \Box\Delta, \Delta_1, [\alpha]^{\lambda_1} \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n, [\alpha]^{\lambda_n} \qquad (\star)}{G \mid H \mid \Box\Gamma, \Gamma_1, \ldots, \Gamma_n, \Sigma_1, \ldots, \Sigma_k \Rightarrow \Box\Delta, \Delta_1, \ldots, \Delta_n, \Pi_1, \ldots, \Pi_k}}{}}{G \mid H \mid \Box\Gamma \Rightarrow \Box\Delta \mid \Gamma_1, \ldots, \Gamma_n, \Sigma_1, \ldots, \Sigma_k \Rightarrow \Delta_1, \ldots, \Delta_n, \Pi_1, \ldots, \Pi_k}$$

If cut-formula is modal:

$$\text{(BSHM)}\ \dfrac{\text{(MS)}\ \dfrac{G \mid \Box\Gamma, \Gamma_1 \Rightarrow \Box\Delta, \Delta_1, [\Box\alpha]^{\lambda_1} \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n, [\Box\alpha]^{\lambda_n}}{G \mid \Box\Gamma \Rightarrow \Box\Delta, [\Box\alpha]^{\lambda_1} \mid \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n, [\Box\alpha]^{\lambda_n}} \qquad (\star\star)}{G \mid H \mid \Box\Gamma \Rightarrow \Box\Delta \mid \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n \mid \Sigma_1 \Rightarrow \Pi_1 \mid \ldots \mid \Sigma_k \Rightarrow \Pi_k}$$

where

$$(\star\star) \text{ is } H \mid [\Box\alpha]^{\mu_1}, \Sigma_1 \Rightarrow \Pi_1 \mid \ldots \mid [\Box\alpha]^{\mu_k}, \Sigma_k \Rightarrow \Pi_k.$$

and after transformation:

$$\text{(MS)}\ \dfrac{\text{(BSHM)}\ \dfrac{G \mid \Box\Gamma, \Gamma_1 \Rightarrow \Box\Delta, \Delta_1, [\Box\alpha]^{\lambda_1} \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n, [\Box\alpha]^{\lambda_n} \qquad (\star\star)}{G \mid H \mid \Box\Gamma, \Gamma_1 \Rightarrow \Box\Delta, \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n \mid \Sigma_1 \Rightarrow \Pi_1 \mid \ldots \mid \Sigma_k \Rightarrow \Pi_k}}{G \mid H \mid \Box\Gamma \Rightarrow \Box\Delta \mid \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n \mid \Sigma_1 \Rightarrow \Pi_1 \mid \ldots \mid \Sigma_k \Rightarrow \Pi_k}$$

In the case of other structural and logical rules transformations are similar and require no more than a permutation of respective rule with (SHM) or (BSHM). For illustration we consider the situation when the main formula is modal.

(c) Consider the application of $(\Box \Rightarrow)$ on the left with $\alpha$ non-modal:

$$\text{(SHM)}\ \dfrac{(\Box\Rightarrow)\ \dfrac{G \mid \beta, \Gamma_1 \Rightarrow \Delta_1, [\alpha]^{\lambda_1} \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n, [\alpha]^{\lambda_n}}{G \mid \Box\beta, \Gamma_1 \Rightarrow \Delta_1, [\alpha]^{\lambda_1} \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n, [\alpha]^{\lambda_n}} \qquad (\star)}{G \mid H \mid \Box\beta, \Gamma_1, \ldots, \Gamma_n, \Sigma_1, \ldots, \Sigma_k \Rightarrow \Delta_1, \ldots, \Delta_n, \Pi_1, \ldots, \Pi_k}$$

which transforms to:

$$(\Box\Rightarrow)\ \dfrac{\text{(SHM)}\ \dfrac{G \mid \beta, \Gamma_1 \Rightarrow \Delta_1, [\alpha]^{\lambda_1} \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n, [\alpha]^{\lambda_n} \qquad (\star)}{G \mid H \mid \beta, \Gamma_1, \ldots, \Gamma_n, \Sigma_1, \ldots, \Sigma_k \Rightarrow \Delta_1, \ldots, \Delta_n, \Pi_1, \ldots, \Pi_k}}{G \mid H \mid \Box\beta, \Gamma_1, \ldots, \Gamma_n, \Sigma_1, \ldots, \Sigma_k \Rightarrow, \Delta_1, \ldots, \Delta_n, \Pi_1, \ldots, \Pi_k}$$

in case $\alpha$ is modal the only difference is that (BSHM) is applied.

(d) Consider the application of $(\Rightarrow\Box)$. If it is performed on the left, then the active sequent belongs just to the context $G$, since the succedent must contain only the main formula. Hence we consider the situation with the rule applied on the right and with cut-formula being some modal formula in the antecedent of active sequent:

$$\dfrac{(\star\star\star) \qquad \dfrac{H \mid [\Box\alpha]^{\mu_1}, \Box\Sigma_1 \Rightarrow \beta \mid \ldots \mid [\Box\alpha]^{\mu_k}, \Sigma_k \Rightarrow \Pi_k}{H \mid [\Box\alpha]^{\mu_1}, \Box\Sigma_1 \Rightarrow \Box\beta \mid \ldots \mid [\Box\alpha]^{\mu_k}, \Sigma_k \Rightarrow \Pi_k}\,(\Rightarrow\Box)}{G \mid H \mid \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n \mid \Box\Sigma_1 \Rightarrow \Box\beta \mid \ldots \mid \Sigma_k \Rightarrow \Pi_k}\,(\text{BSHM})$$

where

$$(\star\star\star) \text{ is } G \mid \Gamma_1 \Rightarrow \Delta_1, [\Box\alpha]^{\lambda_1} \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n, [\Box\alpha]^{\lambda_n}$$

which after transformation looks like this:

$$\dfrac{\dfrac{(\star\star\star) \qquad H \mid [\Box\alpha]^{\mu_1}, \Box\Sigma_1 \Rightarrow \beta \mid \ldots \mid [\Box\alpha]^{\mu_k}, \Sigma_k \Rightarrow \Pi_k}{G \mid H \mid \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n \mid \Box\Sigma_1 \Rightarrow \beta \mid \ldots \mid \Sigma_k \Rightarrow \Pi_k}\,(\text{BSHM})}{G \mid H \mid \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n \mid \Box\Sigma_1 \Rightarrow \Box\beta \mid \ldots \mid \Sigma_k \Rightarrow \Pi_k}\,(\Rightarrow\Box)$$

*Ad* 3. The hardest cases appear if the premises of (SHM) or (BSHM) were introduced by logical rule and the main formula in both premises is cut-formula. We consider four cases as examples:

(a) The cut-formula $\alpha = \neg\beta$

$$(\text{SHM})\,\dfrac{(\Rightarrow\neg)\,\dfrac{G \mid \Gamma_1, \beta \Rightarrow \Delta_1, [\alpha]^{\lambda_1 - 1} \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n, [\alpha]^{\lambda_n}}{G \mid \Gamma_1 \Rightarrow \Delta_1, [\alpha]^{\lambda_1} \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n, [\alpha]^{\lambda_n}} \qquad \dfrac{(*)}{(**)}}{G \mid H \mid \Gamma_1, \ldots, \Gamma_n, \Sigma_1, \ldots, \Sigma_k \Rightarrow \Delta_1, \ldots, \Delta_n, \Pi_1, \ldots, \Pi_k}$$

where

$$(*) \text{ is } H \mid [\alpha]^{\mu_1 - 1}, \Sigma_1 \Rightarrow \beta, \Pi_1 \mid \ldots \mid [\alpha]^{\mu_k}, \Sigma_k \Rightarrow \Pi_k,$$
$$(**) \text{ is } H \mid [\alpha]^{\mu_1}\alpha, \Sigma_1 \Rightarrow \Pi_1 \mid \ldots \mid [\alpha]^{\mu_k}, \Sigma_k \Rightarrow \Pi_k$$

Now we perform two applications of (SHM) of lower height obtaining two proofs $d_l$ and $d_r$.

$d_l$ has the form:

(SHM)

$$\dfrac{G \mid \Gamma_1, \beta \Rightarrow \Delta_1, [\alpha]^{\lambda_1 - 1} \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n, [\alpha]^{\lambda_n} \qquad H \mid [\alpha]^{\mu_1}\alpha, \Sigma_1 \Rightarrow \Pi_1 \mid \ldots \mid [\alpha]^{\mu_k}, \Sigma_k \Rightarrow \Pi_k}{G \mid H \mid \Gamma_1, \ldots, \Gamma_n, \beta, \Sigma_1, \ldots, \Sigma_k \Rightarrow \Delta_1, \ldots, \Delta_n, \Pi_1, \ldots, \Pi_k}$$

$d_r$ has the form:

(SHM)

$$\dfrac{G \mid \Gamma_1 \Rightarrow \Delta_1, [\alpha]^{\lambda_1} \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n, [\alpha]^{\lambda_n} \qquad H \mid [\alpha]^{\mu_1 - 1}, \Sigma_1 \Rightarrow \beta, \Pi_1 \mid \ldots \mid [\alpha]^{\mu_k}, \Sigma_k \Rightarrow \Pi_k}{G \mid H \mid \Gamma_1, \ldots, \Gamma_n, \Sigma_1, \ldots, \Sigma_k \Rightarrow \Delta_1, \ldots, \Delta_n, \beta, \Pi_1, \ldots, \Pi_k}$$

The last Multicut is made on the formula $\beta$ with lower complexity than $\alpha$. In all cases when considered is a formula with lower complexity it is necessary to distinguish the cases in which it is modal or not.

The first case, $\beta$ is not modal formula: see (I) on page 301. The second case, $\beta$ is modal formula: see (II) on page 301. This way the original application of (SHM) is replaced with three new ones; the first and the second with (SHM) of lower height, and the third of lower complexity which means that all are eliminable by induction hypotheses.

(b) The cut-formula $\alpha = \beta \wedge \gamma$: Our application of (SHM) is of the form:

$$(\mathrm{SHM})\ \frac{d_l \qquad d_r}{G \mid H \mid \Gamma_1, \ldots, \Gamma_n, \Sigma_1, \ldots, \Sigma_k \Rightarrow \Delta_1, \ldots, \Delta_n, \Pi_1, \ldots, \Pi_k}$$

where branch $d_l$ has the form (III) on page 301 and branch $d_r$ has the form:

$$\frac{H \mid [\alpha]^{\mu_1 - 1}, \beta, \Sigma_1 \Rightarrow \Pi_1 \mid \ldots \mid [\alpha]^{\mu_k}, \Sigma_k \Rightarrow \Pi_k}{H \mid [\alpha]^{\mu_1}, \Sigma_1 \Rightarrow \Pi_1 \mid \ldots \mid [\alpha]^{\mu_k}, \Sigma_k \Rightarrow \Pi_k}$$

Again first we perform (SHM) twice, both of lower height, obtaining two new proofs: the branch $d_l1$ has the form (IV) on page 301 and the branch $d_r1$ has the form (V) on page 301.

Now we must cut out $\beta$. First, for the case when it is not modal see (VI) on page 302. Second, for the case when the mix formula is modal see (VII) on page 302.

(c) The cut-formula $\alpha = \beta \rightarrow \gamma$. Our application of (SHM) is of the form:

$$(\mathrm{SHM})\ \frac{d_l \qquad d_r}{G \mid H \mid \Gamma_1, \ldots, \Gamma_n, \Sigma_1, \ldots, \Sigma_k \Rightarrow \Delta_1, \ldots, \Delta_n, \Pi_1, \ldots, \Pi_k}$$

where the branch $d_l$ has the form:

$$\frac{G \mid \Gamma_1, \beta \Rightarrow \gamma, \Delta_1, [\alpha]^{\lambda_1 - 1} \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n, [\alpha]^{\lambda_n}}{G \mid \Gamma_1 \Rightarrow \Delta_1, [\alpha]^{\lambda_1} \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n, [\alpha]^{\lambda_n}}$$

and the branch $d_r$ has the form (VIII) on page 302.

Now we must perform first three cross-cuts reducing height.

The branch $d_l1$ has the form (IX) on page 302. The branch $d_r1$ has the form (X) on page 302. The branch $d_r2$ has the form (XI) on page 303.

Now we must cut out all appearances of $\beta$ and $\gamma$ taking into account if they are modal or not.

First, for the case when both formulas $\beta$ and $\gamma$ are not modal see (XII) on page 303. Second, for the case when $\beta$ is modal and $\gamma$ not see (XIII) on page 303.

Proof for modal $\gamma$ and not modal $\beta$ is analogical, as well as the last case when both formulas are modal.

(d) The cut-formula $\alpha = \Box\beta$. Our application of (BSHM) looks like as (XIV) on page 303.

(I)

$$\text{(EC), (IC)}\ \dfrac{\text{(SHM)}\ \dfrac{d_r \qquad\qquad d_l}{G \mid G \mid H \mid H \mid \mathit{\Gamma}_1, \mathit{\Gamma}_1, \dots, \mathit{\Gamma}_n, \mathit{\Gamma}_n, \mathit{\Sigma}_1, \mathit{\Sigma}_1, \dots, \mathit{\Sigma}_k, \mathit{\Sigma}_k \Rightarrow \mathit{\Delta}_1, \mathit{\Delta}_1, \dots, \mathit{\Delta}_n, \mathit{\Delta}_n \mathit{\Pi}_1, \mathit{\Pi}_1, \dots, \mathit{\Pi}_k \mathit{\Pi}_k}}{G \mid H \mid \mathit{\Gamma}_1, \dots, \mathit{\Gamma}_n, \mathit{\Sigma}_1, \dots \mathit{\Sigma}_k, \Rightarrow \mathit{\Delta}_1, \dots, \mathit{\Delta}_n, \mathit{\Pi}_1, \dots, \mathit{\Pi}_k}$$

(II)

$$\text{(EC)}\ \dfrac{\text{(BSHM)}\ \dfrac{d_l \qquad\qquad d_r}{G \mid G \mid H \mid H \mid \mathit{\Gamma}_1, \dots, \mathit{\Gamma}_n, \mathit{\Sigma}_1, ..., \mathit{\Sigma}_k \Rightarrow \mathit{\Delta}_1, ..., \mathit{\Delta}_n, \mathit{\Pi}_1, ..., \mathit{\Pi}_k \mid \mathit{\Gamma}_1, ..., \mathit{\Gamma}_n, \mathit{\Sigma}_1, ..., \mathit{\Sigma}_k \Rightarrow \mathit{\Delta}_1, ..., \mathit{\Delta}_n, \mathit{\Pi}_1, ..., \mathit{\Pi}_k}}{G \mid H \mid \mathit{\Gamma}_1, \dots, \mathit{\Gamma}_n, \mathit{\Sigma}_1, \dots, \mathit{\Sigma}_k \Rightarrow \mathit{\Delta}_1, \dots, \mathit{\Delta}_n, \mathit{\Pi}_1, \dots, \mathit{\Pi}_k}$$

(III)

$$\dfrac{G \mid \mathit{\Gamma}_1 \Rightarrow \mathit{\Delta}_1, [\alpha]^{\lambda_1 - 1}, \beta \mid \dots \mid \mathit{\Gamma}_n \Rightarrow \mathit{\Delta}_n, [\alpha]^{\lambda_n} \qquad G \mid \mathit{\Gamma}_1 \Rightarrow \mathit{\Delta}_1, [\alpha]^{\lambda_1 - 1}, \gamma \mid \dots \mid \mathit{\Gamma}_n \Rightarrow \mathit{\Delta}_n, [\alpha]^{\lambda_n}}{G \mid \mathit{\Gamma}_1 \Rightarrow \mathit{\Delta}_1, [\alpha]^{\lambda} \mid \dots \mid \mathit{\Gamma}_n \Rightarrow \mathit{\Delta}_n, [\alpha]^{\lambda_n}}$$

(IV)

$$\text{(SHM)}\ \dfrac{G \mid \mathit{\Gamma}_1 \Rightarrow \mathit{\Delta}_1, [\alpha]^{\lambda_1 - 1}, \beta \mid \dots \mid \mathit{\Gamma}_n \Rightarrow \mathit{\Delta}_n, [\alpha]^{\lambda_n} \qquad H \mid [\alpha]^{\mu_1}, \mathit{\Sigma}_1 \Rightarrow \mathit{\Pi}_1 \mid \dots \mid [\alpha]^{\mu_k}, \mathit{\Sigma}_k \Rightarrow \mathit{\Pi}_k}{G \mid H \mid \mathit{\Gamma}_1, \dots, \mathit{\Gamma}_n, \mathit{\Sigma}_1, \dots, \mathit{\Sigma}_k \Rightarrow \mathit{\Delta}_1, \dots, \mathit{\Delta}_n, \mathit{\Pi}_1, \dots, \mathit{\Pi}_k, \beta}$$

(V)

$$\text{(SHM)}\ \dfrac{G \mid \mathit{\Gamma}_1 \Rightarrow \mathit{\Delta}_1, [\alpha]^{\lambda} \mid ... \mid \mathit{\Gamma}_n \Rightarrow \mathit{\Delta}_n, [\alpha]^{\lambda_n} \qquad H \mid [\alpha]^{\mu_1 - 1}, \beta, \mathit{\Sigma}_1 \Rightarrow \mathit{\Pi}_1 \mid ... \mid [\alpha]^{\mu_k}, \mathit{\Sigma}_k \Rightarrow \mathit{\Pi}_k}{G \mid H \mid \beta, \mathit{\Gamma}_1, ..., \mathit{\Gamma}_n, \mathit{\Sigma}_1, ..., \mathit{\Sigma}_k \Rightarrow \mathit{\Delta}_1, ..., \mathit{\Delta}_n, \mathit{\Pi}_1, ..., \mathit{\Pi}_k}$$

(VI)

$$\dfrac{\dfrac{d_l1 \qquad d_r1}{G \mid G \mid H \mid H \mid \Gamma_1, \Gamma_1, \ldots, \Gamma_n, \Gamma_n, \Sigma_1, \Sigma_1, \ldots, \Sigma_k, \Sigma_k \Rightarrow \Delta_1, \Delta_1, \ldots, \Delta_n, \Delta_n, \Pi_1, \Pi_1, \ldots, \Pi_k, \Pi_k}\,(\mathrm{SHM})}{G \mid H \mid \Gamma_1, \ldots, \Gamma_n, \Sigma_1, \ldots, \Sigma_k \Rightarrow \Delta_1, \ldots, \Delta_n, \Pi_1, \ldots, \Pi_k}\,(\mathrm{EC}), (\mathrm{IC})$$

(VII)

$$\dfrac{\dfrac{d_l1 \qquad d_r1}{G \mid G \mid H \mid H \mid \Gamma_1, ..., \Gamma_n, \Sigma_1, ..., \Sigma_k \Rightarrow \Delta_1, ..., \Delta_n, \Pi_1, ..., \Pi_k \mid \Gamma_1, ..., \Gamma_n, \Sigma_1, ..., \Sigma_k \Rightarrow \Delta_1, ..., \Delta_n, \Pi_1, ..., \Pi_k}\,(\mathrm{BSHM})}{G \mid H \mid \Gamma_1, ..., \Gamma_n, \Sigma_1, ..., \Sigma_k \Rightarrow \Delta_1, ..., \Delta_n, \Pi_1, ..., \Pi_k}\,(\mathrm{EC})$$

(VIII)

$$\dfrac{H \mid [\alpha]^{\mu_1 - 1}, \Sigma_1 \Rightarrow \Pi_1, \beta \mid \ldots \mid [\alpha]^{\mu_k}, \Sigma_k \Rightarrow \Pi_k \qquad H \mid \gamma, [\alpha]^{\mu_1 - 1}, \Sigma_1 \Rightarrow \Pi_1, \mid \ldots \mid [\alpha]^{\mu_k}, \Sigma_k \Rightarrow \Pi_k}{H \mid [\alpha]^{\mu_1}, \Sigma_1 \Rightarrow \Pi_1, , \mid \ldots \mid [\alpha]^{\mu_k}, \Sigma_k \Rightarrow \Pi_k}$$

(IX)

$$\dfrac{G \mid \Gamma_1, \beta \Rightarrow \gamma, \Delta_1, [\alpha]^{\lambda_1 - 1} \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n, [\alpha]^{\lambda_n} \qquad H \mid [\alpha]^{\mu_1}, \Sigma_1 \Rightarrow \Pi_1, \mid \ldots \mid [\alpha]^{\mu_k}, \Sigma_k \Rightarrow \Pi_k}{G \mid H \mid \Gamma_1, \ldots, \Gamma_n, \Sigma_1, \ldots, \Sigma_k, \beta \Rightarrow \gamma, \Delta_1, \ldots, \Delta_n, \Pi_1, \ldots, \Pi_k}\,(\mathrm{SHM})$$

(X)

$$\dfrac{G \mid \Gamma_1 \Rightarrow \Delta_1, [\alpha]^{\lambda_1} \mid ... \mid \Gamma_n \Rightarrow \Delta_n, [\alpha]^{\lambda_n} \qquad H \mid [\alpha]^{\mu_1 - 1}, \Sigma_1 \Rightarrow \Pi_1, \beta \mid ... \mid [\alpha]^{\mu_k} \Sigma_k \Rightarrow \Pi_k}{G \mid H \mid G \mid H \mid \Gamma_1, ..., \Gamma_n, \Sigma_1, ..., \Sigma_k \Rightarrow \Delta_1, ..., \Delta_n, \Pi_1, ..., \Pi_k, \beta}\,(\mathrm{SHM})$$

(XI)

$$(\mathrm{SHM})\,\dfrac{G \mid \mathit{\Gamma}_1 \Rightarrow \mathit{\Delta}_1, [\alpha]^{\lambda_1} \mid \dots \mid \mathit{\Gamma}_n \Rightarrow \mathit{\Delta}_n, [\alpha]^{\lambda_n} \qquad H \mid \gamma, [\alpha]^{\mu_1 - 1}, \mathit{\Sigma}_1 \Rightarrow \mathit{\Pi}_1 \mid \dots \mid [\alpha]^{\mu_k}, \mathit{\Sigma}_k \Rightarrow \mathit{\Pi}_k}{G \mid H \mid \gamma, \mathit{\Gamma}_1, \dots, \mathit{\Gamma}_n, \mathit{\Sigma}_1, \dots, \mathit{\Sigma}_k \Rightarrow \mathit{\Delta}_1, \dots, \mathit{\Delta}_n, \mathit{\Pi}_1, \dots, \mathit{\Pi}_k}$$

(XII)

$$\dfrac{\dfrac{\dfrac{\dfrac{d_r1 \qquad d_l1}{G \mid G \mid H \mid H \mid \mathit{\Gamma}_1, \mathit{\Gamma}_1, ..., \mathit{\Gamma}_n, \mathit{\Gamma}_n, \mathit{\Sigma}_1, \mathit{\Sigma}_1, ..., \mathit{\Sigma}_k, \mathit{\Sigma}_k \Rightarrow \gamma, \mathit{\Delta}_1, \mathit{\Delta}_1, ..., \mathit{\Delta}_n, \mathit{\Delta}_n, \mathit{\Pi}_1, \mathit{\Pi}_1, ..., \mathit{\Pi}_k, \mathit{\Pi}_k}\,(\mathrm{SHM})}{G \mid H \mid \mathit{\Gamma}_1, ..., \mathit{\Gamma}_n, \mathit{\Sigma}_1, ..., \mathit{\Sigma}_k \Rightarrow \gamma, \mathit{\Delta}_1, ..., \mathit{\Delta}_n, \mathit{\Pi}_1, ..., \mathit{\Pi}_k}\,(\mathrm{IC}), (\mathrm{EC}) \qquad d_r2}{G \mid G \mid H \mid H \mid \mathit{\Gamma}_1, \mathit{\Gamma}_1, ..., \mathit{\Gamma}_n, \mathit{\Gamma}_n, \mathit{\Sigma}_1, \mathit{\Sigma}_1, ..., \mathit{\Sigma}_k, \mathit{\Sigma}_k \Rightarrow \mathit{\Delta}_1, \mathit{\Delta}_1, ..., \mathit{\Delta}_n, \mathit{\Delta}_n, \mathit{\Pi}_1, \mathit{\Pi}_1, ..., \mathit{\Pi}_k, \mathit{\Pi}_k}\,(\mathrm{SHM})}{G \mid H \mid \mathit{\Gamma}_1, ..., \mathit{\Gamma}_n, \mathit{\Sigma}_1, ..., \mathit{\Sigma}_k \Rightarrow \mathit{\Delta}_1, ..., \mathit{\Delta}_n, \mathit{\Pi}_1, ..., \mathit{\Pi}_k}\,(\mathrm{IC}), (\mathrm{EC})$$

(XIII)

$$\dfrac{\dfrac{\dfrac{\dfrac{d_l1 \qquad d_r1}{G \mid G \mid H \mid H \mid \mathit{\Gamma}_1, ..., \mathit{\Gamma}_n, \mathit{\Sigma}_1, ..., \mathit{\Sigma}_k, \Rightarrow \gamma, \mathit{\Delta}_1, ..., \mathit{\Delta}_n, \mathit{\Pi}_1, ..., \mathit{\Pi}_k \mid \mathit{\Gamma}_1, ..., \mathit{\Gamma}_n, \mathit{\Sigma}_1, ..., \mathit{\Sigma}_k \Rightarrow \mathit{\Delta}_1, ..., \mathit{\Delta}_n, \mathit{\Pi}_1, ..., \mathit{\Pi}_k,}\,(\mathrm{BSHM})}{G \mid H \mid \mathit{\Gamma}_1, ..., \mathit{\Gamma}_n, \mathit{\Sigma}_1, ..., \mathit{\Sigma}_k, \Rightarrow \gamma, \mathit{\Delta}_1, ..., \mathit{\Delta}_n, \mathit{\Pi}_1, ..., \mathit{\Pi}_k}\,\mathrm{EC}, \mathrm{IW} \qquad d_r2}{G \mid G \mid H \mid H \mid \mathit{\Gamma}_1, \mathit{\Gamma}_1, ..., \mathit{\Gamma}_n, \mathit{\Gamma}_n, \mathit{\Sigma}_1, \mathit{\Sigma}_1, ..., \mathit{\Sigma}_k, \mathit{\Sigma}_k \Rightarrow \mathit{\Delta}_1, \mathit{\Delta}_1, ..., \mathit{\Delta}_n, \mathit{\Delta}_n, \mathit{\Pi}_1, \mathit{\Pi}_1, ..., \mathit{\Pi}_k, \mathit{\Pi}_k}\,(\mathrm{SHM})}{G \mid H \mid \mathit{\Gamma}_1, ..., \mathit{\Gamma}_n, \mathit{\Sigma}_1, ..., \mathit{\Sigma}_k \Rightarrow \mathit{\Delta}_1, ..., \mathit{\Delta}_n, \mathit{\Pi}_1, ..., \mathit{\Pi}_k}\,\mathrm{IC}, \mathrm{EC}$$

(XIV)

$$(\mathrm{BSHM})\,\dfrac{(\Rightarrow\Box)\,\dfrac{G \mid \Box\mathit{\Gamma}_1 \Rightarrow \beta \mid ... \mid \mathit{\Gamma}_n \Rightarrow \mathit{\Delta}_n, [\alpha]^{\lambda_n}}{G \mid \Box\mathit{\Gamma}_1 \Rightarrow \alpha \mid ... \mid \mathit{\Gamma}_n \Rightarrow \mathit{\Delta}_n, [\alpha]^{\lambda_n}} \qquad \dfrac{H \mid [\alpha]^{\mu_1 - 1}, \beta, \mathit{\Sigma}_1 \Rightarrow \mathit{\Pi}_1 \mid ... \mid [\alpha]^{\mu_k}, \mathit{\Sigma}_k \Rightarrow \mathit{\Pi}_k}{H \mid [\alpha]^{\mu_1}, \mathit{\Sigma}_1 \Rightarrow \mathit{\Pi}_1 \mid [\alpha]^{\mu_k}, \mathit{\Sigma}_k \Rightarrow \mathit{\Pi}_k}\,(\Box\Rightarrow)}{G \mid H \mid \Box\mathit{\Gamma}_1 \Rightarrow \mid ... \mid \mathit{\Gamma}_n \Rightarrow \mathit{\Delta}_n \mid \mathit{\Sigma}_1 \Rightarrow \mathit{\Pi}_1 \mid ... \mid \mathit{\Sigma}_k \Rightarrow \mathit{\Pi}_k}$$

Again we start with cross-cuts of reduced height. $d_l 1$ has the form (XV) on page 305 and $d_r 1$ has the form (XVI) on page 305.

Now we cut out $\beta$. For the case when $\beta$ is not modal see (XVII) on page 305. For the case when $\beta$ is modal see (XVIII) on page 305.

### 5.2. Pottinger's system revisited

The original proof of Avron for his system is rather involved; it requires strong form of Multicut and, moreover, in two forms which leads to multiplication of subcases to consider. One may ask: (a) if it is not possible to deal with only one version of cut; (b) if it is not possible to obtain a proof showing directly elimination or admissibility of (H-Cut)? In fact three such proofs were offered which we will comment on briefly.

As for reduction of the system to having only one rule the key point is that the reduction step for (SHM) fails because (MS) is introducing boxed formula on both sides of the new sequent in the conclusion (see e.g. from the beginning of Section 5.1). But one can easily notice that in order to prove (5) or (B) we do not need so strong form of (MS). In Kurokawa [18] (and similarly in Lahav [19]) a weaker version is used which operates only on the antecedent. For a system with such a rule a special (BSHM) is not required. Original proof of Kurokawa is performed for (WH-Mix) according to the lines of the proof from [9].

Restall [29] provided a proof which applies the global strategy of elimination of cuts in the proof, introduced by Curry [11] and refined by Belnap [5] in the context of display calculus. In general such proofs are based on the 'big' transformation of the whole parts of proofs instead of local 'small' reductions. The solution of Restall is based on the fact that all rules of the system (including modal ones and (Merge)) are regular in the sense of allowing unrestricted permutation with cuts performed on parametric formulae. It is an elegant solution but shown in a very sketchy way which leaves some essential points of necessary transformations open. It seems however that the application of Curry's solution based on inductive definition of the set of ancestors of respective sequent may be adapted here.

Poggiolesi [27] proposed a proof based on Dragalin's strategy which is of local character. In her system (Merge) is height-preserving admissible which simplifies further steps in essential way. We do not enter into details since the proof is described in [27] in an exact way, so we only sketch it. First of all Poggiolesi must prove that axioms in atomic form

(XV)

$$\text{(BSHM)}\ \dfrac{G \mid \Box\Gamma_1 \Rightarrow \beta \mid ... \mid \Gamma_n \Rightarrow \Delta_n, [\alpha]^{\lambda_n} \qquad H \mid [\alpha]^{\mu_1}, \Sigma_1 \Rightarrow \Pi_1 \mid ... \mid [\alpha]^{\mu_k}, \Sigma_k \Rightarrow \Pi_k}{G \mid H \mid \Box\Gamma_1 \Rightarrow \beta \mid ... \mid \Gamma_n \Rightarrow \Delta_n \mid \Sigma_1 \Rightarrow \Pi_1 \mid ... \mid \Sigma_k \Rightarrow \Pi_k}$$

(XVI)

$$\text{(BSHM)}\ \dfrac{G \mid \Box\Gamma_1 \Rightarrow \alpha \mid ... \mid \Gamma_n \Rightarrow \Delta_n, [\alpha]^{\lambda_n} \qquad H \mid [\alpha]^{\mu_1 - 1}, \beta, \Sigma_1 \Rightarrow \Pi_1 \mid ... \mid [\alpha]^{\mu_k}, \Sigma_k \Rightarrow \Pi_k}{G \mid H \mid \Box\Gamma_1 \Rightarrow \ \mid ... \mid \Gamma_n \Rightarrow \Delta_n \mid \beta, \Sigma_1 \Rightarrow \Pi_1 \mid ... \mid \Sigma_k \Rightarrow \Pi_k}$$

(XVII)

$$\text{(EC)}\ \dfrac{\text{(MS)}\ \dfrac{\text{(SHM)}\ \dfrac{d_l1 \qquad d_r1}{G \mid H \mid G \mid H \mid \Box\Gamma_1 \Rightarrow \ \mid \Box\Gamma_1, \Sigma_1 \Rightarrow \Pi_1 \mid ... \mid \Gamma_n \Rightarrow \Delta_n \mid \Gamma_n \Rightarrow \Delta_n \mid \Sigma_1 \Rightarrow \Pi_1 \mid ... \mid \Sigma_k \Rightarrow \Pi_k \mid \Sigma_k \Rightarrow \Pi_k}}{G \mid H \mid G \mid H \mid \Box\Gamma_1 \Rightarrow \ \mid \Box\Gamma_1 \Rightarrow \mid \Sigma_1 \Rightarrow \Pi_1 \mid ... \mid \Gamma_n \Rightarrow \Delta_n \mid \Gamma_n \Rightarrow \Delta_n \mid \Sigma_1 \Rightarrow \Pi_1 \mid ... \mid \Sigma_k \Rightarrow \Pi_k \mid \Sigma_k \Rightarrow \Pi_k}}{G \mid H \mid \Box\Gamma_1 \Rightarrow \mid ... \mid \Gamma_n \Rightarrow \Delta_n \mid \Sigma_1 \Rightarrow \Pi_1 \mid ... \mid \Sigma_k \Rightarrow \Pi_k}$$

(XVIII)

$$\text{(EC)}\ \dfrac{\text{(BSHM)}\ \dfrac{d_l1 \qquad d_r1}{G \mid H \mid G \mid H \mid \Box\Gamma_1 \Rightarrow \ \mid \Box\Gamma_1 \Rightarrow \mid ... \mid \Gamma_n \Rightarrow \Delta_n \mid \Gamma_n \Rightarrow \Delta_n \mid \Sigma_1 \Rightarrow \Pi_1 \mid \Sigma_1 \Rightarrow \Pi_1 \mid ... \mid \Sigma_k \Rightarrow \Pi_k \mid \Sigma_k \Rightarrow \Pi_k}}{G \mid H \mid \Box\Gamma_1 \Rightarrow \mid ... \mid \Gamma_n \Rightarrow \Delta_n \mid \Sigma_1 \Rightarrow \Pi_1 \mid ... \mid \Sigma_k \Rightarrow \Pi_k}$$

may be generalised to arbitrary formula $\alpha$ on both sides. The next step is the proof that (Merge) is height-preserving admissible. From this follows height-preserving admissibility of (IW) and (EW) and height-preserving invertibility of logical rules. The additional machinery of admissible tools allows for smooth proof of (H-Cut).

The price for having fully logical system (i.e., with no structural primitive rules) which admits a simple proof of (H-Cut) admissibility is certain inelegancy in having two rules for box introduction in the antecedent. One can easily note that it is not redundant by observing that in Restall's system we cannot prove (T) without using (Merge) hence this rule must be primitive. On the contrary, in Poggiolesi's system the presence of a special rule corresponding to (T) makes (Merge) redundant.

This leads to the next question: is it possible to provide such a system that only one rule for ($\Box\Rightarrow$) will be enough but (EC) (or (Merge)) will be redundant. One can think of Pottinger's system but this requires some reformulation of rules. Instead of ($\Rightarrow\Box^{\mathrm{P}}$) we will use Poggiolesi's rule ($\Rightarrow\Box^{\mathrm{P}^*}$), and instead of ($\Box\Rightarrow^{\mathrm{P}}$) we will use the following rule:

$$(\Box{\Rightarrow}_{\mathrm{P}})\quad \frac{\alpha, \Box\alpha, \Gamma \Rightarrow \Delta \mid \alpha, \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \alpha, \Gamma_n \Rightarrow \Delta_n}{\Box\alpha, \Gamma \Rightarrow \Delta \mid \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n}$$

We will point out below why these changes are necessary. It is evident that the proposed rule combines both Poggiolesi's rules. In order to be able to follow her proof as closely as possible we also take as a basis Poggiolesi's system, i.e., generalised atomic axioms and all invertible inference rules for Boolean connectives. Both (IW) and (EW) rules, as well as (EC) are not required as primitive but we cannot get rid of (IC). Because of that we will provide a proof of admissibility of (H-Mix) not (H-Cut) as in Poggiolesi. Also, in order to prove height-preserving invertibility of logical rules (which in turn is necessary for proving admissibility of (EC)) we need a generalised form of internal contraction:

$$\frac{\Gamma, \Gamma \Rightarrow \Delta, \Delta \mid G}{\Gamma \Rightarrow \Delta \mid G}$$

One can easily check that for this system, called HCS5P, we obtain usual preliminary results: the generalization of axioms, height-preserving admissibility of (IW) and (EW), and height-preserving invertibility of all logical rules except ($\Rightarrow\Box^{\mathrm{P}^*}$). By straightforward induction on the height of the derivation of the premiss we obtain:

LEMMA 5. (IW) *and* (EW) *are height-preserving admissible in* HCS5P.

LEMMA 6. *All logical rules except* $(\Rightarrow \Box^{P^*})$ *are height-preserving invertible in* HCS5P.

PROOF. By straightforward induction on the height of the derivation of the premisses. The general form of (IC) which allows for contraction of many different formulas in one sequent in one step is necessary for the case when the last rule was obtained by (IC) in order to save the same height in transformed proof. The case of $(\Box \Rightarrow_P)$ goes by height-preserving application of (IW). ⊣

Having all that we can prove:

LEMMA 7. (EC) *is admissible in* HCS5P

The proof is by induction on the height of the derivation and it is similar as in Poggiolesi [27]. The only reason why we cannot prove the stronger result, namely that it is height-preserving admissible is connected with the case of $(\Rightarrow \Box^{P^*})$ as the last rule applied. The situation is like that:

$$\frac{G \mid \Gamma \Rightarrow \Delta, \Box\alpha \mid \Gamma \Rightarrow \Delta \mid \Rightarrow \alpha \quad n-1}{G \mid \Gamma \Rightarrow \Delta, \Box\alpha \mid \Gamma \Rightarrow \Delta, \Box\alpha \quad n}$$

On the right we indicated the height of respective lines in the proof. By the height-preserving admissibility of (IW), induction hypothesis, $(\Rightarrow \Box^{P^*})$ and (IC) we obtain:

$$\cfrac{\cfrac{\cfrac{G \mid \Gamma \Rightarrow \Delta, \Box\alpha \mid \Gamma \Rightarrow \Delta, \Box\alpha \mid \Rightarrow \alpha \quad n-1}{G \mid \Gamma \Rightarrow \Delta, \Box\alpha \mid \Rightarrow \alpha \quad n-1}}{G \mid \Gamma \Rightarrow \Delta, \Box\alpha, \Box\alpha \quad n}}{G \mid \Gamma \Rightarrow \Delta, \Box\alpha \quad n+1}$$

Here we can see that (IC) is necessary and because in the system it is primitive the height of the last line is $n+1$ not $n$. One could think that we should first prove height-preserving admissibility of (IC) instead of having it as primitive. Unfortunately it is not possible because in order to do that we must apply (EC), hence either (IC) or (EC) must be primitive in such a system and we have chosen the former since it admits simpler form of Mix. This example also shows that height-preserving invertibility of $(\Rightarrow \Box^{P^*})$ is not required for the proof of this lemma, in contrast to cases of Boolean rules where height-preserving invertibility of respective rules is necessary. Let us illustrate this point with the case of $(\Rightarrow\vee)$ as the last applied rule:

$$\frac{G \mid \Gamma \Rightarrow \Delta, \alpha \vee \beta \mid \Gamma \Rightarrow \Delta, \alpha, \beta \quad n-1}{G \mid \Gamma \Rightarrow \Delta, \alpha \vee \beta \mid \Gamma \Rightarrow \Delta, \alpha \vee \beta \quad n}$$

this is transformed into:

$$\cfrac{\cfrac{G \mid \Gamma \Rightarrow \Delta, \alpha, \beta \mid \Gamma \Rightarrow \Delta, \alpha, \beta \quad n-1}{G \mid \Gamma \Rightarrow \Delta, \alpha, \beta \quad n-1}}{G \mid \Gamma \Rightarrow \Delta, \alpha \vee \beta \quad n}$$

where the first line is by height-preserving invertibility of $(\Rightarrow\vee)$ and the second by the induction hypothesis. In case of $(\Rightarrow\vee)$ being not height-preserving invertible we would be unable to apply the induction hypothesis so even simple invertibility is not enough here.

Now we are ready to prove the admissibility of (H-Mix).

Theorem 3. (H-Mix) *is admissible in* HCS5P*.*

The proof is similar to the proof provided for Avron's system but simpler since there is only one rule and we are dealing with only one sequent at a time. The cases of one premiss being an axiom or with Mix-formula being parametric are straightforward. As for the cases where in both premises one occurrence of Mix-formula is principal the only difference is connected with the change of one-premiss rules for connectives into multiplicative form. We leave it to the reader. The essentially different point is:

$$(\text{H-Mix})\ \cfrac{\cfrac{G \mid \Gamma \Rightarrow \Delta, [\Box\alpha]^{\lambda} \mid \Rightarrow \alpha}{G \mid \Gamma \Rightarrow \Delta, [\Box\alpha]^{\lambda+1}} \qquad \cfrac{[\Box\alpha]^{\mu}, \alpha, \Sigma_1 \Rightarrow \Pi_1 \mid ... \mid \alpha, \Sigma_n \Rightarrow \Pi_n}{[\Box\alpha]^{\mu}, \Sigma_1 \Rightarrow \Pi_1 \mid ... \mid \Sigma_n \Rightarrow \Pi_n}}{G \mid \Gamma, \Sigma_1 \Rightarrow \Delta, \Pi_1 \mid ... \mid \Sigma_n \Rightarrow \Pi_n}$$

First we must perform two cross-cuts on $\Box\alpha$ reducing the height in both cases:

$$(\text{H-Mix})\ \frac{G \mid \Gamma \Rightarrow \Delta, [\Box\alpha]^{\lambda+1} \qquad [\Box\alpha]^{\mu}, \alpha, \Sigma_1 \Rightarrow \Pi_1 \mid ... \mid \alpha, \Sigma_n \Rightarrow \Pi_n}{G \mid \alpha, \Gamma, \Sigma_1 \Rightarrow \Delta, \Pi_1 \mid ... \mid \alpha, \Sigma_n \Rightarrow \Pi_n}$$

$$(\text{H-Mix})\ \frac{G \mid \Gamma \Rightarrow \Delta, [\Box\alpha]^{\lambda} \mid \Rightarrow \alpha \qquad [\Box\alpha]^{\mu}, \Sigma_1 \Rightarrow \Pi_1 \mid ... \mid \Sigma_n \Rightarrow \Pi_n}{G \mid \Gamma, \Sigma_1 \Rightarrow \Delta, \Pi_1 \mid ... \mid \Sigma_n \Rightarrow \Pi_n \mid \Rightarrow \alpha}$$

Note that if $\lambda = 0$ the second proof is not required.

Now, either $\alpha \in \Sigma_i$ for all $i \leq n$ or not. In the first case we obtain the result by $n$ applications of (IC) on $\alpha$ in every sequent. Otherwise we must apply (H−Mix) on $\alpha$ to both hypersequents derived above (or if $\lambda = 0$ to the left premise of the original Mix). E.g. let $\alpha \notin \Sigma_1$. Then we have:

(H−Mix)

$$\frac{G \mid \Gamma, \Sigma_1 \Rightarrow \Delta, \Pi_1 \mid \ldots \mid \Sigma_n \Rightarrow \Pi_n \mid \Rightarrow \alpha \quad G \mid \alpha, \Gamma, \Sigma_1 \Rightarrow \Delta, \Pi_1 \mid \ldots \mid \alpha, \Sigma_n \Rightarrow \Pi_n}{G \mid G \mid \Gamma, \Sigma_1 \Rightarrow \Delta, \Pi_1 \mid \Gamma, \Sigma_1 \Rightarrow \Delta, \Pi_1 \mid \ldots \mid \Sigma_n \Rightarrow \Pi_n \mid \alpha, \Sigma_n \Rightarrow \Pi_n}$$

We repeat (H-Mix) on this sequent again with the left premise or the conclusion of the previous (H-Mix) systematically to all cases where $\alpha \notin \Sigma_i$. Since we are cutting $\alpha$ all Mixes are eliminable by induction on the complexity. Finally by (EC) we obtain the desired result.

By the way one may notice why the original Pottinger's rules do not work for such a proof. First as for $(\Rightarrow \Box^{\mathrm{P}})$ in case some boxed formulae are in $\Gamma$ we will have $\Gamma^{\Box} \Rightarrow \alpha$ instead of $\Rightarrow \alpha$ and this sequent when mixed with some $\alpha, \Sigma_i \Rightarrow \Pi_i$ yields $\Gamma^{\Box}, \Sigma_i \Rightarrow \Pi_i$. and there seems to be no way to get rid of $\Gamma^{\Box}$ in any such case to obtain the desired result. Second, in $(\Box \Rightarrow^{\mathrm{P}})$ we have $\Box\alpha$ instead of $\alpha$ added to every $\Sigma_i \Rightarrow \Pi_i$ and it is impossible to perform a reduction on cut-formula complexity in the series of steps described above.

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