Definition. The *propositional logic of linear time based on classical logic*, $\mathbf{TL_{PC}}$, is the collection of these models and the semantic consequence relation.

Definition (Fully General Abstraction). Any function $\mathfrak{v}$ from the set of propositional variables to $\{\mathtt{T}, \mathtt{F}\}$ plus any linearly ordered set $\langle \mathsf{T}, < \rangle$ and any function $\mathfrak{t}$ from the set of propositional variables to intervals of T comprises a model.

4. Finite, Infinite, Dense, and Discrete Models

In a model the only instants that enter into evaluations of wffs are the beginning and ending points of the intervals assigned to the atomic propositions. To show this we'll construct reduced models that include only those points. However, since the beginning and ending points of the entire time line can't be in an interval, we have to add two new points.

Definition (Reduced models). Given any model $\mathfrak{M} = \langle \mathfrak{v}, \langle \mathsf{T}, < \rangle, \mathfrak{t} \rangle$, the associated *reduced* model is $\mathfrak{M}_{\mathrm{r}} = \langle \mathfrak{v}_{\mathrm{r}}, \langle \mathsf{T}_{\mathrm{r}}, <_{\mathrm{r}} \rangle, \mathfrak{t}_{\mathrm{r}} \rangle$, where:

$$\mathsf{T}_{\mathrm{r}} = \bigcup \{\{\mathfrak{b}_{\mathrm{p}}, \mathfrak{e}_{\mathrm{p}}\} : \mathrm{p} \text{ is a propositional variable}\} \cup \{\mathfrak{x}, \mathfrak{y}\},$$

where $\mathfrak{x}$ and $\mathfrak{y}$ are letters that do not appear in T,

$$\mathfrak{v}_{\mathrm{r}}(\mathrm{p}) = \mathfrak{v}(\mathrm{p}),$$
$$z <_{\mathrm{r}} w \text{ iff } z, w \notin \{\mathfrak{x}, \mathfrak{y}\} \text{ and } z < w, \text{ or } z = \mathfrak{x}, \text{ or } w = \mathfrak{y},$$
$$\mathfrak{t}_{\mathrm{r}}(\mathrm{p}) = \{\mathfrak{b}_{\mathrm{q}} : \mathfrak{b}_{\mathrm{p}} \leq \mathfrak{b}_{\mathrm{q}} \leq \mathfrak{e}_{\mathrm{p}}\} \cup \{\mathfrak{e}_{\mathrm{q}} : \mathfrak{b}_{\mathrm{p}} \leq \mathfrak{e}_{\mathrm{q}} \leq \mathfrak{e}_{\mathrm{p}}\}.$$

Given a wff A, the associated *reduced model* for A is

$$\mathfrak{M}_{\mathrm{A}} = \langle \mathfrak{v}_{\mathrm{A}}, \langle \mathsf{T}_{\mathrm{A}}, <_{\mathrm{A}} \rangle, \mathfrak{t}_{\mathrm{A}} \rangle,$$

where:

$$\mathsf{T}_{\mathrm{A}} = \bigcup \{\{\mathfrak{b}_{\mathrm{p}}, \mathfrak{e}_{\mathrm{p}}\} : \mathrm{p} \text{ is a propositional variable appearing in A}\} \cup \{\mathfrak{x}, \mathfrak{y}\},$$

$$\mathfrak{v}_{\mathrm{A}}(\mathrm{p}) = \mathfrak{v}(\mathrm{p}),$$
$$z <_{\mathrm{A}} w \text{ iff } z, w \notin \{\mathfrak{x}, \mathfrak{y}\} \text{ and } z < w, \text{ or } z = \mathfrak{x}, \text{ or } w = \mathfrak{y},$$
$$\mathfrak{t}_{\mathrm{A}}(p) = \begin{cases} \{\mathfrak{b}_{\mathrm{q}} : \mathfrak{b}_{\mathrm{p}} \leq \mathfrak{b}_{\mathrm{q}} \leq \mathfrak{e}_{\mathrm{p}}\} \cup \{\mathfrak{e}_{\mathrm{q}} : \mathfrak{b}_{\mathrm{p}} \leq \mathfrak{e}_{\mathrm{q}} \leq \mathfrak{e}_{\mathrm{p}}\} & \text{if p appears in A} \\ \mathsf{T}_{\mathrm{r}} \setminus \{\mathfrak{x}, \mathfrak{y}\} & \text{otherwise} \end{cases}$$

THEOREM 7 (Models and reduced models).

(a) *Given any model* $\mathfrak{M}$, *for all* A, $\mathfrak{M}_r \models A$ *iff* $\mathfrak{M} \models A$.
(b) *Given any model* $\mathfrak{M}$ *and wff* A, $\mathfrak{M}_A \models A$ *iff* $\mathfrak{M} \models A$.

PROOF. We'll let you show that for every p, $\mathsf{t}_r(p)$ is an interval and in $\mathfrak{M}_r$ the beginning point of $\mathsf{t}_r(p)$ is b_p and the ending point is e_p.

The proof is by induction. We'll do just one case as an example and leave the rest, including part (b), to you.

$$\begin{aligned} \mathfrak{v}(p \wedge_{bb} q) = \mathtt{T} &\text{ iff } \mathfrak{v}(p) = \mathfrak{v}(q) = \mathtt{T} \text{ and } \mathsf{t}(p) <_{bb} \mathsf{t}(q) \\ &\text{ iff } \mathfrak{v}(p) = \mathfrak{v}(q) = \mathtt{T} \text{ and } \mathsf{e}_p < \mathsf{e}_q \\ &\text{ iff } \mathfrak{v}_r(p) = \mathfrak{v}_r(q) = \mathtt{T} \text{ and } \mathsf{e}_p <_r \mathsf{e}_q \\ &\text{ iff } \mathfrak{v}_r(p) = \mathfrak{v}_r(q) = \mathtt{T} \text{ and } \mathsf{t}(p) <_{rbb} \mathsf{t}(q) \\ &\text{ iff } \mathfrak{v}_r(p \wedge_{bb} q) = \mathtt{T}. \end{aligned} \dashv$$

We say that a model is *finite* if T is finite; otherwise it is *infinite*. Note that for every wff A and for every $\mathfrak{M}$, $\mathfrak{M}_A$ is finite.

COROLLARY 8. *For every* A, *if for some model* $\mathfrak{M}$, $\mathfrak{M} \not\models A$, *then there is a finite model* $\mathfrak{M}'$ *such that* $\mathfrak{M}' \not\models A$.

COROLLARY 9. $\models A$ *iff for every finite model* $\mathfrak{M}$, $\mathfrak{M} \models A$

COROLLARY 10. $\mathbf{TL_{PC}}$ *is decidable.*

PROOF. Suppose A has exactly n propositional variables. If $\models A$, then A is true in every finite model. If $\not\models A$, then A fails in some model in which there are at most $2n + 2$ instants. There are only a finite number such models, and in each the truth-value of A can be calculated.[7] So A is valid iff it is true in each one of those. $\dashv$

The tautologies of our logic are the wffs true in every finite model. But equally, as we'll show, they are the wffs true in every infinite model. Actually, we'll show something stronger after we make two definitions.

DEFINITION (Dense models). A model $\mathfrak{M} = \langle \mathsf{T}, < \rangle$ is *dense* iff the ordering $<$ on T is dense, i.e., for every a, b in T such that $\mathsf{a} < \mathsf{b}$ there is a c in T such that $\mathsf{a} < \mathsf{c} < \mathsf{b}$.

---

[7] If someone could provide a formula for the number of different ordered sets of $2n + 2$ instants that can serve for a model for n propositional variables we could say exactly many distinct models must be checked.

Note that every dense linear ordering is infinite.

Definition (Discrete models). A model $\mathfrak{M} = \langle \mathsf{T}, < \rangle$ is *discrete* iff the ordering $<$ on T is discrete, i.e., for every a, if there is a b such that $\mathsf{a} < \mathsf{b}$, then there is an immediate successor of a in T and if there is a b such that $\mathsf{b} < \mathsf{a}$, then there is an immediate predecessor of a in T.

Theorem 11. *For every wff* A,

(a) $\models$ A *iff for every discrete model* $\mathfrak{M}$, $\mathfrak{M} \models$ A.
(b) $\models$ A *iff for every dense model* $\mathfrak{M}$, $\mathfrak{M} \models$ A.

Proof. Part (a) comes from Corollary 9, since every finite model is discrete.

For part (b), if $\models$ A, then A is true in every dense model. So suppose that $\not\models$ A. Then by Corollary 8, there is a finite model $\mathfrak{M}$ with $\mathcal{T} = \langle \mathsf{T}, < \rangle$ such that $\mathfrak{M} \not\models$ A. Let the linear ordering for that model be $\mathsf{b}_{\mathcal{T}} < \mathsf{c}_1 < \cdots < \mathsf{c}_n < \mathsf{e}_{\mathcal{T}}$. Since every time-assignment is not empty and since $\mathsf{b}_{\mathcal{T}} \neq \mathsf{e}_{\mathcal{T}}$ and neither $\mathsf{b}_{\mathcal{T}}$ nor $\mathsf{e}_{\mathcal{T}}$ is in any time assignment, we have that $n \geqslant 1$. Define a model $\mathfrak{M}'$ with the same valuation on propositional variables and $\mathsf{T}' = \{\mathrm{q} : \mathrm{q}$ is a rational number and $0 \leqslant \mathrm{q} \leqslant 1\}$, and:

$$\mathsf{t}'(\mathrm{p}) = \{\mathrm{q} : \tfrac{i}{n} \leqslant \mathrm{q} \leqslant \tfrac{j}{n} \text{ where } \mathsf{b}_{\mathrm{p}} = \mathsf{c}_i \text{ and } \mathsf{e}_{\mathrm{p}} = \mathsf{c}_j\}$$

We'll let you show that for any wff B, $\mathfrak{M}' \models$ B iff $\mathfrak{M} \models$ B. Hence, $\mathfrak{M}' \not\models$ A. ⊣

Corollary 12. $\models$ A *iff for every infinite model* $\mathfrak{M}$, $\mathfrak{M} \models$ A.

Valid wffs cannot distinguish between finite and infinite models, nor between dense and discrete models. However, we do have the following.

Theorem 13. *There is a computable collection of wffs* Γ *such that if* $\mathfrak{M} \models \Gamma$ *then* $\mathfrak{M}_{\mathrm{r}}$ *is dense and hence infinite.*

Proof. Let f be a computable function that enumerates the rational numbers between 0 and 1 without repetitions (see [6]. That is, for each n, $f(n) = s$, where s is a rational number such that $0 \leqslant s \leqslant 1$, and for every rational number s there is one and only one n such that $f(n) = s$. Set:

$$\Gamma = \{\neg(\mathrm{p}_n \wedge \neg\mathrm{p}_n) \wedge_{\mathrm{bb}} \neg(\mathrm{p}_m \wedge \neg\mathrm{p}_m) : f(n) < f(m)\}$$
$$\cup \{\neg(\mathrm{p}_n <_{\mathrm{be}} \mathrm{p}_n) \wedge \neg(\mathrm{p}_n <_{\mathrm{eb}} \mathrm{p}_n) : n \geqslant 0\}.$$

Note that $\neg(\mathrm{p}_n \wedge \neg\mathrm{p}_n) \wedge_{\mathrm{bb}} \neg(\mathrm{p}_m \wedge \neg\mathrm{p}_m)$ is true in a model iff $\mathsf{t}(\mathrm{p}_n) <_{\mathrm{bb}} \mathsf{t}(\mathrm{p}_m)$, and $\neg(\mathrm{p}_n <_{\mathrm{be}} \mathrm{p}_n) \wedge \neg(\mathrm{p}_n <_{\mathrm{eb}} \mathrm{p}_n)$ is true in a model iff $\mathsf{b}_{\mathrm{p}} = \mathsf{e}_{\mathrm{p}}$.

We'll let you show that if $\mathfrak{M} \models \Gamma$ then $\langle \mathsf{T}_{\mathrm{r}}, <_{\mathrm{r}} \rangle$ is order isomorphic to the rational numbers between 0 and 1, and hence is dense. ⊣

We can't claim that every model of Γ is dense because we can always add "irrelevant" points to the ordered set of a model to get an elementarily equivalent model that is not dense.

5. An Axiom System for the Logic of Linear Time Based on Classical Logic

Our goal is to define a notion of theorem and syntactic consequence, which we'll denote as ⊢, that is strongly complete for this logic: For any wff A and collection Γ of wffs of the language L, $\Gamma \vdash \mathrm{A}$ iff $\Gamma \models \mathrm{A}$.

We first make the following abbreviations:

$$\begin{aligned}
\mathrm{A} <_{\mathrm{bb}} \mathrm{B} &\equiv_{\mathrm{df}} \neg(\mathrm{A} \wedge \neg\mathrm{A}) \wedge_{\mathrm{bb}} \neg(\mathrm{B} \wedge \neg\mathrm{B})\\
\mathrm{A} <_{\mathrm{ee}} \mathrm{B} &\equiv_{\mathrm{df}} \neg(\mathrm{A} \wedge \neg\mathrm{A}) \wedge_{\mathrm{ee}} \neg(\mathrm{B} \wedge \neg\mathrm{B})\\
\mathrm{A} <_{\mathrm{be}} \mathrm{B} &\equiv_{\mathrm{df}} \neg(\mathrm{A} \wedge \neg\mathrm{A}) \wedge_{\mathrm{be}} \neg(\mathrm{B} \wedge \neg\mathrm{B})\\
\mathrm{A} <_{\mathrm{eb}} \mathrm{B} &\equiv_{\mathrm{df}} \neg(\mathrm{A} \wedge \neg\mathrm{A}) \wedge_{\mathrm{eb}} \neg(\mathrm{B} \wedge \neg\mathrm{B})\\
\mathrm{A} \approx_{\mathrm{bb}} \mathrm{B} &\equiv_{\mathrm{df}} \neg(\mathrm{A} <_{\mathrm{bb}} \mathrm{B}) \wedge \neg(\mathrm{B} <_{\mathrm{bb}} \mathrm{A})\\
\mathrm{A} \approx_{\mathrm{ee}} \mathrm{B} &\equiv_{\mathrm{df}} \neg(\mathrm{A} <_{\mathrm{ee}} \mathrm{B}) \wedge \neg(\mathrm{B} <_{\mathrm{ee}} \mathrm{A})\\
\mathrm{A} \approx_{\mathrm{be}} \mathrm{B} &\equiv_{\mathrm{df}} \neg(\mathrm{A} <_{\mathrm{be}} \mathrm{B}) \wedge \neg(\mathrm{B} <_{\mathrm{eb}} \mathrm{A})
\end{aligned}$$

Lemma 14. *In any model*:

$$\begin{aligned}
\mathfrak{v}(\mathrm{A} <_{bb} \mathrm{B}) = \mathtt{T} \text{ iff } \mathsf{t}(\mathrm{A}) <_{bb} \mathsf{t}(\mathrm{B}),\\
\mathfrak{v}(\mathrm{A} <_{ee} \mathrm{B}) = \mathtt{T} \text{ iff } \mathsf{t}(\mathrm{A}) <_{ee} \mathsf{t}(\mathrm{B}),\\
\mathfrak{v}(\mathrm{A} <_{be} \mathrm{B}) = \mathtt{T} \text{ iff } \mathsf{t}(\mathrm{A}) <_{be} \mathsf{t}(\mathrm{B}),\\
\mathfrak{v}(\mathrm{A} <_{eb} \mathrm{B}) = \mathtt{T} \text{ iff } \mathsf{t}(\mathrm{A}) <_{eb} \mathsf{t}(\mathrm{B}),\\
\mathfrak{v}(\mathrm{A} \approx_{bb} \mathrm{B}) = \mathtt{T} \text{ iff } \mathfrak{b}_{\mathrm{A}} = \mathfrak{b}_{\mathrm{B}},\\
\mathfrak{v}(\mathrm{A} \approx_{ee} \mathrm{B}) = \mathtt{T} \text{ iff } \mathfrak{e}_{\mathrm{A}} = \mathfrak{e}_{\mathrm{B}},\\
\mathfrak{v}(\mathrm{A} \approx_{be} \mathrm{B}) = \mathtt{T} \text{ iff } \mathfrak{b}_{\mathrm{A}} = \mathfrak{e}_{\mathrm{B}}.
\end{aligned}$$

Our base logic will be classical propositional logic, **PC**, using the primitives ¬ and ∧. We take the strongly complete axiom system for that from [2, Chapter II.7], where the schema of that axiom system now range over wffs of $\mathrm{L}(\neg, \wedge, \wedge_{\mathrm{bb}}, \wedge_{\mathrm{ee}}, \wedge_{\mathrm{be}}, \wedge_{\mathrm{eb}}, \mathrm{p}_0, \mathrm{p}_1, \ldots)$:

$$B \supset (A \supset B)$$
$$(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$$
$$(A \wedge B) \supset A$$
$$(A \wedge B) \supset B$$
$$A \supset (B \supset (A \wedge B))$$
$$\neg A \supset (A \supset B)$$
$$(A \supset B) \supset ((\neg A \supset B) \supset B)$$

rule $\dfrac{A, A \supset B}{B}$ *material detachment*

Now we present an axiom system for $\mathbf{TL_{PC}}$ using the defined connectives. It is long and likely to be unclear until you read the proof of strong completeness below, after which we hope it will seem natural. In that proof we mark in boldface the first use of each axiom so you can see why the axiom is adopted.

$\mathbf{TL_{PC}}$ in $L(\neg, \wedge, \wedge_{bb}, \wedge_{ee}, \wedge_{be}, \wedge_{eb}, p_0, p_1, \ldots)$. The axiom schema of **PC** in this language plus the following schema:

1. $\neg(A <_{bb} A)$
2. $\neg(A <_{ee} A)$
3. $(A \approx_{bb} B) \wedge (B \approx_{bb} C) \supset (A \approx_{bb} C)$
4. $(A \approx_{ee} B) \wedge (B \approx_{ee} C) \supset (A \approx_{ee} C)$
5. $(A \approx_{bb} B) \wedge (B \approx_{be} C) \supset (A \approx_{be} C)$
6. $(A \approx_{be} B) \wedge (C \approx_{be} B) \supset (A \approx_{bb} C)$
7. $(A \approx_{be} B) \wedge (B \approx_{ee} C) \supset (A \approx_{be} C)$
8. $(A \approx_{ee} B) \wedge (C \approx_{be} B) \supset (C \approx_{be} A)$
9. $(B \approx_{be} A) \wedge (B \approx_{be} C) \supset (A \approx_{ee} C)$
10. $(B \approx_{be} A) \wedge (B \approx_{bb} C) \supset (C \approx_{be} A)$
11. $[(A <_{bb} B) \wedge (A \approx_{bb} C) \wedge (B \approx_{bb} D)] \supset (C <_{bb} D)$
12. $[(A <_{bb} B) \wedge (A \approx_{bb} C) \wedge (B \approx_{be} D)] \supset (C <_{be} D)$
13. $[(A <_{bb} B) \wedge (A \approx_{be} C) \wedge (B \approx_{bb} D)] \supset (C <_{bb} D)$
14. $[(A <_{bb} B) \wedge (A \approx_{be} C) \wedge (B \approx_{be} D)] \supset (C <_{ee} D)$
15. $[(A <_{be} B) \wedge (A \approx_{bb} C) \wedge (B \approx_{be} D)] \supset (C <_{be} D)$
16. $[(A <_{be} B) \wedge (A \approx_{be} C) \wedge (D \approx_{be} C)] \supset (C <_{eb} D)$
17. $[(A <_{be} B) \wedge (A \approx_{bb} C) \wedge (B \approx_{ee} D)] \supset (C <_{be} D)$

18. $[(\mathrm{A} <_{\mathrm{be}} \mathrm{B}) \wedge (\mathrm{A} \approx_{\mathrm{be}} \mathrm{C}) \wedge (\mathrm{B} \approx_{\mathrm{bb}} \mathrm{D})] \supset (\mathrm{C} <_{\mathrm{bb}} \mathrm{D})$
19. $[(\mathrm{A} <_{\mathrm{eb}} \mathrm{B}) \wedge (\mathrm{A} \approx_{\mathrm{be}} \mathrm{C}) \wedge (\mathrm{B} \approx_{\mathrm{bb}} \mathrm{D})] \supset (\mathrm{C} <_{\mathrm{bb}} \mathrm{D})$
20. $[(\mathrm{A} <_{\mathrm{eb}} \mathrm{B}) \wedge (\mathrm{A} \approx_{\mathrm{ee}} \mathrm{C}) \wedge (\mathrm{B} \approx_{\mathrm{bb}} \mathrm{D})] \supset (\mathrm{C} <_{\mathrm{eb}} \mathrm{D})$
21. $[(\mathrm{A} <_{\mathrm{eb}} \mathrm{B}) \wedge (\mathrm{C} \approx_{\mathrm{be}} \mathrm{C}) \wedge (\mathrm{B} \approx_{\mathrm{be}} \mathrm{D})] \supset (\mathrm{C} <_{\mathrm{be}} \mathrm{D})$
22. $[(\mathrm{A} <_{\mathrm{eb}} \mathrm{B}) \wedge (\mathrm{A} \approx_{\mathrm{ee}} \mathrm{C}) \wedge (\mathrm{B} \approx_{\mathrm{be}} \mathrm{D})] \supset (\mathrm{C} <_{\mathrm{ee}} \mathrm{D})$
23. $[(\mathrm{A} <_{\mathrm{ee}} \mathrm{B}) \wedge (\mathrm{C} \approx_{\mathrm{be}} \mathrm{A}) \wedge (\mathrm{D} \approx_{\mathrm{be}} \mathrm{B})] \supset (\mathrm{C} <_{\mathrm{bb}} \mathrm{D})$
24. $[(\mathrm{A} <_{\mathrm{ee}} \mathrm{B}) \wedge (\mathrm{A} \approx_{\mathrm{ee}} \mathrm{C}) \wedge (\mathrm{D} \approx_{\mathrm{be}} \mathrm{C})] \supset (\mathrm{C} <_{\mathrm{eb}} \mathrm{D})$
25. $[(\mathrm{A} <_{\mathrm{ee}} \mathrm{B}) \wedge (\mathrm{C} \approx_{\mathrm{be}} \mathrm{A}) \wedge (\mathrm{B} \approx_{\mathrm{ee}} \mathrm{D})] \supset (\mathrm{C} <_{\mathrm{be}} \mathrm{D})$
26. $[(\mathrm{A} <_{\mathrm{ee}} \mathrm{B}) \wedge (\mathrm{A} \approx_{\mathrm{ee}} \mathrm{C}) \wedge (\mathrm{B} \approx_{\mathrm{ee}} \mathrm{D})] \supset (\mathrm{C} <_{\mathrm{ee}} \mathrm{D})$
27. $(\mathrm{A} <_{\mathrm{bb}} \mathrm{B}) \supset \neg(\mathrm{B} <_{\mathrm{bb}} \mathrm{A})$
28. $(\mathrm{A} <_{\mathrm{ee}} \mathrm{B}) \supset \neg(\mathrm{B} <_{\mathrm{ee}} \mathrm{A})$
29. $(\mathrm{A} <_{\mathrm{be}} \mathrm{B}) \supset \neg(\mathrm{B} <_{\mathrm{eb}} \mathrm{A})$
30. $(\mathrm{A} <_{\mathrm{eb}} \mathrm{B}) \supset \neg(\mathrm{B} <_{\mathrm{be}} \mathrm{A})$
31. $(\mathrm{A} <_{\mathrm{bb}} \mathrm{B}) \wedge (\mathrm{B} <_{\mathrm{bb}} \mathrm{C}) \supset (\mathrm{A} <_{\mathrm{bb}} \mathrm{C})$
32. $(\mathrm{A} <_{\mathrm{bb}} \mathrm{B}) \wedge (\mathrm{B} <_{\mathrm{be}} \mathrm{C}) \supset (\mathrm{A} <_{\mathrm{be}} \mathrm{C})$
33. $(\mathrm{A} <_{\mathrm{be}} \mathrm{B}) \wedge (\mathrm{B} <_{\mathrm{eb}} \mathrm{C}) \supset (\mathrm{A} <_{\mathrm{bb}} \mathrm{C})$
34. $(\mathrm{A} <_{\mathrm{be}} \mathrm{B}) \wedge (\mathrm{B} <_{\mathrm{ee}} \mathrm{C}) \supset (\mathrm{A} <_{\mathrm{be}} \mathrm{C})$
35. $(\mathrm{A} <_{\mathrm{ee}} \mathrm{B}) \wedge (\mathrm{B} <_{\mathrm{ee}} \mathrm{C}) \supset (\mathrm{A} <_{\mathrm{ee}} \mathrm{C})$
36. $(\mathrm{A} <_{\mathrm{ee}} \mathrm{B}) \wedge (\mathrm{B} <_{\mathrm{eb}} \mathrm{C}) \supset (\mathrm{A} <_{\mathrm{eb}} \mathrm{C})$
37. $(\mathrm{A} <_{\mathrm{eb}} \mathrm{B}) \wedge (\mathrm{B} <_{\mathrm{be}} \mathrm{C}) \supset (\mathrm{A} <_{\mathrm{ee}} \mathrm{C})$
38. $(\mathrm{A} <_{\mathrm{eb}} \mathrm{B}) \wedge (\mathrm{B} <_{\mathrm{bb}} \mathrm{C}) \supset (\mathrm{A} <_{\mathrm{eb}} \mathrm{C})$
39. $\mathrm{A} \wedge_{\mathrm{bb}} \mathrm{B} \equiv (\mathrm{A} \wedge \mathrm{B}) \wedge (\mathrm{A} <_{\mathrm{bb}} \mathrm{B})$
40. $\mathrm{A} \wedge_{\mathrm{ee}} \mathrm{B} \equiv (\mathrm{A} \wedge \mathrm{B}) \wedge (\mathrm{A} <_{\mathrm{ee}} \mathrm{B})$
41. $\mathrm{A} \wedge_{\mathrm{be}} \mathrm{B} \equiv (\mathrm{A} \wedge \mathrm{B}) \wedge (\mathrm{A} <_{\mathrm{be}} \mathrm{B})$
42. $\mathrm{A} \wedge_{\mathrm{eb}} \mathrm{B} \equiv (\mathrm{A} \wedge \mathrm{B}) \wedge (\mathrm{A} <_{\mathrm{eb}} \mathrm{B})$
43. $(\mathrm{A} <_{\mathrm{bb}} \mathrm{B}) \equiv \bigvee_{\mathrm{p\ in\ A}}(\bigwedge_{\mathrm{q\ in\ B}}(\mathrm{p} <_{\mathrm{bb}} \mathrm{q}))$
44. $(\mathrm{A} <_{\mathrm{ee}} \mathrm{B}) \equiv \bigvee_{\mathrm{q\ in\ B}}(\bigwedge_{\mathrm{p\ in\ A}}(\mathrm{p} <_{\mathrm{ee}} \mathrm{q}))$
45. $(\mathrm{A} <_{\mathrm{eb}} \mathrm{B}) \equiv \bigvee_{\mathrm{p\ in\ A}} \bigvee_{\mathrm{q\ in\ B}}[\bigwedge_{\mathrm{p'\ in\ A}}(\mathrm{p'} <_{\mathrm{ee}} \mathrm{p} \vee [\neg(\mathrm{p} <_{\mathrm{ee}} \mathrm{p'}) \wedge \neg(\mathrm{p'} <_{\mathrm{ee}} \mathrm{p})]) \wedge \bigwedge_{\mathrm{q'\ in\ B}}(\mathrm{q} <_{\mathrm{bb}} \mathrm{q'} \vee [\neg(\mathrm{q} <_{\mathrm{bb}} \mathrm{q'}) \wedge \neg(\mathrm{q'} <_{\mathrm{bb}} \mathrm{q})]) \wedge \mathrm{p} <_{\mathrm{eb}} \mathrm{q}]$
46. $(\mathrm{A} <_{\mathrm{be}} \mathrm{B}) \equiv \bigvee_{\mathrm{p\ in\ A}} \bigvee_{\mathrm{q\ in\ B}}[\bigwedge_{\mathrm{p'\ in\ A}}(p <_{\mathrm{bb}} \mathrm{p'} \vee [\neg(\mathrm{p} <_{\mathrm{bb}} \mathrm{p'}) \wedge \neg(p' <_{\mathrm{bb}} \mathrm{p})] \wedge \bigwedge_{\mathrm{q'\ in\ B}}(\mathrm{q'} <_{\mathrm{ee}} \mathrm{q} \vee [\neg(\mathrm{q} <_{\mathrm{ee}} \mathrm{q'}) \wedge \neg(\mathrm{q'} <_{\mathrm{ee}} \mathrm{q})]) \wedge \mathrm{p} <_{\mathrm{be}} \mathrm{q}]$

rule $\dfrac{\mathrm{A}, \mathrm{A} \supset \mathrm{B}}{\mathrm{B}}$

We adopt the usual definitions of completeness, consistency, and theory as for **PC**:

Γ is *consistent* iff for no A does $\Gamma \vdash A$ and $\Gamma \vdash \neg A$.
Γ is *complete* iff for every A, $\Gamma \vdash A$ or $\Gamma \vdash \neg A$.
Γ is a *theory* iff for every A, if $\Gamma \vdash A$ then A is in Γ.

Using the methods of [2, Chapter II] it is routine to prove the following.

LEMMA 15. (a) (The Deduction Theorem) $\Gamma \cup \{A\} \vdash B$ *iff* $\Gamma \vdash A \supset B$.
(b) *If* Γ *is complete and consistent, then* Γ *is a theory.*
(c) *If* $\Gamma \nvdash A$, *then there is some complete and consistent* $\Sigma \supseteq \Gamma$ *such that* $A \notin \Sigma$.
(d) *If* A *has the form of a valid wff in* **PC**, *then* $\vdash A$.

In the proofs in this section we write "by **PC**" to mean that the wff under consideration is **PC**-valid and so by Lemma 15 is a theorem of this system.

The proof of strong completeness now reduces to proving the following theorem. We include all the details in the proof because, though after reading the proof many steps may seem routine and easy, we have found that it is difficult to formulate them correctly.

THEOREM 16. *If* Γ *is a complete and consistent collection of wffs of* L, *then* Γ *has a model.*

PROOF. Given Γ we will construct a model $\langle \mathfrak{v}, \langle \mathsf{T}, < \rangle, \mathsf{t} \rangle$ such that $\mathfrak{v}(A) = \mathsf{T}$ iff $A \in \Gamma$.

First note that by Lemma 15, Γ is a theory, and $\Gamma \supseteq$ **PC**.

In order to construct a linearly ordered set, we first define:

$$\mathsf{S} = \{\mathsf{x}_p, \mathsf{y}_p : p \text{ is a propositional variable}\} \cup \{\mathsf{b}, \mathsf{e}\}.$$

The letters x_{p_0}, y_{p_0}, x_{p_1}, y_{p_1}, ..., b, e do not stand for anything; they are not symbols but simply letters.

We define a relation $<$ on S. For all propositional variables p, q,

$\mathsf{b} < \mathsf{x}_p$	
$\mathsf{b} < \mathsf{y}_p$	$\mathsf{x}_p < \mathsf{x}_q$ iff $(p <_{bb} q) \in \Gamma$
$\mathsf{x}_p < \mathsf{e}$	$\mathsf{y}_p < \mathsf{y}_q$ iff $(p <_{ee} q) \in \Gamma$
$\mathsf{y}_p < \mathsf{e}$	$\mathsf{x}_p < \mathsf{y}_q$ iff $(p <_{be} q) \in \Gamma$
$\mathsf{b} < \mathsf{e}$	$\mathsf{y}_p < \mathsf{x}_q$ iff $(p <_{eb} q) \in \Gamma$

The points x_p, y_p are meant to be the beginning and ending points of the interval assigned to the propositional variable p. But for distinct p, q we may have that $(p \approx_{bb} q) \in \Gamma$, yet $x_p \neq y_p$ since those are different letters. So we define an equivalence relation on S and take the equivalence classes as the instants of our models.

$$b \sim b$$
$$e \sim e$$
$$x_p \sim x_q \text{ iff } (p \approx_{bb} q) \in \Gamma$$
$$y_p \sim y_q \text{ iff } (p \approx_{ee} q) \in \Gamma$$
$$x_p \sim y_q \text{ iff } (p \approx_{be} q) \in \Gamma$$
$$y_q \sim x_p \text{ iff } (p \approx_{be} q) \in \Gamma$$

Note that the defining condition for the last two equivalences are the same. Below we will note the use of an axiom that justifies a step.

Lemma A. *The relation $\sim$ is an equivalence relation.*

Proof. We must show that $\sim$ is reflexive, symmetric, and transitive.
Reflexive For all z, $z \sim z$.

$x_p \sim x_p$ $\qquad \neg(A <_{bb} A)$

$y_p \sim y_p$ $\qquad \neg(A <_{ee} A)$

$b \sim b$ and $e \sim e$ by definition

Symmetric For all z, w, if $z \sim w$, then $w \sim z$.
If $z = w$, this follows by reflexivity.
If z is b, then $w = b$, and we are done by reflexivity.
If z is e, then $w = e$, and we are done by reflexivity.

$$\begin{aligned} x_p \sim x_q &\text{ iff } (p \approx_{bb} q) \in \Gamma && \text{by definition} \\ &\text{ iff } \neg(p <_{bb} q) \wedge \neg(q <_{bb} p) \in \Gamma && \text{by } \mathbf{PC} \\ &\text{ iff } \neg(q <_{bb} p) \wedge \neg(p <_{bb} q) \in \Gamma && \text{by } \mathbf{PC} \\ &\text{ iff } (q \approx_{bb} p) \in \Gamma \\ &\text{ iff } x_q \sim x_p \end{aligned}$$

If $y_p \sim y_q$, then $y_p \sim y_q$ follows similarly.

$$\begin{aligned} y_p \sim x_q &\text{ iff } (p \approx_{be} q) \in \Gamma \\ &\text{ iff } x_p \sim y_q \end{aligned}$$

Transitive For all z, w, u, if $z \sim w$ and $w \sim u$, then $z \sim u$.
If $z = w$ or $w = u$, we are done.
If z, w, or u is b, then all three are, and we are done.
If z, w, or u is e, then all three are, and we are done.

1. If $x_p \sim x_q$ and $x_q \sim x_r$ then $x_p \sim x_r$
$$(A \approx_{bb} B) \wedge (B \approx_{bb} C) \supset (A \approx_{bb} C).$$
2. If $x_p \sim x_q$ and $x_q \sim y_r$ then $x_p \sim y_r$
$$(A \approx_{bb} B) \wedge (B \approx_{be} C) \supset (A \approx_{be} C).$$
3. If $x_p \sim y_q$ and $y_q \sim x_r$ then $x_p \sim x_r$
$$(A \approx_{be} B) \wedge (C \approx_{be} B) \supset (A \approx_{bb} C).$$
4. If $x_p \sim y_q$ and $y_q \sim y_r$ then $x_p \sim y_r$
$$(A \approx_{be} B) \wedge (B \approx_{ee} C) \supset (A \approx_{be} C).$$
5. If $y_p \sim y_q$ and $y_q \sim y_r$ then $y_p \sim y_r$
$$(A \approx_{ee} B) \wedge (B \approx_{ee} C) \supset (A \approx_{ee} C).$$
6. If $y_p \sim y_q$ and $y_q \sim x_r$ then $y_p \sim x_r$
$$(A \approx_{ee} B) \wedge (C \approx_{be} B) \supset (C \approx_{be} A).$$
7. If $y_p \sim x_q$ and $x_q \sim y_r$ then $y_p \sim y_r$
$$(B \approx_{be} A) \wedge (B \approx_{be} C) \supset (A \approx_{ee} C).$$
8. If $y_p \sim x_q$ and $x_q \sim x_r$ then $y_p \sim x_r$
$$(B \approx_{be} A) \wedge (B \approx_{bb} C) \supset (C \approx_{be} A).$$

This completes the proof of Lemma A. ⊣

Lemma B. *The relation* $\sim$ *respects* $<$. *That is,*

$$\textit{if } z < w \textit{ and } z \sim u \textit{ and } w \sim v, \textit{ then } u < v.$$

Proof. If $b < w$ and $b \sim u$ and $w \sim v$, then u is b, and $w \neq b$. Hence $v \neq b$, so $b \sim v$. We'll leave to you the other cases when any of z, w, u, or v is either b or e.

1. If $x_p < x_q$ and $x_p \sim x_r$ and $x_q \sim x_s$, then $x_r < x_s$
$$[(A <_{bb} B) \wedge (A \approx_{bb} C) \wedge (B \approx_{bb} D)] \supset (C <_{bb} D).$$
2. If $x_p < x_q$ and $x_p \sim x_r$ and $x_q \sim y_s$, then $x_r < y_s$
$$[(A <_{bb} B) \wedge (A \approx_{bb} C) \wedge (B \approx_{be} D)] \supset (C <_{be} D).$$
3. If $x_p < x_q$ and $x_p \sim y_r$ and $x_q \sim x_s$, then $y_r < x_s$
$$[(A <_{bb} B) \wedge (A \approx_{be} C) \wedge (B \approx_{bb} D)] \supset (C <_{eb} D).$$
4. If $x_p < x_q$ and $x_p \sim y_r$ and $x_q \sim y_s$, then $y_r < y_s$
$$[(A <_{bb} B) \wedge (A \approx_{be} C) \wedge (B \approx_{be} D)] \supset (C <_{ee} D).$$

5. If $x_p < y_q$ and $x_p \sim x_r$ and $y_q \sim x_s$, then $x_r < x_s$
$$[(A <_{be} B) \wedge (A \approx_{bb} C) \wedge (D \approx_{be} B)] \supset (C <_{bb} D).$$
6. If $x_p < y_q$ and $x_p \sim y_r$ and $y_q \sim x_s$, then $y_r < x_s$
$$[(A <_{be} B) \wedge (A \approx_{be} C) \wedge (D \approx_{be} B)] \supset (C <_{eb} D).$$
7. If $x_p < y_q$ and $x_p \sim y_r$ and $y_q \sim y_s$, then $x_r < y_s$
$$[(A <_{be} B) \wedge (A \approx_{bb} C) \wedge (B \approx_{ee} D)] \supset (C <_{be} D).$$
8. If $x_p < y_q$ and $x_p \sim x_r$ and $y_q \sim y_s$, then $x_r < y_s$
$$[(A <_{be} B) \wedge (A \approx_{bb} C) \wedge (B \approx_{ee} D)] \supset (C <_{be} D).$$
9. If $y_p < x_q$ and $y_p \sim x_r$ and $x_q \sim x_s$, then $x_r < x_s$
$$[(A <_{eb} B) \wedge (C \approx_{be} A) \wedge (B \approx_{bb} D)] \supset (C <_{bb} D).$$
10. If $y_p < x_q$ and $y_p \sim y_r$ and $x_q \sim x_s$, then $y_r < x_s$
$$[(A <_{eb} B) \wedge (A \approx_{ee} C) \wedge (B \approx_{bb} D)] \supset (C <_{eb} D).$$
11. If $y_p < x_q$ and $y_p \sim x_r$ and $x_q \sim y_s$, then $x_r < y_s$
$$[(A <_{eb} B) \wedge (C \approx_{be} C) \wedge (D \approx_{be} B)] \supset (C <_{be} D).$$
12. If $y_p < x_q$ and $y_p \sim y_r$ and $x_q \sim y_s$, then $y_r < y_s$
$$[(A <_{eb} B) \wedge (A \approx_{ee} C) \wedge (B \approx_{be} D)] \supset (C <_{ee} D).$$
13. If $y_p < y_q$ and $y_p \sim x_r$ and $y_q \sim x_s$, then $x_r < x_s$
$$[(A <_{ee} B) \wedge (C \approx_{be} A) \wedge (D \approx_{be} B)] \supset (C <_{bb} D).$$
14. If $y_p < y_q$ and $y_p \sim x_r$ and $y_q \sim y_s$, then $x_r < y_s$
$$[(A <_{ee} B) \wedge (C \approx_{be} A) \wedge (B \approx_{ee} D)] \supset (C <_{be} D).$$
15. If $y_p < y_q$ and $y_p \sim y_r$ and $y_q \sim x_s$, then $y_r < x_s$
$$[(A <_{ee} B) \wedge (A \approx_{ee} C) \wedge (D \approx_{be} C)] \supset (C <_{eb} D).$$
16. If $y_p < y_q$ and $y_p \sim y_r$ and $y_q \sim y_s$, then $y_r < y_s$
$$[(A <_{ee} B) \wedge (A \approx_{ee} C) \wedge (B \approx_{ee} D)] \supset (C <_{ee} D).$$

This completes the proof of Lemma B. ⊣

For $z \in S$, denote the equivalence class of z by $[z]$. Note that the $[b] = \{b\}$ and $[e] = \{e\}$.

Define $T = \{[z] : z \in S\}$ and $[z] < [w]$ iff $z < w$. By Lemma B this is well-defined.

LEMMA C. $\langle T, < \rangle$ *is a linearly ordered set with endpoints.*

PROOF. We first show that it has endpoints.

For every $z \neq b$, $b < z$ and not $b < b$. Hence, for every $[z] \neq [b]$, $[b] < [z]$, and also not $[b] < [b]$. And similarly, for every $[z] \neq [e]$, $[z] < [e]$, while not $[e] < [e]$.

We now show that $<$ is anti-reflexive and transitive.

Anti-reflexive For any $[z]$, not $[z] < [z]$.

If $[z] = [\mathfrak{b}]$ or $[z] = [\mathfrak{e}]$ we are done by what we have just noted above. Otherwise, for some p, $[z] = [x_p]$ or $[z] = [y_p]$. We have not $x_p < x_p$ by Axiom (schema) 1, and we have not $y_p < y_p$ by Axiom 2.

Transitive If $[z] < [w]$ and $[w] < [v]$, then $[z] < [v]$.

We'll leave to you the cases when any one of $[z]$, $[w]$, or $[v]$ is $[\mathfrak{b}]$ or $[\mathfrak{e}]$. We'll abbreviate $[z] < [w]$ and $[w] < [v]$ as $[z] < [w] < [v]$.

1. If $[x_p] < [x_q] < [x_r]$ then $[x_p] < [x_r]$
$$(A <_{bb} B) \wedge (B <_{bb} C) \supset (A <_{bb} C).$$
2. If $[x_p] < [x_q] < [y_r]$ then $[x_p] < [y_r]$
$$(A <_{bb} B) \wedge (B <_{be} C) \supset (A <_{be} C).$$
3. If $[x_p] < [y_q] < [x_r]$ then $[x_p] < [x_r]$
$$(A <_{be} B) \wedge (B <_{eb} C) \supset (A <_{bb} C).$$
4. If $[x_p] < [y_q] < [y_r]$ then $[x_p] < [y_r]$
$$(A <_{be} B) \wedge (B <_{ee} C) \supset (A <_{be} C).$$
5. If $[y_p] < [y_q] < [y_r]$ then $[y_p] < [y_r]$
$$(A <_{ee} B) \wedge (B <_{ee} C) \supset (A <_{ee} C).$$
6. If $[y_p] < [y_q] < [x_r]$ then $[y_p] < [x_r]$
$$(A <_{ee} B) \wedge (B <_{eb} C) \supset (A <_{eb} C).$$
7. If $[y_p] < [x_q] < [y_r]$ then $[y_p] < [y_r]$
$$(A <_{eb} B) \wedge (B <_{be} C) \supset (A <_{ee} C).$$
8. If $[y_p] < [x_q] < [x_r]$ then $[y_p] < [x_r]$
$$(A <_{eb} B) \wedge (B <_{bb} C) \supset (A <_{eb} C).$$

This completes the proof of Lemma C. ⊣

Now we define the valuation $\mathfrak{v}$ and time-assignment t for our model.

$$\mathfrak{v}(p) = \mathsf{T} \text{ iff } p \in \Gamma,$$
$$\mathsf{t}(p) = \{[z] : [x_p] \le [z] \le [y_p]\}.$$

Then t is extended to all wffs by the standard condition for models:

$$\mathsf{t}(A) = \bigcup\{\mathsf{t}(p) : p \text{ appears in } A\}.$$

And $\mathfrak{v}$ is extended to all wffs by the tables for the connectives.

In this ordering, b_p is $[x_p]$, e_p is $[y_p]$, and for all p, $\mathfrak{b} \notin \mathsf{t}(p)$ and $\mathfrak{e} \notin \mathsf{t}(p)$.

LEMMA D. $\mathfrak{v}(A) = \mathsf{T}$ *iff* $A \in \Gamma$.

PROOF. If A has length 1, A is p and the lemma follows by definition.

If A has length 2 and A is $p \wedge q$ or $\neg p$, then the proof follows as for **PC** (see [2]). Otherwise, we have:

$$\begin{aligned}
\mathfrak{v}(p \wedge_{bb} q) = \mathtt{T} &\text{ iff } \mathfrak{v}(p) = \mathfrak{v}(q) = \mathtt{T} \text{ and } \mathtt{t}(p) <_{bb} \mathtt{t}(q) \\
&\text{ iff } p \in \Gamma \text{ and } q \in \Gamma \text{ and } [\mathfrak{x}_p] < [\mathfrak{x}_q] \\
&\text{ iff } (p \wedge q) \in \Gamma \text{ and } [\mathfrak{x}_p] < [\mathfrak{x}_q] && \text{by } \textbf{PC} \\
&\text{ iff } (p \wedge q) \in \Gamma \text{ and } \mathfrak{x}_p < \mathfrak{x}_q \\
&\text{ iff } (p \wedge q) \in \Gamma \text{ and } (p <_{bb} q) \in \Gamma && \text{by construction} \\
&\text{ iff } (p \wedge_{bb} q) && A \wedge_{bb} B \equiv (A \wedge B) \wedge (A <_{bb} B)
\end{aligned}$$

$$\begin{aligned}
\mathfrak{v}(p \wedge_{ee} q) = \mathtt{T} &\text{ iff } \mathfrak{v}(p) = \mathfrak{v}(q) = \mathtt{T} \text{ and } \mathtt{t}(p) <_{ee} \mathtt{t}(q) \\
&\text{ iff } (p \wedge q) \in \Gamma \text{ and } [\mathfrak{y}_p] < [\mathfrak{y}_q] \\
&\text{ iff } (p \wedge q) \in \Gamma \text{ and } \mathfrak{y}_p < \mathfrak{y}_q \\
&\text{ iff } (p \wedge q) \in \Gamma \text{ and } (p <_{ee} q) \in \Gamma && \text{by construction} \\
&\text{ iff } (p \wedge_{bb} q) && A \wedge_{ee} B \equiv (A \wedge B) \wedge (A <_{ee} B)
\end{aligned}$$

$$\begin{aligned}
\mathfrak{v}(p \wedge_{be} q) = \mathtt{T} &\text{ iff } \mathfrak{v}(p) = \mathfrak{v}(q) = \mathtt{T} \text{ and } \mathtt{t}(p) <_{be} \mathtt{t}(q) \\
&\text{ iff } (p \wedge q) \in \Gamma \text{ and } [\mathfrak{x}_p] < [\mathfrak{y}_q] \\
&\text{ iff } (p \wedge q) \in \Gamma \text{ and } \mathfrak{x}_p < \mathfrak{y}_q \\
&\text{ iff } (p \wedge q) \in \Gamma \text{ and } (p <_{be} q) \in \Gamma && \text{by construction} \\
&\text{ iff } (p \wedge_{bb} q) && A \wedge_{be} B \equiv (A \wedge B) \wedge (A <_{be} B)
\end{aligned}$$

$$\begin{aligned}
\mathfrak{v}(p \wedge_{eb} q) = \mathtt{T} &\text{ iff } \mathfrak{v}(p) = \mathfrak{v}(q) = \mathtt{T} \text{ and } \mathtt{t}(p) <_{eb} \mathtt{t}(q) \\
&\text{ iff } (p \wedge q) \in \Gamma \text{ and } [\mathfrak{y}_p] < [\mathfrak{x}_q] \\
&\text{ iff } (p \wedge q) \in \Gamma \text{ and } \mathfrak{y}_p < \mathfrak{x}_q \\
&\text{ iff } (p \wedge q) \in \Gamma \text{ and } (p <_{eb} q) \in \Gamma && \text{by construction} \\
&\text{ iff } (p \wedge_{bb} q) && A \wedge_{eb} B \equiv (A \wedge B) \wedge (A <_{eb} B)
\end{aligned}$$

Now suppose that A has length 3 or greater and the lemma is true for all shorter wffs. If A is of the form $A \wedge B$ or $\neg A$, then the proof follows as for **PC**. Otherwise, we have:

$$\begin{aligned}
\mathfrak{v}(A \wedge_{bb} B) = \mathtt{T} &\text{ iff } \mathfrak{v}(A) = \mathfrak{v}(B) = \mathtt{T} \text{ and } \mathtt{t}(A) <_{bb} \mathtt{t}(B) \\
&\text{ iff } A \in \Gamma \text{ and } B \in \Gamma \text{ and } \mathtt{t}(A) <_{bb} \mathtt{t}(B) \\
&\text{ iff } (A \wedge B) \in \Gamma \text{ and for some } p \text{ in } A, \text{ all } q \text{ in } B, \\
&\qquad \mathtt{t}(p) <_{bb} \mathtt{t}(q) && \text{by Lemma 7}
\end{aligned}$$

iff $(\mathrm{A} \wedge \mathrm{B}) \in \Gamma$ and for some p in A, all q in B, $(\mathrm{p} <_{\mathrm{bb}} \mathrm{q}) \in \Gamma$ by construction

iff $(\mathrm{A} \wedge \mathrm{B}) \in \Gamma$ and $(\mathrm{A} <_{\mathrm{bb}} \mathrm{B}) \in \Gamma$

$(\mathrm{A} <_{\mathrm{bb}} \mathrm{B}) \equiv \bigvee_{\mathrm{p\ in\ A}}(\bigwedge_{\mathrm{q\ in\ B}}(\mathrm{p} <_{\mathrm{bb}} \mathrm{q}))$ and Axiom 39 as in the evaluation of $\mathfrak{v}(\mathrm{p} \wedge_{\mathrm{bb}} \mathrm{q})$

The other cases are done similarly using Lemma 5 and the following axioms:

$$(\mathrm{A} <_{\mathrm{ee}} \mathrm{B}) \equiv \bigvee_{\mathrm{q\ in\ B}}(\bigwedge_{\mathrm{p\ in\ A}}(\mathrm{p} <_{\mathrm{ee}} \mathrm{q}))$$

$$\begin{aligned}(\mathrm{A} <_{\mathrm{eb}} \mathrm{B}) \equiv \bigvee_{\mathrm{p\ in\ A}} \bigvee_{\mathrm{q\ in\ B}}[\bigwedge_{\mathrm{p'\ in\ A}}(\mathrm{p'} <_{\mathrm{ee}} \mathrm{p} \vee \\ [\neg(\mathrm{p} <_{\mathrm{ee}} \mathrm{p'}) \wedge \neg(\mathrm{p'} <_{\mathrm{ee}} \mathrm{p})]) \\ \wedge \bigwedge_{\mathrm{q'\ in\ B}}(\mathrm{q} <_{\mathrm{bb}} \mathrm{q'} \vee [\neg(\mathrm{q} <_{\mathrm{bb}} \mathrm{q'}) \wedge \neg(\mathrm{q'} <_{\mathrm{bb}} \mathrm{q})]) \wedge \mathrm{p} <_{\mathrm{eb}} \mathrm{q}]\end{aligned}$$

$$\begin{aligned}(\mathrm{A} <_{\mathrm{be}} \mathrm{B}) \equiv \bigvee_{\mathrm{p\ in\ A}} \bigvee_{\mathrm{q\ in\ B}}[\bigwedge_{\mathrm{p'\ in\ A}}(\mathrm{p} <_{\mathrm{bb}} \mathrm{p'} \vee \\ [\neg(\mathrm{p} <_{\mathrm{bb}} \mathrm{p'}) \wedge \neg(\mathrm{p'} <_{\mathrm{bb}} \mathrm{p})]) \\ \wedge \bigwedge_{\mathrm{q'\ in\ B}}(\mathrm{q'} <_{\mathrm{ee}} \mathrm{q} \vee [\neg(\mathrm{q} <_{\mathrm{ee}} \mathrm{q'}) \wedge \neg(\mathrm{q'} <_{\mathrm{ee}} \mathrm{q})]) \wedge \mathrm{p} <_{\mathrm{be}} \mathrm{q}]\end{aligned}$$

This completes the proof of Lemma D. ⊣

Lemma D concludes the proof of Theorem 16. ⊣

THEOREM 17 (Strong Completeness). *For any set* $\Gamma \cup \{\mathrm{A}\}$ *of wffs*: $\Gamma \vdash \mathrm{A}$ *iff* $\Gamma \models \mathrm{A}$.

PROOF. We'll leave to you to check that each of the axioms is true in every model and that the rule preserves truth in a model. Hence, if $\Gamma \vdash \mathrm{A}$ then $\Gamma \models \mathrm{A}$.

In the other direction, suppose $\Gamma \nvdash \mathrm{A}$. Then by Lemma 15(c) there is a complete and consistent Σ such that $\Sigma \supseteq \Gamma$ and $\mathrm{A} \notin \Sigma$. By Theorem 16 there is a model $\mathfrak{M}$ such that for all wffs B, $\mathfrak{M} \models \mathrm{B}$ iff $\mathrm{B} \in \Sigma$. Hence, $\mathfrak{M} \not\models \mathrm{A}$, and so $\Sigma \not\models \mathrm{A}$, and hence $\Gamma \not\models \mathrm{A}$. ⊣

COROLLARY 18. (a) *If* $\Gamma \nvdash \mathrm{A}$, *then there is a model* $\mathfrak{M}$ *in which every time assignment is a closed interval and* $\mathfrak{M} \not\models \mathrm{A}$.

(b) *If* $\Gamma \nvdash \mathrm{A}$, *then there is a model* $\mathfrak{M}$ *in which every time assignment is an open interval and* $\mathfrak{M} \not\models \mathrm{A}$.

PROOF. (a) Every time assignment in the model constructed in Theorem 16 is closed.

(b) We can modify the construction of the model in Theorem 16 to have only open intervals as time assignments. We just have to ensure that the time assignments are not empty since we may have that for no z is $x_p < z < y_p$. For the construction of S we add the following step:

For any p such that for no z, $x_p < z < y_p$ add a point m_p to S with the conditions:

$$z < m_p \text{ iff } z \leq x_p$$

$$m_p < z \text{ iff } y_p \leq z$$

Then in the construction of the model set $t(p) = \{[z] : [x_p] < [z] < [y_p]\}$. ⊣

COROLLARY 19. (a) $\Gamma \vdash A$ *iff for every model* $\mathfrak{M}$ *that validates* Γ *and in which every time-assignment is a closed interval,* $\mathfrak{M} \models A$.

(b) $\Gamma \vdash A$ *iff for every model* $\mathfrak{M}$ *that validates* Γ *and in which every time-assignment is an open interval,* $\mathfrak{M} \models A$.

It might be possible to simplify the axiom system for $\mathbf{TL_{PC}}$ by using the definitions of the defined connectives and noting that the following are valid:

$$A <_{eb} B \supset A <_{ee} B \qquad A <_{eb} B \supset A <_{be} B$$

$$A <_{eb} B \supset A <_{bb} B \qquad A <_{bb} B \supset A <_{be} B$$

6. Time as a Subjective Ordering of Experience

We set out to formalize how to take time into account in our reasoning. That sounds as if we're assuming that time has a reality in the world distinct from us. Viewing time as a linear ordering seemed to confirm that we were viewing time as objective.

But nothing we've done depends on our taking time to be real beyond our experience of before and after. Our experiences, at least those we can communicate and investigate together, are expressed in linguistic propositions. The import of the proof of Theorem 16 is that we can construct "time" from a complete and consistent ordering of propositions in terms of when we say their descriptions are meant to begin and to end. In reasoning about time in this logic we need not assume that time is anything more than our ordering of experiences, and to use this logic together we need only agree on that ordering.

The intersubjectivity of our ordering of experiences may be due to the reality of time external to us which we perceive or are forced to acknowledge. But equally this conception allows for that intersubjective ordering to be based solely on our sharing subjective evaluations. For example, Zoe may "know" that "Dick yelled at Spot" is a true description of a time after "Spot was barking" because that's how it seems to her. If Dick agrees that "Dick yelled at Spot" is a true description of a time after "Spot was barking," then their orderings are the same for that part of their experience. If lots of people agree that "Dick yelled at Spot" is a true description of a time after "Spot was barking," then they have an intersubjective ordering of time for those experiences. On the other hand, that Zoe knows that "Zoe forgot her keys" came after "Zoe decided to go to Suzy's house" is a subjective ordering that others can only infer, though, again, it could be that her memory is grounded in a reality of time external to her that is shared by us all.[8]

7. Examples of Formalization

1. *Spot barked and then Dick yelled.*

Spot barked $\wedge_{\mathrm{eb}}$ Dick yelled

Analysis Unless context suggests otherwise, let's formalize "and then" to mean that the first proposition describes a time completely before the second.

[8] A subjective conception of time could also allow for a metric. An experience may be the recording of an oscillation of an atom. Then the recording of a successive oscillation is another experience, which could give us a standard minimal difference in times. Accumulating those, projecting those backwards and forwards, we have not only a linear ordering but a metric. One trillion trillion of those "experiences" would be equal — if we were to have them — to the time between when Spot barked and Dick yelled. Time, when we put a metric on it, is then not only our actual experiences but what we take to be our possible experiences. To say that time is dense in this subjective conception of time is to say that given any two experiences, it is possible to have another experience that comes between them. Compare J. F. Allen in [1]:

> There seems to be a strong intuition that, given an event, we can always "turn up the magnification" and look at its structure. This has certainly been the experience so far in physics. Since the only times we consider will be times of events, it appears that we can always decompose times into subparts. Thus the formal notion of a time point, which would not be decomposable, is not useful. [1, p. 834]

2. *Spot barked. Then Dick yelled.*

   Same as for Example 1

Analysis The word "then" is used to relate two propositions as with "and then."

3. *Dick yelled after Spot barked.*

   Same as for Example 1.

Analysis The order of the propositions is reversed in the formalization, understanding "after" to be the reverse of "and then."

4. *Suzy took off her clothes and went to bed.*

   Suzy took off her clothes $\wedge_{\mathrm{eb}}$ Suzy went to bed

Analysis We understand "and" here to mean "and then."

5. *If Suzy went to bed, then she off her clothes.*

   ¬(Suzy went to bed ∧ ¬(Suzy took off her clothes))

Analysis The "then" here is not meant temporally but only as part of the conditional. Indeed, in normal circumstances if this is true, then "Suzy took off her clothes" would be about a time before that of "Suzy went to bed." But that understanding is not part of what is asserted here.

6. *Spot barked before Dick yelled.*

Analysis As noted in Section 2.1, the use of "before" as a connective in English is ambiguous. We have at least two choices:

   Spot started barking before Dick began to yell.
   Spot barked $\wedge_{\mathrm{bb}}$ Dick yelled.
   Spot finished barking before Dick began to yell.
   Spot barked $\wedge_{\mathrm{eb}}$ Dick yelled.

In the latter case, we do not need to add that Spot started barking before Dick began to yell, since $(A \wedge_{\mathrm{eb}} B) \supset (A \wedge_{\mathrm{bb}} B)$ is valid. Nor do we need to add that Spot started barking before Dick ended yelling, since $(A \wedge_{\mathrm{eb}} B) \supset (A \wedge_{\mathrm{be}} B)$ is valid.

7. *Spot barked at 7:12 am May 5th, 2005.*
   Spot barked around 7:12 am May 5th, 2005.
   Spot barked at exactly 7:12 am May 5th, 2005.

Analysis We can distinguish these only by what time-assignment we give to "Spot barked." To distinguish them syntactically we would have to be able to refer to specific times in the formal language.

8. *Sometime after Dick finished eating, Spot began to bark.*

Dick ate $\wedge_{\mathrm{eb}}$ Spot barked

Analysis Despite the apparent quantification over times in the example, we can formalize it without quantifying. Note that we modify the sentences in the original to eliminate "finished" and "began."

9. *Dick yelled at the same time as Spot barked.*

(Spot barked $\wedge$ Dick yelled) $\wedge$ (Spot barked $\approx_{\mathrm{bb}}$ Dick yelled)
$\wedge$ (Spot barked $\approx_{\mathrm{ee}}$ Dick yelled)

Analysis We need no quantification or reference to a time in the formalization.

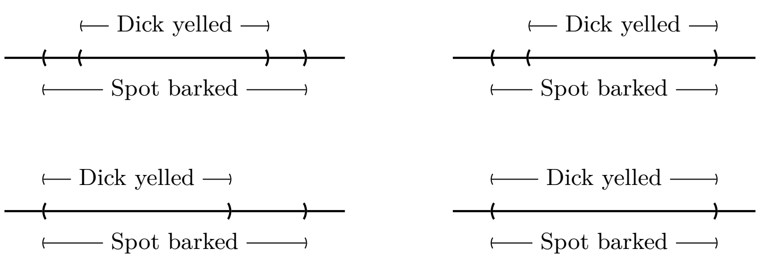
We can define a connective to formalize at the same time as:

$$\mathrm{A} \wedge_{\approx} \mathrm{B} \;\equiv_{\mathrm{df}}\; (\mathrm{A} \wedge \mathrm{B}) \wedge (\mathrm{A} \approx_{\mathrm{bb}} \mathrm{B}) \wedge (\mathrm{A} \approx_{\mathrm{ee}} \mathrm{B})$$

10. *Dick yelled within the time that Spot was barking.*

(Spot barked $\wedge$ Dick yelled) $\wedge\, \neg$(Dick yelled $\wedge_{\mathrm{bb}}$ Spot barked)
$\wedge\, \neg$(Spot barked $\wedge_{\mathrm{ee}}$ Dick yelled)

Analysis We have four possibilities for the times assigned to "Spot barked" and "Dick yelled" that can make this true:



All are covered by the formalization here. We can define a connective to formalize within the time that:

$$\mathrm{A} \wedge_{\mathrm{in}} \mathrm{B} \;\equiv_{\mathrm{df}}\; (\mathrm{A} \wedge \mathrm{B}) \wedge \neg(\mathrm{A} \wedge_{\mathrm{bb}} \mathrm{B}) \wedge \neg(\mathrm{B} \wedge_{\mathrm{ee}} \mathrm{A})$$

11. *Dick yelled during the time that Spot barked.*

ANALYSIS This is ambiguous. We have (at least) two choices:

Dick yelled within the time that Spot barked.

Formalization as for Example 10.

Dick yelled through all the time that Spot yelled, beginning before and ending after.

(Spot barked $\wedge$ Dick yelled) $\wedge$ (Dick yelled $\wedge_{\text{bb}}$ Spot barked)
$\wedge$ (Spot barked $\wedge_{\text{ee}}$ Dick yelled)

12. *There was a time when both Dick yelled and Spot barked.*

(Spot barked $\wedge$ Dick yelled) $\wedge$ [(Dick yelled $<_{\text{bb}}$ Spot barked) $\wedge$ (Spot barked $<_{\text{be}}$ Dick yelled)] $\vee$ [(Spot barked $<_{\text{bb}}$ Dick yelled) $\wedge$ (Dick yelled $<_{\text{be}}$ Spot barked)] $\vee$ (Spot barked $\wedge_{\text{in}}$ Dick yelled) $\vee$ (Dick yelled $\wedge_{\text{in}}$ Spot barked)

ANALYSIS Though the ordinary English involves a quantification over time, we need no quantification in the formalization to formalize overlapping times.

13. *Dick yelled, then Spot began barking, and then Spot finished barking before Dick stopped yelling.*

(Dick yelled $\wedge_{\text{bb}}$ Spot barked) $\wedge$ (Spot barked $\wedge_{\text{ee}}$ Dick yelled)

ANALYSIS It seems that we have four propositions here:

Dick yelled.
Spot began barking.
Spot finished barking.
Dick stopped yelling.

But we're treating "began," "finished," "stopped," and other words like those as modifiers of verbs, incorporated into the connectives rather than within the propositions since their role is to relate times of propositions. Moreover, we understand the second and third to be modifications, in this context, of the same proposition, "Spot barked."

14. *Dick yelled while Spot was barking.*

ANALYSIS The use of "while" as a connective in English is ambiguous. We have (at least) the following choices:

There was a time when both Dick yelled and Spot barked.
  Formalized as in Example 12.
Dick yelled at the same time that Spot barked.
  Formalized as in Example 9.
Dick yelled within the time that Spot barked.
  Formalized as in Example 10.
Dick yelled through all the time that Spot barked.

(Dick yelled $\wedge$ Spot barked) $\wedge$ $\neg$(Spot barked $\wedge_{\text{bb}}$ Dick yelled)
$\wedge$ $\neg$(Spot $\wedge_{\text{ee}}$ Dick yelled)

15. *Dick will cook dinner when Zoe arrives.*

Zoe will arrive $\wedge_{\text{eb}}$ Dick will cook dinner

ANALYSIS We use the present tense in "Zoe arrives" to talk about some future time, formalizing the example to mean that Dick will start cooking only after Zoe arrives.

16. *Dick will start cooking dinner the moment when Zoe arrives.*

(Dick will cook dinner $\wedge$ Zoe will arrive) $\wedge$
(Zoe will arrive $\approx_{\text{eb}}$ Dick will cook dinner)

ANALYSIS This works regardless of whether the times assigned to the propositions are open or closed intervals.

17. *Dick will begin cooking as soon as Zoe arrives home.*

Formalization the same as for Example 16.

ANALYSIS Even in ordinary speech we need not talk of quantification over times.

18. *Sometime between when Zoe arrived and Dick started cooking, Spot ran away.*

((Zoe arrived) $\wedge_{\text{eb}}$ (Spot ran away)) $\wedge$ (Spot ran away $\wedge_{\text{eb}}$
Dick cooked)

ANALYSIS We can formalize the time of a proposition being strictly between the times of two other propositions by defining a new connective:

$$\mathsf{SB}(\mathrm{A}, \mathrm{B}, \mathrm{C}) \equiv_{\text{df}} (\mathrm{A} <_{\text{eb}} \mathrm{B}) \wedge (\mathrm{B} <_{\text{eb}} \mathrm{C})$$

Alternatively, we can formalize that the time of a proposition comes between the times of two other propositions though not strictly by defining a new connective:

$$\mathsf{B}(\mathrm{A},\mathrm{B},\mathrm{C}) \;\equiv_{\mathrm{df}}\; \neg(\mathrm{B} <_{\mathrm{eb}} \mathrm{A}) \wedge \neg(\mathrm{C} <_{\mathrm{eb}} \mathrm{B})$$

19. *There was some time between when Zoe arrived and Dick started cooking.*

    Not formalizable?

Analysis Unless we know a proposition that is true whose time is between that of the time assigned to “Zoe arrived” and the time assigned to “Dick cooked” we can’t point to any time that comes between those. We can only say that the first is true of a time ending before than the start of the second with “(Zoe arrived) $\wedge_{\mathrm{eb}}$ (Dick cooked)”. Whether there are any “times” between those — not just in our model but in “reality” — is connected to the question of whether we can point to such a time with a proposition that describes a (possible) experience, as discussed in Section 6.

20. *Spot barked and then Dick yelled. But before that, Zoe shouted. This was all after Suzy arrived and before Tom arrived.*

    ((Suzy arrived) $\wedge_{\mathrm{eb}}$ ((Zoe shouted) $\wedge_{\mathrm{eb}}$ (Spot barked $\wedge_{\mathrm{eb}}$ Dick yelled))) $\wedge_{\mathrm{eb}}$ (Tom arrived)

Analysis We can take account of the relative times of any number of propositions.

21. *If Dick studied, then Zoe will cook.*

Analysis We can read “if ... then ...” classically to formalize the example as:

Dick studied $\supset$ Zoe will cook

Alternatively, we can read “then” in “if ... then ...” as indicating later in time. In that case we can formalize the example:

(Dick studies $\supset$ Zoe cooks) $\wedge$ (Dick studies $<_{\mathrm{eb}}$ Zoe cooks)

We can formalize “if ... then (later) ...” by introducing a new connective:

$$\mathrm{A} \supset_{\mathrm{eb}} \mathrm{B} \;\equiv_{\mathrm{df}}\; (\mathrm{A} \supset \mathrm{B}) \wedge (\mathrm{A} <_{\mathrm{eb}} \mathrm{B})$$

22. *The temperatures in New Mexico in 2012 were the highest they have been since the last ice age.*

    Not formalizable.

Analysis To formalize comparisons of times ("the last") as well as temperatures we need a more ample language, such as that of predicate logic.

23. *When Zoe cooks, Dick washes up.*

    Not formalizable.

Analysis The example is meant to be true of all times. That is, "Zoe cooks $\supset$ Dick washes up" is always true. But we don't want all of time to be assigned to the union of the times assigned to "Zoe cooks" and "Dick washes up." It seems that to formalize this example we need to quantify over times, which we can do in predicate logic. We can make the implicit quantification explicit by rewriting the example as "Whenever Zoe cooks, Dick washes up."

24. *Tom shot, skinned, and butchered a deer.*

    Not formalizable.

Analysis The first comma and the word "and" here are meant as "and then." But we can't formalize this as "(Tom shot a deer $\wedge_{\text{eb}}$ Tom skinned a deer) $\wedge_{\text{eb}}$ Tom butchered a deer" because the example clearly means that it's the same deer that Tom shot, skinned, and butchered. We need a way to formalize cross-referencing for this example, which is possible in predicate logic even though no quantification is involved.

25. *Dick talked only when Spot wasn't barking.*

    Not formalizable.

Analysis We can take "Spot barked" as an atomic proposition and assign a time to it. Then "$\neg$(Spot barked)" is meant as its contradictory and hence is about the same time. We can talk of times when Spot wasn't barking by considering propositions that are true of times before and times after the time when Spot barked. But "Dick talked" is an atomic proposition, and hence we can only assert either "Dick talked $\wedge_{\text{eb}}$ Spot barked," in which case we're asserting that Dick talked before Spot barked, or "Spot barked $\wedge_{\text{be}}$ Dick talked," in which case we're asserting that Dick talked after Spot barked. Or we could take "(Dick talked)$_1$" and "(Dick talked)$_2$" to be distinct propositions and assert:

$$((\text{Dick talked})_1 \wedge_{\text{eb}} \text{Spot barked}) \vee \text{Spot barked} \wedge_{\text{be}} (\text{Dick talked})_2$$

But that doesn't formalize that we intend "Dick talked" to be about lots of times, perhaps some before Spot barked and some after Spot barked.

The problem is that the complement of the time assigned to "Spot barked" or of "Spot barked ∧ Dick talked" need not be an interval or a union of intervals. In Section 9 we'll look at how we might extend our logic to allow for a temporal negation.

26. *Spot will bark and then Dick yelled.*

Spot will bark $\wedge_{\text{eb}}$ Dick yelled

ANALYSIS This can be true in a model since we can assign an interval of time to "Spot will bark" that comes before the time assigned to "Dick yelled." Yet the example can't be true, since the tenses indicate that "Dick yelled" is about a time in the past and "Spot will bark" is about a time in the future. To ensure that this is formalized as an anti-tautology we should take account of tenses directly in the language.[9]

Alternatively, we can stipulate an informal convention that if A is in the past or present tense and B is in the future tense, then we use $A \wedge_{\text{eb}} B$ as part of a formalization involving A and B, with similar conditions for other tenses. Let's look at how we might do that.

8. Past, Present, and Future

Part of our conception of time is of the past, present, and future. The present is *right now*. As you are reading this perhaps you'll understand that to mean the present is exactly as we were writing this. Or perhaps you'll understand it to mean that the present is exactly as you are reading this, which would have been in the future as we were writing. In any case, both those "times" are past now.

We have a subjective sense of the present or *right now*. But in our reasoning we talk of the present as any time that's used to split propositions into those meant to be about a time before and those meant to be about a time after. The present when we are writing this, the present

[9] A way to do that is sketched in [4] and is developed in [5].

of an essay written in 1774, the present of a science-fiction novel set in 2095, the present since the last ice age, all can be understood by us as "the present" in a conversation.

Given some collection of atomic propositions with which we want to reason and the semi-formal language of those, there are various ways we can talk about the world at a particular time, whether that's what we want to call the present or simply some other time we want to distinguish. We do so by giving a description that is meant to be about that time, say "Spot barked $\wedge$ Dick yelled." Then the fullest description we can give of that time would be the collection of atomic propositions that are true of that time: $\{p : p$ is atomic, and $\mathfrak{v}(p) = \mathtt{T}$, and $\mathsf{t}(p) \subseteq \mathsf{t}($Spot barked $\wedge$ Dick yelled$)\}$. If we can choose one particular proposition, whether compound or atomic, as a description of the world at "the present," then we can divide all propositions into those that are meant to be about times before that and those that are meant to be about times after that.

Explicitly, given a realization let N be a wff of the semi-formal language that we choose to be a description of the "the whole" present. Then we can define:

Past(A) $\equiv_{\text{df}}$ A $<_{\text{eb}}$ N
  In a model, $\mathfrak{v}(\text{Past(A)}) = \mathtt{T}$ iff $\mathfrak{v}(\text{A}) = \mathtt{T}$ and $\mathsf{t}(\text{A}) <_{\text{eb}} \mathsf{t}(\text{N})$

Present(A) $\equiv_{\text{df}}$ $\neg(\text{A} <_{\text{bb}} \text{N}) \wedge \neg(\text{N} <_{\text{ee}} \text{A})$
  In a model, $\mathfrak{v}(\text{Pres(A)}) = \mathtt{T}$ iff $\mathsf{t}(\text{A}) \subseteq \mathsf{t}(\text{N})$

Future(A) $\equiv_{\text{df}}$ N $<_{\text{eb}}$ A
  In a model, $\mathfrak{v}(\text{Future(A)}) = \mathtt{T}$ iff $\mathsf{t}(\text{N}) <_{\text{eb}} \mathsf{t}(\text{A})$

Note that Past(A) means A is about some time in the past. Past(A) does not mean A is about all of the past. When we say "the past" or "the future" we needn't be construed as talking about all of the past or all of the future as if those were some things. Similarly, when we say "the present" we needn't be talking of all of the time we consider to be the present.

27. *Tom had a dog.*

Tom had a dog $\wedge$ Past(Tom had a dog)

Analysis Because of the tensed form of "to have" we know that this is meant to be about some time in the past. In a narrative we can usually identify one or several propositions as being about the "now" of it. Suppose, in this case, we know that "Spot is Dick and Zoe's dog," and "Suzy loves Tom," and "Ralph is a dog" are all about the present and are ample enough together to cover the entire present of the narrative. Then we can take:

$$\text{N} \equiv_{\text{df}} \text{Spot is Dick and Zoe's dog} \wedge \text{Suzy loves Tom} \wedge \text{Ralph is a dog}$$

We needn't agree that N is true. Then we can formalize the example as an assertion not only that the proposition is true but is also about the past.

28. *Tom will have a dog.*

$$\text{Tom will have a dog} \wedge \text{Future(Tom will have a dog)}$$

ANALYSIS Taking N to be as in the previous example, we have the formalization.

29. *Tom has a dog.*

$$\text{Tom has a dog} \wedge \text{Present(Tom has a dog)}$$

ANALYSIS We formalize with the same assumption about the time of the present as in the previous two examples.

30. *Spot will bark and then Dick yelled.*

$$(\text{Spot will bark} \wedge_{\text{eb}} \text{Dick yelled}) \wedge \text{Future(Spot will bark)}$$
$$\wedge \text{Past(Dick yelled)}$$

ANALYSIS Using the same conventions about the time of the present, the formalization here is an anti-tautology. In a model, we can't have $\mathtt{t}(\text{N}) < \mathtt{t}(\text{Spot will bark})$, and $\mathtt{t}(\text{Dick yelled}) < \mathtt{t}(\text{N})$, and $\mathtt{t}(\text{Spot will bark}) < \mathtt{t}(\text{Dick yelled})$, since $<$ is transitive and linear.

We can take account of tenses in our formalizations by first agreeing that the times of a particular (small) collection of atomic propositions cover the present. Then relative to the conjunction of those we can define Past(A), Present(A), and Future(A).[10] We could, perhaps, be more subtle in devising ways to distinguish, say, the past perfect from the past. But all these conventions are about how to formalize certain kinds of ordinary language propositions. They are not directly formalizations of

[10] These bear a superficial resemblance to operators used in the tradition of Arthur Prior, but they are quite different, as explained in [4].

tenses. In English tenses are modifiers of verbs, and to deal directly with them we need to look at the internal structure of atomic propositions.[11]

9. Timeless Propositions

Some say that there are propositions about no time at all. They say that numbers, for example, are abstract objects existing outside space and time. Hence, "$2+2=4$" is not true of any time at all. It is not true of all times. It is timelessly true.

We can extend our temporal propositional logic to accomodate reasoning based on the assumption that there are such propositions. We add a unary connective $\text{\O}$ to the language of our logic, where $\text{\O}\,\mathrm{A}$ is meant to be an assertion that A is true and A is about no time. We modify our models to allow that the empty set is also an interval, so that $\mathtt{t}(\mathrm{A}) = \varnothing$ is allowed. We take the valuation:

$$\mathfrak{v}(\text{\O}\,\mathrm{A}) = \mathtt{T} \text{ iff } \mathfrak{v}(\mathrm{A}) = \mathtt{T} \text{ and } \mathtt{t}(\mathrm{A}) = \varnothing$$

We can define a further connective:

$$\underline{\text{\O}}\,\mathrm{A} \;\equiv_{\mathrm{df}}\; \text{\O}(\neg(\mathrm{A} \wedge \neg\mathrm{A}))$$
$$\mathfrak{v}(\underline{\text{\O}}\,\mathrm{A}) = \mathtt{T} \text{ iff } \mathtt{t}(\mathrm{A}) = \varnothing$$

Call this system $\mathbf{TL_{PC}}+\text{\O}$. With it we can formalize some talk of abstract things, such as numbers and beauty, along with talk about dogs and cats. We'll leave to those who have developed a greater intuition than we have for the nature of abstract things to provide examples and analyses.

It might seem simple to axiomatize this logic. We can modify each scheme A of the axiomatization of $\mathbf{TL_{PC}}$ other than those for **PC** itself to read:

$$(\bigwedge_{\mathrm{p\ in\ A}} \neg\,\underline{\text{\O}}\,\mathrm{p}) \supset \mathrm{A}$$

We can also add an axiom scheme:

$$\underline{\text{\O}}\,\mathrm{A} \supset [\neg(\mathrm{A} <_{\mathrm{bb}} \mathrm{B}) \wedge \neg(\mathrm{B} <_{\mathrm{bb}} \mathrm{A}) \wedge \neg(\mathrm{A} <_{\mathrm{ee}} \mathrm{B}) \wedge \neg(\mathrm{B} <_{\mathrm{ee}} \mathrm{A})$$
$$\wedge \neg(\mathrm{A} <_{\mathrm{be}} \mathrm{B}) \wedge \neg(\mathrm{B} <_{\mathrm{be}} \mathrm{A}) \wedge \neg(\mathrm{A} <_{\mathrm{eb}} \mathrm{B}) \wedge \neg(\mathrm{B} <_{\mathrm{eb}} \mathrm{A})$$
$$\wedge \neg(\mathrm{A} \approx_{\mathrm{bb}} \mathrm{B}) \wedge \neg(\mathrm{A} \approx_{\mathrm{ee}} \mathrm{B}) \wedge \neg(\mathrm{A} \approx_{\mathrm{be}} \mathrm{B}) \wedge \neg(\mathrm{B} \approx_{\mathrm{be}} \mathrm{A})]$$

However, the axiomatization of $\mathbf{TL_{PC}}$ uses in an essential way that $\approx_{\mathrm{bb}}$, $\approx_{\mathrm{be}}$, $\approx_{\mathrm{ee}}$ are defined connectives.

[11] See the discussion in [4].

10. Omnitemporal Propositions and Temporal Negation

10.1. Omnitemporal propositions

A proposition such as "Electrons have spin" is meant to be a true description of all times. At any time whatsoever, every electron has spin. We call propositions meant to be true of all times *omnitemporal*.

Suppose we have a realization where A is meant to be a true description of every time. Then we can assign $\mathsf{t}(\mathrm{A}) = \{z : \mathfrak{b}_{\mathcal{T}} < z < \mathfrak{e}_{\mathcal{T}}\}$, which is the collection of all times in our model, since the endpoints of the entire time line are meant as markers only. We then have for every p in the model:

not: $\mathsf{t}(\mathrm{p}) <_{\mathrm{bb}} \mathsf{t}(\mathrm{A})$
not: $\mathsf{t}(\mathrm{A}) <_{\mathrm{ee}} \mathsf{t}(\mathrm{p})$

We would like to pick out such propositions in our reasoning. We could do so if we introduce the following connective:

omnitemporal om
$\mathfrak{v}(\mathsf{om}\,\mathrm{A}) = \mathtt{T}$ iff $\mathfrak{v}(\mathrm{A}) = \mathtt{T}$ and $\mathsf{t}(\mathrm{A}) = \{z : \mathfrak{b}_{\mathcal{T}} < z < \mathfrak{e}_{\mathcal{T}}\}$

This cannot be defined in $\mathbf{TL_{PC}}$ because given any scheme $\mathsf{S}(\mathrm{A})$, even with parameters, $\mathsf{t}(\mathsf{S}(\mathrm{A}))$ will be the union of only a finite number of intervals, those assigned to the propositional variables in $\mathsf{S}(\mathrm{A})$, yet another propositional variable could be assigned an interval that lies outside those.

10.2. Temporal negation

We take ¬A to be the contradictory of A and hence to be about the same time as A. So "Spot barked" and "¬(Spot barked)" are about the same time. But then how can we formalize the following?

Dick talked only during the time that Spot didn't bark.

We need a way to talk about all times other than those in the interval of the time assigned to "Spot barked." We could do so if we introduce the following connective, where "−" indicates the set-theoretic complement of the given set:

temporal negation n
$\mathsf{t}(\mathsf{n}(\mathrm{A})) = -\mathsf{t}(\mathrm{A})$
$\mathfrak{v}(\mathsf{n}(\mathrm{A})) = \mathtt{T}$ iff $\mathfrak{v}(\mathrm{A}) = \mathtt{F}$

This can't be defined in $\mathbf{TL_{PC}}$ because given any scheme S(A) with parameters $q_1, \ldots, q_n$,

$$\mathsf{t}(\mathrm{S}(\mathrm{A})) = \bigcup\{\mathsf{t}(\mathrm{p}) : \mathrm{p} \text{ appears in } \mathrm{A}\} \cup \{\mathsf{t}(\mathrm{q}_i) : i \geqslant 1\} \supseteq \mathsf{t}(\mathrm{A}).$$

10.3. Temporal intersection

Consider:

Dick talked only during the time that both Zoe was talking and Spot was sleeping.

For this to be true we need that each of "Dick talked," "Zoe was talking," and "Spot was sleeping" are true and the time assigned to "Dick talked" is contained in both the time assigned to "Zoe was talking" and "Spot was sleeping." We could formalize this if we introduce in addition to the temporal negation connective the following connective:

temporal intersection X

$\mathsf{t}(\mathrm{A} \,\mathsf{X}\, \mathrm{B}) = \mathsf{t}(\mathrm{A}) \cap \mathsf{t}(\mathrm{B})$

$\mathfrak{v}(\mathrm{A} \,\mathsf{X}\, \mathrm{B}) = \mathtt{T}$ iff $\mathfrak{v}(\mathrm{A}) = \mathfrak{v}(\mathrm{B}) = \mathtt{T}$

Just as for the connective n, this cannot be defined in $\mathbf{TL_{PC}}$.

10.4. Extending $\mathbf{TL_{PC}}$?

We have seen limitations on what we can formalize with $\mathbf{TL_{PC}}$. It might seem that we could simply add the connectives om, on, and X to the language and then use the evaluations given above. Then we could define any connective whose evaluation or time assignment is given in terms of the truth-values of the atomic propositions in it and any combination using union, intersection, and complement of the times assigned to the atomic propositions in it. But if we do, then the time assigned to a compound wff need no longer be a union of intervals, so that comparing the times of compound propositions will not be straightforward.

Summary

We set out to formalize how to reason taking account of time when the only structure of propositions we would pay attention to is how they are built from other propositions using sentential connectives. We take a proposition to be a sentence that is a description of the world at some particular time. Thus, a proposition has two semantic values: a truth-value and a time it is meant to be about. We saw how to extend those semantic values to compound propositions that use the standard propositional connectives as well as connectives that formalize the "before" relation in terms of the beginning point and ending point of the intervals of times assigned to their compounds. This allowed us to give an axiomatization for which any complete and consistent collection of propositions has a model. From that we have that our logic can be understood as formalizing a subjective conception of time in which the timeline can be derived from an ordering of propositions describing our experiences. A series of examples of formalizing ordinary language sentences showed how the logic can be applied. Those also showed limitations on what we can formalize using this logic. We saw that we could formalize some reasoning with tenses by taking a particular proposition as being about the present and comparing others to it. We saw how we could extend our logic to allow for reasoning about timeless propositions. We considered also connectives that formalize omnitemporal propositions, temporal negation, and temporal intersection, but it was not clear how to modify our formal system to incorporate those.

This is a work in progress, a beginning only.

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RICHARD L. EPSTEIN
Advanced Reasoning Forum
Dogshine, New Mexico, USA
rle@ARFbooks.org

ESPERANZA BUITRAGO-DÍAZ
ebuitragod@gmail.com