T-ABF	T-DBF	T-TBF
$Ea, w_j t_k$ $rw_i w_j t_k$ $\downarrow$ $Ea, w_i t_k$	$Ea, w_j t_k$ $sw_i w_j t_k$ $\downarrow$ $Ea, w_i t_k$	$Ea, w_i t_k$ $t_j < t_k$ $\downarrow$ $Ea, w_i t_j$
T-ACBF	T-DCBF	T-TCBF
$Ea, w_i t_k$ $rw_i w_j t_k$ $\downarrow$ $Ea, w_j t_k$	$Ea, w_i t_k$ $sw_i w_j t_k$ $\downarrow$ $Ea, w_j t_k$	$Ea, w_i t_j$ $t_j < t_k$ $\downarrow$ $Ea, w_i t_k$

Table 9. Domain-inclusion (Barcan) rules

T-R=	T-S=	T-N=	T-D=
$*$ $\downarrow$ $t = t, w_i t_j$ for every t on the branch	$s = t, w_i t_j$ $A[s/x], w_i t_j$ $\downarrow$ $A[t/x], w_i t_j$ where A is atomic	$a = b, w_i t_j$ $\downarrow$ $a = b, w_k t_l$ for any w_k and t_l	$*$ $\downarrow$ $c = \alpha, w_i t_j$ where c is new to the branch for every α, w_i and t_j on the branch

Table 10. Identity rules and the descriptor rule

Moreover, in Table 10 (T-R=) and (T-S=) can be applied to both rigid constants and descriptors. (T-N=) is applied only if both terms, a and b, are rigid constants. In (T-D=) c is a rigid constant and α is a descriptor. (T-S=) is applied only within “world-moment pairs”, and we only apply the rule when A is atomic. However, in systems with necessary identity, it is in general true that $A[b/x]$ follows from $a = b$ and $A[a/x]$. And in systems with contingent identity this is true when x is not in the scope of a modal operator.[4]

4.2. Tableau systems

The concepts of tableau system, temporal alethic-deontic tableau system etc. are essentially defined as in [37], with the exception that every temporal alethic-deontic tableau system now also includes the rules A, $\neg\mathsf{A}$, S, and $\neg\mathsf{S}$.

[4] In the identity rules and the descriptor rule “R” stands for “reflexive”, “S” for “substitution (of identities)”, “N” for “necessary identity” and “D” for “descriptor”.

Let S be a temporal alethic-deontic tableau system. Then *CS*, a *constant domain quantified temporal alethic-deontic tableau system*, is S augmented by the rules for the possibilist quantifiers; *VS*, a *variable domain quantified temporal alethic-deontic tableau system*, is S augmented by the rules for the actualist quantifiers; and *CVS*, a *constant and variable domain quantified temporal alethic-deontic tableau system*, is S augmented by the rules for the possibilist and actualist quantifiers.

Any tableau system that includes T-C also includes Id(I) and Id(II) (see [37]). Any subset of the domain-inclusion (Barcan) rules may also be added to our systems.

Let S be a quantified temporal alethic-deontic tableau system without identity. Then *SNI* is S augmented by the rules for necessary identity, i.e., (T-R=), (T-S=) and (T-N=); *SCI* is S augmented by the rules for contingent identity, i.e., (T-R=) and (T-S=); *SNID* is *SNI* augmented by the rule for descriptors, i.e., (T-D=); and *SCID* is *SCI* augmented by (T-D=).

For example, *CVaTdDadMOOCt4CadtSPABFACBFNID* is the constant and variable domain quantified temporal alethic-deontic tableau system with necessary identity and descriptors that includes (T-aT), (T-dD), (T-MO), (T-OC), (T-t4), (T-SP), Id(I), Id(II) (introduced in [37]), (T-C), (T-ABF), (T-ACBF), the rules for the possibilist and actualist quantifiers, (T-R=), (T-S=), (T-N=) and (T-D=).

4.3. Some proof-theoretical concepts

The concepts of proof, theorem, derivation, consistency, inconsistency in a system, the logic of a tableau system etc. are defined as in [37]. An arbitrary formula A (a schema) is a theorem in the system S just in case every (closed) instance of A is a theorem in S. We also speak about theorems, derivations etc. with assumptions. These concepts were used in [37] but never explicitly defined. Here are the definitions.

Let $\triangleleft$ be $<$ or $=$. Then A is a theorem in S with the assumptions $v(t_1) \triangleleft v(t_2)$, $v(t_3) \triangleleft v(t_4)$, ... iff there is a closed S-tableau whose initial list comprises $t_1 \triangleleft t_2$, $t_3 \triangleleft t_4$, ... and $\neg A, w_0 t_0$.

A derivation in the system S of B from the (finite) set of formulas Γ with the assumptions $v(t_1) \triangleleft v(t_2)$, $v(t_3) \triangleleft v(t_4)$, ..., is a closed S-tableau whose initial list comprises $t_1 \triangleleft t_2$, $t_3 \triangleleft t_4$, ..., $A, w_0 t_0$ for every $A \in \Gamma$ and $\neg B, w_0 t_0$. Etc.

Obviously, we can also speak about validity of sentences and arguments with assumptions in models in a similar sense. Our soundness and completeness theorems can then be extended so that they include theorems and derivations with assumptions in a straightforward way.

Some theorems with assumptions were mentioned in [37]. For example, $\mathsf{R}t'\Box\mathsf{R}tA \to \mathsf{R}t\Box\mathsf{R}tA$ is a theorem in the system *adtSP* given that $v(t') < v(t)$ (for a proof, see [38, p. 299]). Since this system is sound with respect to the class of all models that satisfy C-SP, $\mathsf{R}t'\Box\mathsf{R}tA \to \mathsf{R}t\Box\mathsf{R}tA$ is valid on this class given that $v(t') < v(t)$. We can in fact prove something slightly stronger, namely that $\mathsf{R}t'\Box\mathsf{R}t''A \to \mathsf{R}t\Box\mathsf{R}t''A$ is a theorem in the system *adtSP* given that $v(t') < v(t)$. According to this theorem, if it is necessary that at time $v(t'')$ it is the case that A, then at every later time it is also necessary that at $v(t'')$ A. E.g. if it is now (today) (historically) necessary that I was in Stockholm yesterday, then at any time after today, it will be (historically) necessary that I was in Stockholm on that day (see [15]).

5. Examples of theorems

In this section we will consider some theorems in some systems. If not otherwise stated S will denote a constant and variable domain system that includes all definitions of all non-primitive concepts. All proofs are omitted; in most cases they are straightforward.

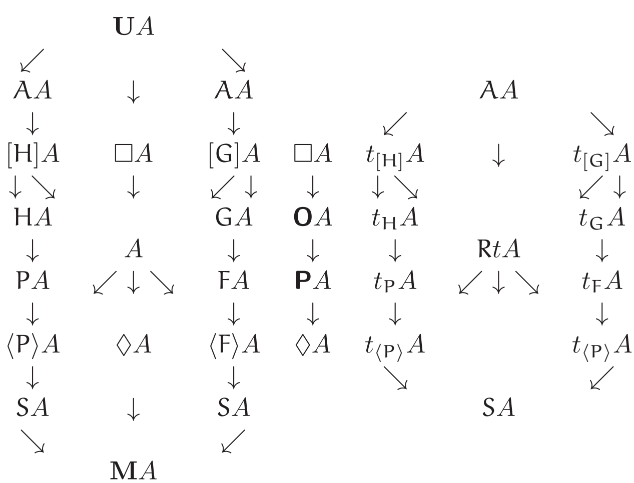

Figure 1.

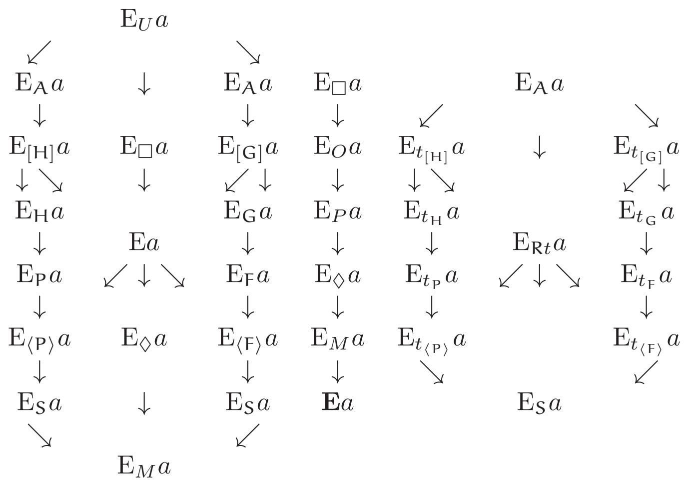

Figure 2.

THEOREM 1. (i) $\mathsf{H}A$ *implies* $\mathsf{P}A$ *and* $t_\mathsf{H}A$ *implies* $t_\mathsf{P}A$ *in every* S *that includes* T-PD. $\mathsf{G}A$ *implies* $\mathsf{F}A$ *and* $t_\mathsf{G}A$ *implies* $t_\mathsf{F}A$ *in every* S *that includes* T-FD. $\Box A$ *implies* A *and* A *implies* $\Diamond A$ *in every* S *that includes* T-aT. $\Box A$ *implies* $\mathbf{O}A$ *and* $\mathbf{P}A$ *implies* $\Diamond A$ *in every* S *that includes* T-MO. $\mathbf{O}A$ *implies* $\mathbf{P}A$ *in every* S *that includes* T-dD. *All other implications in Figure 1 hold in every* S.

(ii) *All implications in Figure 2 hold in* S *with similar provisos as in the part* (i).

(iii) $\forall xA$ *implies* $\forall_\Box xA$, $\forall_\Diamond xA$ *implies* $\forall xA$, $\exists_\Box xA$ *implies* $\exists xA$, *and* $\exists xA$ *implies* $\exists_\Diamond xA$ *in every* S *that includes* T-aT. *If* S *includes* T-FD, *then all of the following implications hold in* S: $\forall_\mathsf{F} xA \to \forall_\mathsf{G} xA$, $\exists_\mathsf{G} xA \to \exists_\mathsf{F} xA$, $\forall_{t_\mathsf{F}} xA \to \forall_{t_\mathsf{G}} xA$, $\exists_{t_\mathsf{G}} xA \to \exists_{t_\mathsf{F}} xA$. *If* S *includes* T-PD, *then all of the following implications hold in* S: $\forall_\mathsf{P} xA \to \forall_\mathsf{H} xA$, $\exists_\mathsf{H} xA \to \exists_\mathsf{P} xA$, $\forall_{t_\mathsf{P}} xA \to \forall_{t_\mathsf{H}} xA$, $\exists_{t_\mathsf{H}} xA \to \exists_{t_\mathsf{P}} xA$. *If* S *includes* T-MO, *then all of the following implications hold in* S: $\exists_\Box xA \to \exists_O xA$, $\exists_P xA \to \exists_\Diamond xA$, $\forall_\Diamond xA \to \forall_P xA$ *and* $\forall_O xA \to \forall_\Box xA$. *If* S *includes* T-dD, *then the following implications hold in* S: $\exists_O xA \to \exists_P xA$ *and* $\forall_P xA \to \forall_O xA$. *All other implications in Figures 3 and 4 hold in every* S.

THEOREM 2. (i) *Let* $\odot$ *be a positive modal operator or empty.* (*When* $\odot$ *is empty,* $\forall_\odot xFx = \forall xFx$ *and* $\exists_\odot xFx = \exists xFx$.) *Then all formulas in Table 11 are theorems in* S.

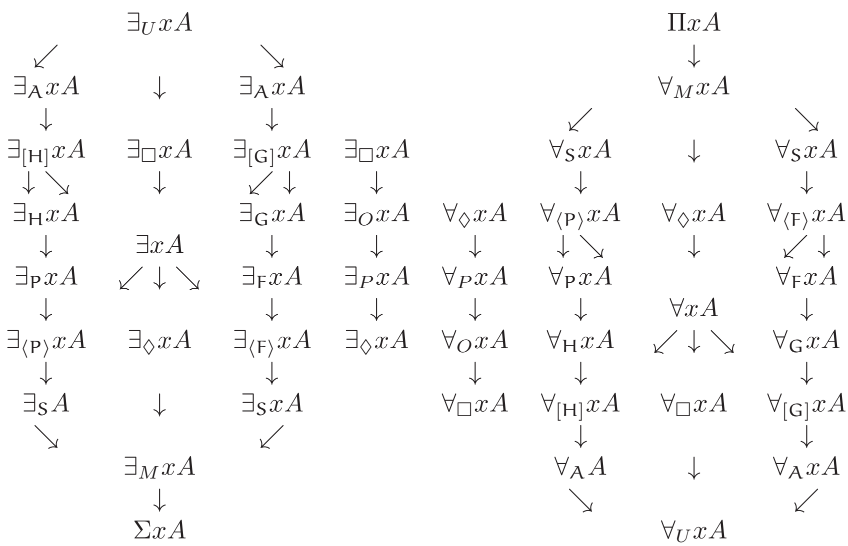

Figure 3.

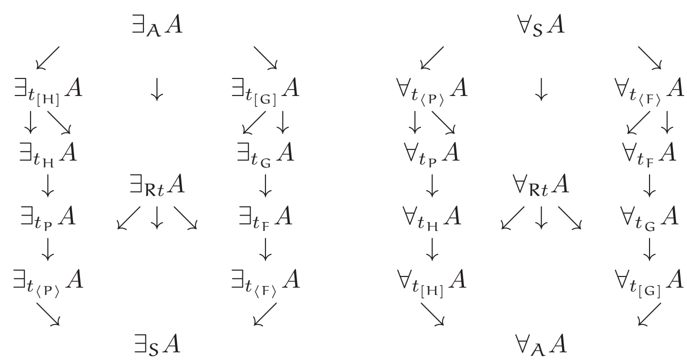

Figure 4.

(ii) *Not all instances of the schemas in Table 12 are theorems in every S.*

(iii) *Replace every occurrence of $\forall_{\odot}$ with Π and every occurrence of $\exists_{\odot}$ with Σ in the schemas in Tables 11 and 12. Then the resulting formulas are theorems in every S. (Π and Σ behave as classical quantifiers, and $\forall_{\odot}$ and $\exists_{\odot}$ as quantifiers in positive free logic.)*

THEOREM 3. *Let $\forall_S$ and $\forall_W$ be two universal quantifiers such that $\forall_S$ is stronger than $\forall_W$ in S (i.e., $\forall_S A$ implies $\forall_W A$ in S), and let $\exists_S$ and $\exists_W$*

$\forall_{\odot}xFx \leftrightarrow \neg\exists_{\odot}x\neg Fx$	$\exists_{\odot}xFx \leftrightarrow \neg\forall_{\odot}x\neg Fx$
$\forall_{\odot}x\neg Fx \leftrightarrow \neg\exists_{\odot}xFx$	$\exists_{\odot}x\neg Fx \leftrightarrow \neg\forall_{\odot}xFx$
$(\forall_{\odot}xFx \land E_{\odot}a) \rightarrow Fa$	$(Fa \land E_{\odot}a) \rightarrow \exists_{\odot}xFx$
$\forall_{\odot}x(Fx \rightarrow Gx) \leftrightarrow \neg\exists_{\odot}x(Fx \land \neg Gx)$	$\forall_{\odot}x(Fx \rightarrow \neg Gx) \leftrightarrow \neg\exists_{\odot}x(Fx \land Gx)$
$\exists_{\odot}x(Fx \land Gx) \leftrightarrow \neg\forall_{\odot}x(Fx \rightarrow \neg Gx)$	$\exists_{\odot}x(Fx \land \neg Gx) \leftrightarrow \neg\forall_{\odot}x(Fx \rightarrow Gx)$
$\forall_{\odot}x(Fx \leftrightarrow Fx)$	$\forall_{\odot}x(Fx \lor \neg Fx)$
$\neg\exists_{\odot}x(Fx \land \neg Fx)$	$\forall_{\odot}x\neg(Fx \land \neg Fx)$
$\neg\exists_{\odot}xFx \rightarrow \forall_{\odot}x(Fx \rightarrow Gx)$	$\exists_{\odot}x(Fx \rightarrow Gx) \rightarrow (\forall_{\odot}xFx \rightarrow \exists_{\odot}xGx)$
$\forall_{\odot}xFx \leftrightarrow \forall_{\odot}yFy$	$\exists_{\odot}xFx \leftrightarrow \exists_{\odot}yFy$
$\forall_{\odot}x\forall_{\odot}yFxy \leftrightarrow \forall_{\odot}y\forall_{\odot}xFxy$	$\exists_{\odot}x\exists_{\odot}yFxy \leftrightarrow \exists_{\odot}y\exists_{\odot}xFxy$
$\forall_{\odot}x(Fx \land Gx) \leftrightarrow (\forall_{\odot}xFx \land \forall_{\odot}xGx)$	$\exists_{\odot}x(Fx \land Gx) \rightarrow (\exists_{\odot}xFx \land \exists_{\odot}xGx)$
$\exists_{\odot}x(Fx \lor Gx) \leftrightarrow (\exists_{\odot}xFx \lor \exists_{\odot}xGx)$	$(\forall_{\odot}xFx \lor \forall_{\odot}xGx) \rightarrow \forall_{\odot}x(Fx \lor Gx)$
$\forall_{\odot}x(Fx \rightarrow Gx) \rightarrow (\forall_{\odot}xFx \rightarrow \forall_{\odot}xGx)$	$\forall_{\odot}x(Fx \rightarrow Gx) \rightarrow (\exists_{\odot}xFx \rightarrow \exists_{\odot}xGx)$
$\forall_{\odot}x(Fx \leftrightarrow Gx) \rightarrow (\forall_{\odot}xFx \leftrightarrow \forall_{\odot}xGx)$	$\forall_{\odot}x(Fx \leftrightarrow Gx) \rightarrow (\exists_{\odot}xFx \leftrightarrow \exists_{\odot}xGx)$

Table 11.

$\forall_{\odot}xFx \rightarrow Fa$	$Fa \rightarrow \exists_{\odot}xFx$
$\forall_{\odot}xFx \rightarrow \exists_{\odot}xFx$	$\exists_{\odot}xFx \lor \exists_{\odot}x\neg Fx$
$\neg(\forall_{\odot}xFx \land \forall_{\odot}x\neg Fx)$	$\forall_{\odot}x\neg Fx \rightarrow \neg\forall_{\odot}xFx$
$(\forall_{\odot}xFx \land \forall_{\odot}xGx) \rightarrow \exists_{\odot}x(Fx \land Gx)$	$\forall_{\odot}x(Fx \land Gx) \rightarrow (\exists_{\odot}xFx \land \exists_{\odot}xGx)$
$(\forall_{\odot}xFx \lor \forall_{\odot}xGx) \rightarrow \exists_{\odot}x(Fx \lor Gx)$	$\forall_{\odot}x(Fx \lor Gx) \rightarrow (\exists_{\odot}xFx \lor \exists_{\odot}xGx)$
$(\exists_{\odot}xFx \rightarrow \exists_{\odot}xGx) \rightarrow \exists_{\odot}x(Fx \rightarrow Gx)$	$(\forall_{\odot}xFx \rightarrow \forall_{\odot}xGx) \rightarrow \exists_{\odot}x(Fx \rightarrow Gx)$

Table 12.

$\forall_S x(A \land B) \rightarrow (\forall_W A \land \forall_W B)$	$\forall_S x(A \leftrightarrow B) \rightarrow (\forall_W A \leftrightarrow \forall_W B)$
$(\forall_S xA \lor \forall_S xB) \rightarrow \forall_W(A \lor B)$	$\forall_S x(A \leftrightarrow B) \rightarrow (\exists_W xA \leftrightarrow \exists_W xB)$
$(\forall_S xA \land \forall_S xB) \rightarrow \forall_W(A \land B)$	$\forall_S x(A \leftrightarrow B) \rightarrow (\neg\exists_W xA \leftrightarrow \neg\exists_W xB)$
$\exists_W x(A \land B) \rightarrow (\exists_S xA \land \exists_S xB)$	$(\forall_W xA \land \forall_S x(A \rightarrow B)) \rightarrow \forall_W xB$
$\exists_W x(A \lor B) \rightarrow (\exists_S xA \lor \exists_S xB)$	$\forall_S x(A \rightarrow B) \rightarrow (\forall_W xA \rightarrow \forall_W xB)$
$(\exists_W xA \lor \exists_W xB) \rightarrow \exists_S x(A \lor B)$	$(\exists_W xA \land \forall_S x(A \rightarrow B)) \rightarrow \exists_W xB$
$\neg\exists_S x(A \lor B) \rightarrow (\neg\exists_W xA \land \neg\exists_W xB)$	$\forall_S x(A \rightarrow B) \rightarrow (\exists_W xA \rightarrow \exists_W xB)$
$(\neg\exists_S xA \lor \neg\exists_S xB) \rightarrow \neg\exists_W x(A \land B)$	$(\neg\exists_W xB \land \forall_S x(A \rightarrow B)) \rightarrow \neg\exists_W xA$
$(\neg\exists_S xA \land \neg\exists_S xB) \rightarrow \neg\exists_W x(A \lor B)$	$\forall_S x(A \rightarrow B) \rightarrow (\neg\exists_W xB \rightarrow \neg\exists_W xA)$
$\forall_S x(A \rightarrow B) \rightarrow (\forall_S xA \rightarrow \forall_W xB)$	$\forall_S x(A \rightarrow B) \rightarrow (\exists_W xA \rightarrow \exists_S xB)$
$\forall_S x(A \rightarrow B) \rightarrow (\neg\exists_S xB \rightarrow \neg\exists_W xA)$	$(\forall_W x(A \lor B) \land \neg\exists_S xB) \rightarrow \forall_W xA$

Table 13.

be the duals of $\forall_S$ *and* $\forall_W$*, respectively. Then all formulas in Tables* 13 *and* 14 *are theorems in* S*.*

THEOREM 4. (i) *All formulas in Tables* 15, 16 *and* 17 *are theorems in* S*.*

(ii) *Let* A *be a formula in Table* 15, 16 *or* 17*. Let* B *be the result of replacing every occurrence of* $\forall$ *in* A *by any universal quantifier,* Q*, every occurrence of* $\exists$ *with the dual of* Q*, every occurrence of* $\Box$ *by any*

$\forall_S x((A \vee B) \to C) \to ((\forall_W xA \vee \forall_W xB) \to \forall_W xC)$
$\forall_S x((A \vee B) \to C) \to ((\exists_W xA \vee \exists_W xB) \to \exists_W xC)$
$\forall_S x((A \vee B) \to C) \to (\neg\exists_W xC \to (\neg\exists_W xA \wedge \neg\exists_W xB))$
$\forall_S x(A \to (B \vee C)) \to (\exists_W xA \to (\exists_W xB \vee \exists_W xC))$
$\forall_S x(A \to (B \vee C)) \to ((\neg\exists_W xB \wedge \neg\exists_W xC) \to \neg\exists_W xA)$
$\forall_S x((A \wedge B) \to C) \to ((\forall_W xA \wedge \forall_W xB) \to \forall_W xC)$
$\forall_S x(A \to (B \wedge C)) \to (\forall_W xA \to (\forall_W xB \wedge \forall_W xC))$
$\forall_S x(A \to (B \wedge C)) \to (\exists_W xA \to (\exists_W xB \wedge \exists_W xC))$
$\forall_S x(A \to (B \wedge C)) \to ((\neg\exists_W xB \vee \neg\exists_W xC) \to \neg\exists_W xA)$
$(\forall_W x(A \vee B) \wedge (\forall_S x(A \to C) \wedge \forall_S x(B \to C))) \to \forall_W xC$
$(\forall_W x(A \vee B) \wedge (\forall_S x(A \to C) \wedge \forall_S x(B \to D))) \to \forall_W x(C \vee D)$
$(\forall_W xA \wedge (\forall_S x(A \to B) \wedge \forall_S x(A \to C))) \to (\forall_W xB \wedge \forall_W xC)$
$(\forall_W x(A \wedge B) \wedge (\forall_S x(A \to C) \vee \forall_S x(B \to D))) \to \forall_W x(C \vee D)$
$(\forall_W xA \wedge (\forall_S x(A \to B) \vee \forall_S x(A \to C))) \to \forall_W x(B \vee C)$
$(\forall_W x(A \wedge B) \wedge (\forall_S x(A \to C) \wedge \forall_S x(B \to D))) \to (\forall_W xC \wedge \forall_W xD)$

Table 14.

$\Box\forall x(A \to B) \leftrightarrow \diamondminus\exists x(A \wedge \neg B)$	$\Box\forall x(A \to \neg B) \leftrightarrow \diamondminus\exists x(A \wedge B)$
$\Box\exists x(A \wedge B) \leftrightarrow \diamondminus\forall x(A \to \neg B)$	$\Box\exists x(A \wedge \neg B) \leftrightarrow \diamondminus\forall x(A \to B)$
$\forall x\Box(A \wedge B) \leftrightarrow \forall x(\Box A \wedge \Box B)$	$\exists x\Diamond(A \vee B) \leftrightarrow \exists x(\Diamond A \vee \Diamond B)$
$\forall x\Diamond(A \vee B) \leftrightarrow \forall x(\Diamond A \vee \Diamond B)$	$\exists x\Box(A \wedge B) \leftrightarrow \exists x(\Box A \wedge \Box B)$
$\forall x\Diamond(A \wedge B) \to \forall x(\Diamond A \wedge \Diamond B)$	$\exists x(\Box A \vee \Box B) \to \exists x\Box(A \vee B)$
$\exists x\Diamond(A \wedge B) \to \exists x(\Diamond A \wedge \Diamond B)$	$\forall x(\Box A \vee \Box B) \to \forall x\Box(A \vee B)$
$\forall x\diamondminus(A \vee B) \leftrightarrow \forall x(\diamondminus A \wedge \diamondminus B)$	$\exists x\diamondminus(A \vee B) \leftrightarrow \exists x(\diamondminus A \wedge \diamondminus B)$
$\forall x(\diamondminus A \vee \diamondminus B) \to \forall x\diamondminus(A \wedge B)$	$\exists x(\diamondminus A \vee \diamondminus B) \to \exists x\diamondminus(A \wedge B)$
$\forall x\Box(A \to B) \to \forall x(\Box A \to \Box B)$	$\forall x\Box(A \to B) \to \forall x(\Diamond A \to \Diamond B)$
$\forall x\Box(A \leftrightarrow B) \to \forall x(\Box A \leftrightarrow \Box B)$	$\forall x\Box(A \leftrightarrow B) \to \forall x(\Diamond A \leftrightarrow \Diamond B)$
$\forall x\Box(A \to B) \to \forall x(\diamondminus B \to \diamondminus A)$	$\forall x\Box(A \leftrightarrow B) \to \forall x(\diamondminus A \leftrightarrow \diamondminus B)$

Table 15.

positive necessity-like modal operator, L, every occurrence of $\Diamond$ by the dual, M, of L, $\boxminus$ by $\neg L$ and $\diamondminus$ by $\neg M$. Then B is a theorem in S.

THEOREM 5. *Let $\Box$ and $\mathbf{O}$ be two (positive) necessity-like modal operators such that $\Box$ is stronger than $\mathbf{O}$ in S (i.e., $\Box A \to \mathbf{O}A$ is a theorem in S), let $\Diamond$ and $\mathbf{P}$ be the duals of $\Box$ and $\mathbf{O}$, respectively, and let $\forall$ be any universal quantifier, and $\exists$ the dual of $\forall$. Then all of the formulas in Tables 18 and 19 hold in S.*

THEOREM 6. (i) *If S includes* T-ABF, *then $\forall x\Box A \to \Box\forall xA$ and $\Diamond\exists xA \to \exists x\Diamond A$ are theorems in S. If it includes* T-ACBF, *then $\Box\forall xA \to \forall x\Box A$, $\exists x\Diamond A \to \Diamond\exists xA$, $\Diamond\forall xA \to \forall x\Diamond A$, and $\exists x\Box A \to \Box\exists xA$ are theorems in S.*

$\Box\forall x(A \wedge B) \leftrightarrow (\Box\forall xA \wedge \Box\forall xB)$	$\Diamond\exists x(A \vee B) \leftrightarrow (\Diamond\exists xA \vee \Diamond\exists xB)$
$\Diamond\forall x(A \wedge B) \rightarrow (\Diamond\forall xA \wedge \Diamond\forall xB)$	$(\Box\exists xA \vee \Box\exists xB) \rightarrow \Box\exists x(A \vee B)$
$\forall x\Box(A \wedge B) \leftrightarrow (\forall x\Box A \wedge \forall x\Box B)$	$\exists x\Diamond(A \vee B) \leftrightarrow (\exists x\Diamond A \vee \exists x\Diamond B)$
$\forall x\Diamond(A \wedge B) \rightarrow (\forall x\Diamond A \wedge \forall x\Diamond B)$	$(\exists x\Box A \vee \exists x\Box B) \rightarrow \exists x\Box(A \vee B)$
$(\Box\forall xA \vee \Box\forall xB) \rightarrow \Box\forall x(A \vee B)$	$\Diamond\exists x(A \wedge B) \rightarrow (\Diamond\exists xA \wedge \Diamond\exists xB)$
$(\Diamond\forall xA \vee \Diamond\forall xB) \rightarrow \Diamond\forall x(A \vee B)$	$\Box\exists x(A \wedge B) \rightarrow (\Box\exists xA \wedge \Box\exists xB)$
$(\forall x\Box A \vee \forall x\Box B) \rightarrow \forall x\Box(A \vee B)$	$\exists x\Diamond(A \wedge B) \rightarrow (\exists x\Diamond A \wedge \exists x\Diamond B)$
$(\forall x\Diamond A \vee \forall x\Diamond B) \rightarrow \forall x\Diamond(A \vee B)$	$\exists x\Box(A \wedge B) \rightarrow (\exists x\Box A \wedge \exists x\Box B)$
$\forall x\diamondminus(A \vee B) \leftrightarrow (\forall x\diamondminus A \wedge \forall x\diamondminus B)$	$(\exists x \boxminus A \vee \exists x \boxminus B) \leftrightarrow \exists x \boxminus (A \wedge B)$
$\exists x\diamondminus(A \vee B) \rightarrow (\exists x\diamondminus A \wedge \exists x\diamondminus B)$	$(\forall x \boxminus A \vee \forall x \boxminus B) \rightarrow \forall x \boxminus (A \wedge B)$
$(\exists x\diamondminus A \vee \exists x\diamondminus B) \rightarrow \exists x\diamondminus(A \wedge B)$	$\forall x \boxminus (A \vee B) \rightarrow (\forall x \boxminus A \wedge \forall x \boxminus B)$
$(\forall x\diamondminus A \vee \forall x\diamondminus B) \rightarrow \forall x\diamondminus(A \wedge B)$	$\exists x \boxminus (A \vee B) \rightarrow (\exists x \boxminus A \wedge \exists x \boxminus B)$

Table 16.

$\Box\forall x(A \rightarrow B) \rightarrow (\Box\forall xA \rightarrow \Box\forall xB)$	$\Box\forall x(A \rightarrow B) \rightarrow (\Diamond\forall xA \rightarrow \Diamond\forall xB)$
$\Box\forall x(A \rightarrow B) \rightarrow (\Box\exists xA \rightarrow \Box\exists xB)$	$\Box\forall x(A \rightarrow B) \rightarrow (\Diamond\exists xA \rightarrow \Diamond\exists xB)$
$\forall x\Box(A \rightarrow B) \rightarrow (\forall x\Box A \rightarrow \forall x\Box B)$	$\forall x\Box(A \rightarrow B) \rightarrow (\forall x\Diamond A \rightarrow \forall x\Diamond B)$
$\forall x\Box(A \rightarrow B) \rightarrow (\exists x\Box A \rightarrow \exists x\Box B)$	$\forall x\Box(A \rightarrow B) \rightarrow (\exists x\Diamond A \rightarrow \exists x\Diamond B)$
$\Box\forall x(A \rightarrow B) \rightarrow (\diamondminus\forall xB \rightarrow \diamondminus\forall xA)$	$\Box\forall x(A \rightarrow B) \rightarrow (\diamondminus\exists xB \rightarrow \diamondminus\exists xA)$
$\forall x\Box(A \rightarrow B) \rightarrow (\forall x\diamondminus B \rightarrow \forall x\diamondminus A)$	$\forall x\Box(A \rightarrow B) \rightarrow (\exists x\diamondminus B \rightarrow \exists x\diamondminus A)$
$\Box\forall x(A \leftrightarrow B) \rightarrow (\Box\forall xA \leftrightarrow \Box\forall xB)$	$\Box\forall x(A \leftrightarrow B) \rightarrow (\Diamond\forall xA \leftrightarrow \Diamond\forall xB)$
$\Box\forall x(A \leftrightarrow B) \rightarrow (\Box\exists xA \leftrightarrow \Box\exists xB)$	$\Box\forall x(A \leftrightarrow B) \rightarrow (\Diamond\exists xA \leftrightarrow \Diamond\exists xB)$
$\forall x\Box(A \leftrightarrow B) \rightarrow (\forall x\Box A \leftrightarrow \forall x\Box B)$	$\forall x\Box(A \leftrightarrow B) \rightarrow (\forall x\Diamond A \leftrightarrow \forall x\Diamond B)$
$\forall x\Box(A \leftrightarrow B) \rightarrow (\exists x\Box A \leftrightarrow \exists x\Box B)$	$\forall x\Box(A \leftrightarrow B) \rightarrow (\exists x\Diamond A \leftrightarrow \exists x\Diamond B)$
$\Box\forall x(A \leftrightarrow B) \rightarrow (\diamondminus\forall xA \leftrightarrow \diamondminus\forall xB)$	$\Box\forall x(A \leftrightarrow B) \rightarrow (\diamondminus\exists xA \leftrightarrow \diamondminus\exists xB)$
$\forall x\Box(A \leftrightarrow B) \rightarrow (\forall x\diamondminus A \leftrightarrow \forall x\diamondminus B)$	$\forall x\Box(A \leftrightarrow B) \rightarrow (\exists x\diamondminus A \leftrightarrow \exists x\diamondminus B)$

Table 17.

$\forall x\Box(A \wedge B) \rightarrow (\forall x\mathbf{O}A \wedge \forall x\mathbf{O}B)$	$\forall x\Box(A \leftrightarrow B) \rightarrow (\forall x\mathbf{O}A \leftrightarrow \forall x\mathbf{O}B)$
$(\forall x\Box A \vee \forall x\Box B) \rightarrow \forall x\mathbf{O}(A \vee B)$	$\forall x\Box(A \leftrightarrow B) \rightarrow (\forall x\mathbf{P}A \leftrightarrow \forall x\mathbf{P}B)$
$(\forall x\Box A \wedge \forall x\Box B) \rightarrow \forall x\mathbf{O}(A \wedge B)$	$\forall x\Box(A \leftrightarrow B) \rightarrow (\forall x\neg\mathbf{P}A \leftrightarrow \forall x\neg\mathbf{P}B)$
$\forall x\mathbf{P}(A \wedge B) \rightarrow (\forall x\Diamond A \wedge \forall x\Diamond B)$	$(\forall x\mathbf{O}A \wedge \forall x\Box(A \rightarrow B)) \rightarrow \forall x\mathbf{O}B$
$\exists x\mathbf{P}(A \vee B) \rightarrow (\exists x\Diamond A \vee \exists x\Diamond B)$	$\forall x\Box(A \rightarrow B) \rightarrow (\forall x\mathbf{O}A \rightarrow \forall x\mathbf{O}B)$
$(\exists x\mathbf{P}A \vee \exists x\mathbf{P}B) \rightarrow \exists x\Diamond(A \vee B)$	$(\forall x\mathbf{P}A \wedge \forall x\Box(A \rightarrow B)) \rightarrow \forall x\mathbf{P}B$
$\forall x\neg\Diamond(A \vee B) \rightarrow (\forall x\neg\mathbf{P}A \wedge \forall x\neg\mathbf{P}B)$	$\forall x\Box(A \rightarrow B) \rightarrow (\forall x\mathbf{P}A \rightarrow \forall x\mathbf{P}B)$
$(\exists x\neg\Diamond A \vee \exists x\neg\Diamond B) \rightarrow \exists x\neg\mathbf{P}(A \wedge B)$	$(\forall x\neg\mathbf{P}B \wedge \forall x\Box(A \rightarrow B)) \rightarrow \forall x\neg\mathbf{P}A$
$(\forall x\neg\Diamond A \wedge \forall x\neg\Diamond B) \rightarrow \forall x\neg\mathbf{P}(A \vee B)$	$\forall x\Box(A \rightarrow B) \rightarrow (\forall x\neg\mathbf{P}B \rightarrow \forall x\neg\mathbf{P}A)$
$\forall x\Box(A \rightarrow B) \rightarrow (\forall x\Box A \rightarrow \forall x\mathbf{O}B)$	$\forall x\Box(A \rightarrow B) \rightarrow (\forall x\mathbf{P}A \rightarrow \forall x\Diamond B)$
$\forall x\Box(A \rightarrow B) \rightarrow (\forall x\neg\Diamond B \rightarrow \forall x\neg\mathbf{P}A)$	$(\forall x\mathbf{O}(A \vee B) \wedge \forall x\neg\Diamond B) \rightarrow \forall x\mathbf{O}A$

Table 18.

(ii) *If S includes* T-DBF, *then* $\forall x\mathbf{O}A \rightarrow \mathbf{O}\forall xA$ *and* $\mathbf{P}\exists xA \rightarrow \exists x\mathbf{P}A$ *are theorems in S. If it includes* T-DCBF, *then* $\mathbf{O}\forall xA \rightarrow \forall x\mathbf{O}A$,

$\forall x\Box((A \vee B) \to C) \to ((\exists x\mathbf{O}A \vee \exists x\mathbf{O}B) \to \exists x\mathbf{O}C)$
$\forall x\Box((A \vee B) \to C) \to ((\exists x\mathbf{P}A \vee \exists x\mathbf{P}B) \to \exists x\mathbf{P}C)$
$\forall x\Box((A \vee B) \to C) \to (\forall x\neg\mathbf{P}C \to (\forall x\neg\mathbf{P}A \wedge \forall x\neg\mathbf{P}B))$
$\forall x\Box(A \to (B \vee C)) \to (\exists x\mathbf{P}A \to (\exists x\mathbf{P}B \vee \exists x\mathbf{P}C))$
$\forall x\Box(A \to (B \vee C)) \to ((\forall x\neg\mathbf{P}B \wedge \forall x\neg\mathbf{P}C) \to \forall x\neg\mathbf{P}A)$
$\forall x\Box((A \wedge B) \to C) \to ((\forall x\mathbf{O}A \wedge \forall x\mathbf{O}B) \to \forall x\mathbf{O}C)$
$\forall x\Box(A \to (B \wedge C)) \to (\forall x\mathbf{O}A \to (\forall x\mathbf{O}B \wedge \forall x\mathbf{O}C))$
$\forall x\Box(A \to (B \wedge C)) \to (\forall x\mathbf{P}A \to (\forall x\mathbf{P}B \wedge \forall x\mathbf{P}C))$
$\forall x\Box(A \to (B \wedge C)) \to ((\exists x\neg\mathbf{P}B \vee \exists x\neg\mathbf{P}C) \to \exists x\neg\mathbf{P}A)$
$(\forall x\mathbf{O}(A \vee B) \wedge (\forall x\Box(A \to C) \wedge \forall x\Box(B \to C))) \to \forall x\mathbf{O}C$
$(\forall x\mathbf{O}(A \vee B) \wedge (\forall x\Box(A \to C) \wedge \forall x\Box(B \to D))) \to \forall x\mathbf{O}(C \vee D)$
$(\forall x\mathbf{O}A \wedge (\forall x\Box(A \to B) \wedge \forall x\Box(A \to C))) \to (\forall x\mathbf{O}B \wedge \forall x\mathbf{O}C)$
$(\forall x\mathbf{O}(A \wedge B) \wedge \forall x(\Box(A \to C) \vee \Box(B \to D))) \to \forall x\mathbf{O}(C \vee D)$
$(\forall x\mathbf{O}A \wedge \forall x(\Box(A \to B) \vee \Box(A \to C))) \to \forall x\mathbf{O}(B \vee C)$
$(\forall x\mathbf{O}(A \wedge B) \wedge (\forall x\Box(A \to C) \wedge \forall x\Box(B \to D))) \to (\forall x\mathbf{O}C \wedge \forall x\mathbf{O}D)$

Table 19.

$\exists x\mathbf{P}A \to \mathbf{P}\exists xA$, $\mathbf{P}\forall xA \to \forall x\mathbf{P}A$, *and* $\exists x\mathbf{O}A \to \mathbf{O}\exists xA$ *are theorems in* S.

(iii) *If* S *includes* T-TBF, *then the following sentences are theorems in* S: $\forall x\mathsf{G}A \to \mathsf{G}\forall xA$, $\mathsf{F}\exists xA \to \exists x\mathsf{F}A$, $\mathsf{H}\forall xA \to \forall x\mathsf{H}A$, $\exists x\mathsf{P}A \to \mathsf{P}\exists xA$, $\forall x[\mathsf{G}]A \to [\mathsf{G}]\forall xA$, $\langle\mathsf{F}\rangle\exists xA \to \exists x\langle\mathsf{F}\rangle A$, $[\mathsf{H}]\forall xA \to \forall x[\mathsf{H}]A$, $\exists x\langle\mathsf{P}\rangle A \to \langle\mathsf{P}\rangle\exists xA$, $\mathsf{P}\forall xA \to \forall x\mathsf{P}A$, $\exists x\mathsf{H}A \to \mathsf{H}\exists xA$, $\langle\mathsf{P}\rangle\forall xA \to \forall x\langle\mathsf{P}\rangle A$, $\exists x[\mathsf{H}]A \to [\mathsf{H}]\exists xA$.

(iv) *If* S *includes* T-TCBF, *then the following sentences are theorems in* S: $\mathsf{G}\forall xA \to \forall x\mathsf{G}A$, $\exists x\mathsf{F}A \to \mathsf{F}\exists xA$, $\forall x\mathsf{H}A \to \mathsf{H}\forall xA$, $\mathsf{P}\exists xA \to \exists x\mathsf{P}A$, $[\mathsf{G}]\forall xA \to \forall x[\mathsf{G}]A$, $\exists x\langle\mathsf{F}\rangle A \to \langle\mathsf{F}\rangle\exists xA$, $\forall x[\mathsf{H}]A \to [\mathsf{H}]\forall xA$, $\langle\mathsf{P}\rangle\exists xA \to \exists x\langle\mathsf{P}\rangle A$, $\mathsf{F}\forall xA \to \forall x\mathsf{F}A$, $\exists x\mathsf{G}A \to \mathsf{G}\exists xA$, $\langle\mathsf{F}\rangle\forall xA \to \forall x\langle\mathsf{F}\rangle A$, $\exists x[\mathsf{G}]A \to [\mathsf{G}]\exists xA$.

(v) *If* S *doesn't contain any domain-inclusion rules, then the actualist quantifiers do not "commute" with any modal operators.*

THEOREM 7. *The following equivalences hold in every* S.

(i) $\Pi x\mathbf{O}\mathsf{A}B \leftrightarrow \mathbf{O}\Pi x\mathsf{A}B \leftrightarrow \mathbf{O}\mathsf{A}\Pi xB \leftrightarrow \neg\Sigma x\mathbf{P}\mathsf{S}\neg B \leftrightarrow \neg\mathbf{P}\Sigma x\mathsf{S}\neg B \leftrightarrow \neg\mathbf{P}\mathsf{S}\Sigma x\neg B$.

(ii) $\Pi x\mathsf{A}\mathbf{O}B \leftrightarrow \mathsf{A}\Pi x\mathbf{O}B \leftrightarrow \mathsf{A}\mathbf{O}\Pi xB \leftrightarrow \neg\Sigma x\mathsf{S}\mathbf{P}\neg B \leftrightarrow \neg\mathsf{S}\Sigma x\mathbf{P}\neg B \leftrightarrow \neg\mathsf{S}\mathbf{P}\Sigma x\neg B$.

(iii) $\Pi x\mathbf{O}\mathsf{G}B \leftrightarrow \mathbf{O}\Pi x\mathsf{G}B \leftrightarrow \mathbf{O}\mathsf{G}\Pi xB \leftrightarrow \neg\Sigma x\mathbf{P}\mathsf{F}\neg B \leftrightarrow \neg\mathbf{P}\Sigma x\mathsf{F}\neg B \leftrightarrow \neg\mathbf{P}\mathsf{F}\Sigma x\neg B$.

(iv) $\Pi x\mathsf{G}\mathbf{O}B \leftrightarrow \mathsf{G}\Pi x\mathbf{O}B \leftrightarrow \mathsf{G}\mathbf{O}\Pi xB \leftrightarrow \neg\Sigma x\mathsf{F}\mathbf{P}\neg B \leftrightarrow \neg\mathsf{F}\Sigma x\mathbf{P}\neg B \leftrightarrow \neg\mathsf{F}\mathbf{P}\Sigma x\neg B$.

(v) *The formulas in* (i) *are not logically equivalent with the formulas in* (ii), *and the formulas in* (iii) *are not logically equivalent with the formulas in* (iv).

Let A be atomic. Then $\mathbf{O}\mathsf{A}A \leftrightarrow (\mathsf{H}A \land A \land \mathbf{O}\mathsf{G}A)$, $\mathbf{O}[\mathsf{G}]A \leftrightarrow (A \land \mathbf{O}\mathsf{G}A)$, $\mathbf{O}\langle\mathsf{F}\rangle A \leftrightarrow (A \lor \mathbf{O}\mathsf{F}A)$, $\mathbf{O}\mathsf{S}A \leftrightarrow (\mathsf{P}A \lor A \lor \mathbf{O}\mathsf{F}A)$ hold in *aTB4dD5adMOtCadtSPBTFT*.

So, it is problematic to symbolize a sentence such as "Everyone ought always to be honest" as $\Pi x\mathbf{O}\mathsf{A}Hx$ in some logics, since this is false if someone in the past wasn't honest or someone now isn't honest. In these systems, which are plausible if we assume that the past and present are settled, a norm is "reasonable" only when it is future oriented (see [37]). But we can get a similar result by using G instead of A. "Everyone ought always (in the future) to be honest" can then be symbolized as $\Pi x\mathbf{O}\mathsf{G}Hx$. The interesting norms in these systems often have one of the following forms: $\mathbf{O}\mathsf{G}A$, $\mathbf{O}\mathsf{F}A$, $\mathbf{O}\mathsf{R}tA$, $\mathbf{O}t_{\mathsf{P}}A$, $\mathbf{O}t_{\mathsf{F}}A$, $\mathbf{O}t_{\langle\mathsf{P}\rangle}A$, $\mathbf{O}t_{\langle\mathsf{F}\rangle}A$, where $v(t)$ lies in the future.

THEOREM 8. (i) *Let* $\blacksquare = \mathbf{U}$, A, $[\mathsf{G}]$, G, $\Box$, O, $[\mathsf{H}]$, H, $\mathsf{R}t$, $t_{[\mathsf{H}]}$, t_{H}, $t_{[\mathsf{G}]}$ *or* t_{H}. *Let* $\blacklozenge = \mathbf{M}$, S, $\langle\mathsf{F}\rangle$, F, $\Diamond$, P, $\langle\mathsf{P}\rangle$, P, $\mathsf{R}t$, $t_{\langle\mathsf{P}\rangle}$, t_{P}, $t_{\langle\mathsf{F}\rangle}$ *or* t_{F}. *Then every formula in Table 20 is a theorem in S.*

(ii) *Let* $\blacksquare = \mathsf{A}$, $[\mathsf{G}]$, G, $[\mathsf{H}]$, H, $\mathsf{R}t$, $t_{[\mathsf{H}]}$, t_{H}, $t_{[\mathsf{G}]}$ *or* t_{H}. *Let* $\blacklozenge = \mathsf{S}$, $\langle\mathsf{F}\rangle$, F, $\langle\mathsf{P}\rangle$, P, $\mathsf{R}t$, $t_{\langle\mathsf{P}\rangle}$, t_{P}, $t_{\langle\mathsf{F}\rangle}$ *or* t_{F}. *Then every formula in Table 21 is a theorem in S.*

$\Pi x\blacksquare A \leftrightarrow \blacksquare\Pi xA$	$\Sigma x\blacklozenge A \leftrightarrow \blacklozenge\Sigma xA$
$\Sigma x\blacksquare A \rightarrow \blacksquare\Sigma xA$	$\blacklozenge\Pi xA \rightarrow \Pi x\blacklozenge A$
$\forall_M x\blacksquare A \leftrightarrow \blacksquare\forall_M xA$	$\exists_M x\blacklozenge A \leftrightarrow \blacklozenge\exists_M xA$
$\exists_M x\blacksquare A \rightarrow \blacksquare\exists_M xA$	$\blacklozenge\forall_M xA \rightarrow \forall_M x\blacklozenge A$
$\forall_U x\blacksquare A \leftrightarrow \blacksquare\forall_U xA$	$\exists_U x\blacklozenge A \leftrightarrow \blacklozenge\exists_U xA$
$\exists_U x\blacksquare A \rightarrow \blacksquare\exists_U xA$	$\blacklozenge\forall_U xA \rightarrow \forall_U x\blacklozenge A$

Table 20.

$\forall_{\mathsf{S}} x\blacksquare A \leftrightarrow \blacksquare\forall_{\mathsf{S}} xA$	$\exists_{\mathsf{S}} x\blacklozenge A \leftrightarrow \blacklozenge\exists_{\mathsf{S}} xA$
$\exists_{\mathsf{S}} x\blacksquare A \rightarrow \blacksquare\exists_{\mathsf{S}} xA$	$\blacklozenge\forall_{\mathsf{S}} xA \rightarrow \forall_{\mathsf{S}} x\blacklozenge A$
$\forall_{\mathsf{A}} x\blacksquare A \leftrightarrow \blacksquare\forall_{\mathsf{A}} xA$	$\exists_{\mathsf{A}} x\blacklozenge A \leftrightarrow \blacklozenge\exists_{\mathsf{A}} xA$
$\exists_{\mathsf{A}} x\blacksquare A \rightarrow \blacksquare\exists_{\mathsf{A}} xA$	$\blacklozenge\forall_{\mathsf{A}} xA \rightarrow \forall_{\mathsf{A}} x\blacklozenge A$
$\forall_{\mathsf{R}t} x\blacksquare A \leftrightarrow \blacksquare\forall_{\mathsf{R}t} xA$	$\exists_{\mathsf{R}t} x\blacklozenge A \leftrightarrow \blacklozenge\exists_{\mathsf{R}t} xA$
$\exists_{\mathsf{R}t} x\blacksquare A \rightarrow \blacksquare\exists_{\mathsf{R}t} xA$	$\blacklozenge\forall_{\mathsf{R}t} xA \rightarrow \forall_{\mathsf{R}t} x\blacklozenge A$

Table 21.

THEOREM 9. *Let S be any system with necessary identity and let $\odot$ be any necessity-like positive modal operator. Then $\forall x \forall y(x = y \to \odot(x = y))$ and $\forall x \forall y(\neg x = y \to \odot\neg(x = y))$ are theorems in S.*

THEOREM 10. *Let $\blacksquare = \mathsf{R}t$, A, H, G, $[\mathsf{H}]$ or $[\mathsf{G}]$, let $\blacklozenge$ be the dual of $\blacksquare$, and let $\odot = t_{\mathsf{P}}, t_{\mathsf{H}}, t_{\langle\mathsf{P}\rangle}, t_{[\mathsf{H}]}, t_{\mathsf{F}}, t_{\mathsf{G}}, t_{\langle\mathsf{F}\rangle}$ or $t_{[\mathsf{G}]}$. Then the following hold in every S: $\forall_{\odot} x \blacksquare A \leftrightarrow \blacksquare \forall_{\odot} x A$, $\exists_{\odot} x \blacklozenge A \leftrightarrow \blacklozenge \exists_{\odot} x A$, $\exists_{\odot} x \blacksquare A \to \blacksquare \exists_{\odot} x A$, $\blacklozenge \forall_{\odot} x A \to \forall_{\odot} x \blacklozenge A$.*

6. Soundness and completeness theorems

We are now in a position to prove that every system in this essay is sound and complete with respect to its semantics. The concepts of soundness and completeness are defined as usual (see e.g. [37]).[5]

6.1. Constant domain logics

Let us first consider all systems with a constant domain. We start with the weakest logic, *CS*, and then take a look at modifications required for stronger systems. At this stage we assume that our language doesn't contain the identity predicate or any descriptors.

LEMMA 11 (Locality). *Let $\mathcal{M}_1 = \langle D, W, T, <, R, S, v_1\rangle$, $\mathcal{M}_2 = \langle D, W, T, <, R, S, v_2\rangle$ be two (constant domain) models. The language of the two, which we call $\mathcal{L}$, is the same, for they have the same domain. Let A be any closed formula of $\mathcal{L}$ such that v_1 and v_2 agree on the denotations of all the predicates and constants in it. Then for all $\omega \in W$ and $\tau \in T$: $v_{1\omega\tau}(A) = v_{2\omega\tau}(A)$.*

PROOF. The proof is by recursion on the sentences in our language. "IH" refers to the induction hypothesis.

Atomic formulas. $v_{1\omega\tau}(Pa_1 \dots a_n) = 1$ iff $\langle v_1(a_1), \dots, v_1(a_n)\rangle \in v_{1\omega\tau}(P)$ iff $\langle v_2(a_1), \dots, v_2(a_n)\rangle \in v_{2\omega\tau}(P)$ iff $v_{2\omega\tau}(Pa_1 \dots a_n) = 1$.

Truth-functional connectives. Straightforward.

($\Box$) $v_{1\omega\tau}(\Box B) = 1$ iff for all ω' such that $R\omega\omega'\tau$, $v_{1\omega'\tau}(B) = 1$ iff for all ω' such that $R\omega\omega'\tau$, $v_{2\omega'\tau}(B) = 1$ (IH) iff $v_{2\omega\tau}(\Box B) = 1$.

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[5] The proofs in this section combine techniques from [36] and [37]. In all the soundness and completeness theorems, the new steps for our new temporal rules are straightforward and omitted.

(G) $v_{1\omega\tau}(\mathsf{G}B) = 1$ iff for all τ' such that $\tau < \tau'$, $v_{1\omega\tau'}(B) = 1$ iff for all τ' such that $\tau < \tau'$, $v_{2\omega\tau'}(B) = 1$ (IH) iff $v_{2\omega\tau}(\mathsf{G}B) = 1$.

The cases for the other primitive positive modal operators are similar.

(Π) $v_{1\omega\tau}(\Pi x B) = 1$ iff for all $d \in D$, $v_{1\omega\tau}(B[k_d/x]) = 1$ iff for all $d \in D$, $v_{2\omega\tau}(B[k_d/x]) = 1$ ((IH), and the fact that $v_{1\omega\tau}(k_d) = v_{2\omega\tau}(k_d) = d$) iff $v_{2\omega\tau}(\Pi x B) = 1$.

The case for the particular quantifier is similar. ⊣

Lemma 12 (Denotation). *Let $\mathcal{M} = \langle D, W, T, <, R, S, v\rangle$ be any (constant domain) model. Let A be any formula of $\mathcal{L}(\mathcal{M})$ with at most one free variable, x, and a and b be any two (non-temporal) constants such that $v(a) = v(b)$. Then for any $\omega \in W$ and $\tau \in T$: $v_{\omega\tau}(A[a/x]) = v_{\omega\tau}(A[b/x])$.*

Proof. The proof is by induction on the complexity of A.

Atomic formulas. (To illustrate, we assume that the formula has one occurrence of "a", distinct from each a_i.) $v_{\omega\tau}(Pa_1...a...a_n) = 1$ iff $\langle v(a_1), ..., v(a), ..., v(a_n)\rangle \in v_{\omega\tau}(P)$ iff $\langle v(a_1), ..., v(b), ..., v(a_n)\rangle \in v_{\omega\tau}(P)$ iff $v_{\omega\tau}(Pa_1...b...a_n) = 1$.

Truth-functional connectives. Straightforward.

(□) $v_{\omega\tau}(\Box B[a/x]) = 1$ iff for all ω' such that $R\omega\omega'\tau$, $v_{\omega'\tau}(B[a/x]) = 1$ iff for all ω' such that $R\omega\omega'\tau$, $v_{\omega'\tau}(B[b/x]) = 1$ (IH) iff $v_{\omega\tau}(\Box B[b/x]) = 1$.

(G) $v_{\omega\tau}(\mathsf{G}B[a/x]) = 1$ iff for all τ' such that $\tau < \tau'$, $v_{\omega\tau'}(B[a/x]) = 1$ iff for all τ' such that $\tau < \tau'$, $v_{\omega\tau'}(B[b/x]) = 1$ (IH) iff $v_{\omega\tau}(\mathsf{G}B[b/x]) = 1$.

Note that $(\Box B)[a/x] = \Box(B[a/x])$. So, the ambiguity in $\Box B[a/x]$ is harmless. The same goes for the ambiguity in $\mathsf{G}B[a/x]$. The arguments for the other primitive positive modal operators are similar.

(Π) Let A be of the form $\Pi y B$. If $x = y$, then $A[a/x] = A[b/x] = A$, so the result is trivial. Accordingly, suppose that x and y are distinct. Then, $(\Pi y B)[b/x] = \Pi y(B[b/x])$ and $(B[b/x])[a/y] = (B[a/y])[b/x]$. $v_{\omega\tau}((\Pi y B)[a/x]) = 1$ iff $v_{\omega\tau}(\Pi y(B[a/x])) = 1$ iff for all $d \in D$, $v_{\omega\tau}((B[a/x])[k_d/y]) = 1$ iff for all $d \in D$, $v_{\omega\tau}((B[k_d/y])[a/x]) = 1$ iff for all $d \in D$, $v_{\omega\tau}((B[k_d/y])[b/x]) = 1$ (IH) iff for all $d \in D$, $v_{\omega\tau}((B[b/x])[k_d/y]) = 1$ iff $v_{\omega\tau}(\Pi y(B[b/x])) = 1$ iff $v_{\omega\tau}((\Pi y B)[b/x]) = 1$.

The case for the particular quantifier is similar. ⊣

6.1.1. Soundness theorem

Let $\mathcal{M}$ be any (constant domain) model and $\mathcal{B}$ any branch of a tableau. Then $\mathcal{B}$ is satisfiable in $\mathcal{M}$ iff there is a function f from $w_0, w_1, w_2, \ldots$ to W, and a function g from $t_0, t_1, t_2, \ldots$ to T such that (i) A is true in $f(w_i)$

at $g(t_j)$ in $\mathcal{M}$, for every node $A, w_i t_j$ on $\mathcal{B}$, (ii) if $rw_i w_j t_k$ is on $\mathcal{B}$, then $Rf(w_i)f(w_j)g(t_k)$ in $\mathcal{M}$, (iii) if $sw_i w_j t_k$ is on $\mathcal{B}$, then $Sf(w_i)f(w_j)g(t_k)$ in $\mathcal{M}$, (iv) if $t_i < t_j$ is on $\mathcal{B}$, then $g(t_i) < g(t_j)$ in $\mathcal{M}$, (v) if $t_i = t_j$ is on $\mathcal{B}$, then $g(t_i) = g(t_j)$ in $\mathcal{M}$. If these conditions are fulfilled, we say that f and g show that $\mathcal{B}$ is satisfiable in $\mathcal{M}$.

Lemma 13 (Soundness Lemma). *Let $\mathcal{B}$ be any branch of a tableau and $\mathcal{M}$ be any (constant domain) model. If $\mathcal{B}$ is satisfiable in $\mathcal{M}$ and a tableau rule is applied to it, then there is a (constant domain) model $\mathcal{M}'$ and an extension of $\mathcal{B}$, $\mathcal{B}'$, such that $\mathcal{B}'$ is satisfiable in $\mathcal{M}'$.*

Proof. As usual the proof is an induction. Let f and g be functions that show that the branch $\mathcal{B}$ is satisfiable in $\mathcal{M}$.

Connectives and the modal operators. See [37].

($\neg\Pi$) Since $\mathcal{B}$ is satisfiable in $\mathcal{M}$, $\neg\Pi x A$ is true in $f(w_i)$ at $g(t_j)$ in $\mathcal{M}$. Thus, $\Pi x A(x)$ is false in $f(w_i)$ at $g(t_j)$ in $\mathcal{M}$. Accordingly, there is some $d \in D$ such that $A[k_d/x]$ is false in $f(w_i)$ at $g(t_j)$ in $\mathcal{M}$, i.e., $\neg A[k_d/x]$ is true in $f(w_i)$ at $g(t_j)$ in $\mathcal{M}$. Consequently, $\Sigma x \neg A$ is true in $f(w_i)$ at $g(t_j)$ in $\mathcal{M}$. We can therefore take $\mathcal{M}'$ to be $\mathcal{M}$. The argument for ($\neg\Sigma$) is similar.

(Π) $\mathcal{M}$ makes $\Pi x A$ true in $f(w_i)$ at $g(t_j)$. For $\mathcal{B}$ is satisfiable in $\mathcal{M}$. Hence, $A[k_d/x]$ is true in $f(w_i)$ at $g(t_j)$ in $\mathcal{M}$, for all $d \in D$. Let d be such that $v(a) = v(k_d)$. By the Denotation Lemma, $A[a/x]$ is true in $f(w_i)$ at $g(t_j)$ in $\mathcal{M}$. Accordingly we can take $\mathcal{M}'$ to be $\mathcal{M}$.

(Σ) Since $\mathcal{B}$ is satisfiable in $\mathcal{M}$, $\Sigma x A$ is true in $f(w_i)$ at $g(t_j)$ in $\mathcal{M}$. Hence, there is some $d \in D$ such that $\mathcal{M}$ makes $A[k_d/x]$ true in $f(w_i)$ at $g(t_j)$. Let $\mathcal{M}' = \langle D, W, T, <, R, S, v' \rangle$ be the same as $\mathcal{M}$ except that $v'(c) = d$. Since c does not occur in $A[k_d/x]$, $A[k_d/x]$ is true in $f(w_i)$ at $g(t_j)$ in $\mathcal{M}'$, by the Locality Lemma. By the Denotation Lemma and the fact that $v'(c) = d = v'(k_d)$, $A[c/x]$ is true in $f(w_i)$ at $g(t_j)$ in $\mathcal{M}'$. Furthermore, $\mathcal{M}'$ makes all other formulas on the branch true at their respective world-moment pairs as well, by the Locality Lemma. For c does not occur in any other formula on the branch. ⊣

Theorem 14 (Soundness Theorem). *CS is strongly sound with respect to its semantics.*

Proof. The proof is as in [37]. ⊣

6.1.2. Completeness theorem

DEFINITION 15 (Induced Model). Let $\mathcal{B}$ be an open complete branch of a tableau, let t_i, t_j and t_k be temporal constants, and let I be the set of numbers on $\mathcal{B}$ immediately preceded by "t" in a temporal constant. We shall say that $i \rightleftharpoons j$ just in case $i = j$, or "$t_i = t_j$" or "$t_j = t_i$" occur on $\mathcal{B}$. $\rightleftharpoons$ is an equivalence relation and $[i]$ is the equivalence class of i. Furthermore, let C be the set of all non-temporal, rigid constants on $\mathcal{B}$. The (constant domain) model, $\mathcal{M} = \langle D, W, T, < R, S, v\rangle$, induced by $\mathcal{B}$ is defined as follows. $D = \{o_a : a \in C\}$ (or if C is empty, $D = \{o\}$, for some arbitrary o). For all non-temporal, rigid constants, a, on $\mathcal{B}$, $v(a) = o_a$. For every n-place predicate on $\mathcal{B}$ $\langle o_{a_1}, \ldots, o_{a_n}\rangle \in v_{\omega_i \tau_{[j]}}(P)$ iff $Pa_1 \ldots a_n, w_i t_j$ is on $\mathcal{B}$. (o is not in the extension of anything.) $W = \{\omega_i : w_i \textit{ occurs on } \mathcal{B}\}$, $T = \{\tau_{[i]} : i \in I\}$, $\tau_{[i]} < \tau_{[j]}$ iff $t_i < t_j$ occurs on $\mathcal{B}$, $R\omega_i\omega_j\tau_{[k]}$ iff $rw_iw_jt_k$ occurs on $\mathcal{B}$, $S\omega_i\omega_j\tau_{[k]}$ iff $sw_iw_jt_k$ occurs on $\mathcal{B}$. If a temporal constant, t_i, occurs on $\mathcal{B}$, then $v(t_i) = \tau_{[i]}$. If our tableau system neither includes T-FC, T-PC nor T-C, $\rightleftharpoons$ is reduced to identity and $[i] = i$. Hence, in such systems, we may take T to be $\{\tau_i : t_i \textit{ occurs on } \mathcal{B}\}$ and dispense with the equivalence classes.

LEMMA 16 (Completeness Lemma). *Let $\mathcal{B}$ be an open branch in a complete tableau and let $\mathcal{M}$ be a (constant domain) model induced by $\mathcal{B}$. Then, for every formula A:*
(i) If $A, w_i t_j$ is on $\mathcal{B}$ then $v_{\omega_i \tau_{[j]}}(A) = 1$, and (ii) If $\neg A, w_i t_j$ is on $\mathcal{B}$ then $v_{\omega_i \tau_{[j]}}(A) = 0$.

PROOF. The proof is by induction on the complexity of A.

Atomic formulas. $Pa_1 ... a_n, w_i t_j$ is on $\mathcal{B} \Rightarrow \langle o_{a_1}, \ldots, o_{a_n}\rangle \in v_{\omega_i \tau_{[j]}}(P)$ $\Rightarrow \langle v(a_1), \ldots, v(a_n)\rangle \in v_{\omega_i \tau_{[j]}}(P) \Rightarrow v_{\omega_i \tau_{[j]}}(Pa_1 \ldots a_n) = 1$.

($\neg$) $\neg Pa_1 \ldots a_n, w_i t_j$ is on $\mathcal{B} \Rightarrow Pa_1 \ldots a_n, w_i t_j$ is not on $\mathcal{B}$ ($\mathcal{B}$ open) $\Rightarrow \langle o_{a_1}, \ldots, o_{a_n}\rangle \notin v_{\omega_i \tau_{[j]}}(P) \Rightarrow \langle v(a_1), \ldots, v(a_n)\rangle \notin v_{\omega_i \tau_{[j]}}(P)$ $\Rightarrow v_{\omega_i \tau_{[j]}}(Pa_1 \ldots a_n) = 0$.

Other truth-functional connectives and modal operators. The argument is as in [37].

(Σ) Suppose that $\Sigma x A, w_i t_j$ is on the branch. Since the tableau is complete (Σ) has been applied. Accordingly, for some c, $A[c/x], w_i t_j$ is on the branch. Hence, $v_{\omega_i \tau_{[j]}}(A[c/x]) = 1$, by (IH). For some $d \in D$, $v(c) = d$. However, $v(k_d) = d$. Consequently, $v_{\omega_i \tau_{[j]}}(A[k_d/x]) = 1$, by the Denotation Lemma. It follows that $v_{\omega_i \tau_{[j]}}(\Sigma x A) = 1$. Suppose that $\neg \Sigma x A, w_i t_j$ is on the branch. Since the tableau is complete ($\neg\Sigma$) has been applied. So, $\Pi x \neg A, w_i t_j$ is on the branch. Again, since the tableau is complete (Π) has been applied. Thus, for all $c \in C$, $\neg A[c/x], w_i t_j$ is on the branch. Consequently, $v_{\omega_i \tau_{[j]}}(A[c/x]) = 0$ for all $c \in C$, by (IH). If $d \in D$, then for some $c \in C$, $v(c) = v(k_d)$. By the Denotation Lemma, for all $d \in D$, $v_{\omega_i \tau_{[j]}}(A[k_d/x]) = 0$. Consequently, $v_{\omega_i \tau_{[j]}}(\Sigma x A) = 0$.

The case for Π is similar. ⊣

THEOREM 17 (Completeness Theorem). *CS is strongly complete with respect to its semantics.*

PROOF. The proof is as in [37]. ⊣

6.1.3. General correctness theorem

THEOREM 18 (General Correctness Theorem). *Let S be any of the constant domain tableau systems discussed in this essay (without identity). Then S is (strongly) sound and complete with respect to its semantics.*

PROOF. The proof is as for *CS*, with some minor modifications. There are new cases for the various accessibility rules in the Soundness Lemma. In the proof of the Completeness Theorem, we have to check that the induced model is a model of the appropriate kind. This is as in [37]. ⊣

6.2. Variable domain logics

In this section, we will prove soundness and completeness theorems for variable domain systems. We start with the weakest system, *VS*, and then consider all stronger systems. We also consider the addition of the domain-inclusion (Barcan) rules.

LEMMA 19 (Locality). *The same as in the constant domain case, except that we replace "constant domain" with "variable domain".*

PROOF. We use the actualist quantifiers. However, the proof is essentially as in the constant domain case. The only difference is that clauses of the form "$d \in D$" are replaced by ones of the form "$d \in D_{\omega\tau}$". ⊣

LEMMA 20 (Denotation). *The same as in the constant domain case, except that we replace "constant domain" with "variable domain".*

PROOF. Again, we use the actualist quantifiers. And again, the proof is essentially as in the constant domain case. The only difference is that clauses of the form "$d \in D$" are replaced by ones of the form "$d \in D_{\omega\tau}$". ⊣

6.2.1. Soundness theorem

Theorem 21 (Soundness Theorem). *VS is strongly sound with respect to its semantics.*

Proof. The proof is as in the constant domain case. However, now we are using the actualist quantifiers. So, we have to add steps for them in the Soundness Lemma. Let f and g be functions that show that the branch $\mathcal{B}$ is satisfiable in $\mathcal{M}$.

($\forall$) Since $\mathcal{B}$ is satisfiable in $\mathcal{M}$, $\forall xA$ is true in $f(w_i)$ at $g(t_j)$ in $\mathcal{M}$. Hence, $A[k_d/x]$ is true in $f(w_i)$ at $g(t_j)$ in $\mathcal{M}$, for all $d \in D_{f(w_i)g(t_j)}$. Consequently, for any $d \in D$, either $\neg Ek_d$ or $A[k_d/x]$ is true in $f(w_i)$ at $g(t_j)$ in $\mathcal{M}$. Let d be such that $v(a) = v(k_d)$. By the Denotation Lemma, either $\neg Ea$ or $A[a/x]$ is true in $f(w_i)$ at $g(t_j)$ in $\mathcal{M}$. Consequently, at least one branch is satisfiable in $\mathcal{M}$, and we can take $\mathcal{M}'$ to be $\mathcal{M}$.

($\exists$) Since $\mathcal{B}$ is satisfiable in $\mathcal{M}$, $\exists xA$ is true in $f(w_i)$ at $g(t_j)$ in $\mathcal{M}$. Accordingly, for some $d \in D_{f(w_i)g(t_j)}$, $A[k_d/x]$ is true in $f(w_i)$ at $g(t_j)$ in $\mathcal{M}$. Hence, $\mathcal{M}$ makes $\mathrm{E}k_d$ and $A[k_d/x]$ true in $f(w_i)$ at $g(t_j)$. Let $\mathcal{M}' = \langle D, W, T, <, R, S, v'\rangle$ be the same as $\mathcal{M}$, except that $v'(c) = d$. By the Locality Lemma, $\mathcal{M}'$ makes $\mathrm{E}k_d$ and $A[k_d/x]$ true in $f(w_i)$ at $g(t_j)$, since c does not occur in $A[k_d/x]$. Furthermore, $v'(c) = d = v'(k_d)$. So, $\mathrm{E}c$ and $A[c/x]$ are true in $f(w_i)$ at $g(t_j)$ in $\mathcal{M}'$, by the Denotation Lemma. By the Locality Lemma, $\mathcal{M}'$ makes all other formulas on the branch true in $f(w_i)$ at $g(t_j)$ as well. For c does not occur in any other formula on the branch. ⊣

6.2.2. Completeness theorem

Theorem 22 (Completeness Theorem). *VS is strongly complete with respect to its semantics.*

Proof. The proof is a modification of that for *CS*. In the Completeness Lemma we add steps for the actualist quantifiers. The Completeness Theorem then follows from the Completeness Lemma as usual. Furthermore, in the induced model $D_{\omega_i\tau_{[j]}} = v(\omega_i\tau_{[j]}) = v_{\omega_i\tau_{[j]}}(\mathrm{E}) = \{o_a : \mathrm{E}a, w_it_j \text{ occurs on } \mathcal{B}\}$.

($\exists$) Assume that $\exists xA, w_it_j$ is on the branch. Since the tableau is complete ($\exists$) has been applied. Hence, for some $c \in C$, $\mathrm{E}c, w_it_j$ and $A[c/x]$, w_it_j are on the branch. By (IH), $v_{\omega_i\tau_{[j]}}(\mathrm{E}c) = 1$ and $v_{\omega_i\tau_{[j]}}(A[c/x]) = 1$. For some $d \in D$, $v(c) = d = v(k_d)$. Accordingly, by the Denotation Lemma, $v_{\omega_i\tau_{[j]}}(\mathrm{E}k_d) = v_{\omega_i\tau_{[j]}}(A[k_d/x]) = 1$. It follows that

$v_{\omega_i\tau_{[j]}}(\exists xA) = 1$. Suppose that $\neg\exists xA, w_it_j$ is on the branch. Since the tableau is complete $(\neg\exists)$ has been applied. So, $\forall x\neg A, w_it_j$ is on the branch. Again, since the tableau is complete $(\forall)$ has been applied. Hence, for every $c \in C$, either $\neg Ec, w_it_j$ or $\neg A[c/x], w_it_j$ is on the branch. Accordingly, for all $c \in C$, if $\mathrm{E}c, w_it_j$ is on the branch, then $\neg A[c/x], w_it_j$ is so too. For the branch is open. By (IH), if $v_{\omega_i\tau_{[j]}}(Ec) = 1$, $v_{\omega_i\tau_{[j]}}(A[c/x]) = 0$. If $d \in D$, then for some $c \in C$, $v(c) = v(k_d)$. By Denotation Lemma, for all $d \in D$, such that $v_{\omega_i\tau_{[j]}}(Ek_d) = 1$ $(d \in D_{\omega_i\tau_{[j]}})$, $v_{\omega_i\tau_{[j]}}(A[k_d/x]) = 0$. It follows that $v_{\omega_i\tau_{[j]}}(\exists xA) = 0$. The case for $\forall$ is similar. ⊣

6.2.3. Systems with domain-inclusion (Barcan) rules

THEOREM 23 (Soundness with Barcan Rules). *VS + any subset of the domain-inclusion (Barcan) rules is (strongly) sound with respect to its semantics.*

PROOF. In the Soundness Lemma we have some new cases. Assume that f and g show that the branch $\mathcal{B}$ is satisfiable in $\mathcal{M}$.

(T-ACBF) Suppose that $\mathcal{B}$ contains $rw_iw_jt_k$ and $\mathrm{E}a, w_it_k$. Then $Rf(w_i)f(w_j)g(t_k)$ and $v(a) \in v_{\omega_i\tau_{[k]}}(\mathrm{E}) = D_{\omega_i\tau_{[k]}}$. By (C-ACBF), $D_{f(w_i)g(t_k)} \subseteq D_{f(w_j)g(t_k)}$. Hence, $v(a) \in D_{\omega_j\tau_{[k]}} = v_{\omega_j\tau_{[k]}}(\mathrm{E})$. Consequently, $\mathrm{E}a$ is true in $f(w_j)$ at $g(t_k)$ in $\mathcal{M}$, and we can take $\mathcal{M}'$ to be $\mathcal{M}$.

(T-TBF) Suppose that $\mathcal{B}$ contains $t_i < t_j$ and $\mathrm{E}a, w_it_j$. Then $g(t_i) < g(t_j)$ and $v(a) \in v_{\omega_i\tau_{[j]}}(\mathrm{E}) = D_{\omega_i\tau_{[j]}}$. By (C-TBF), $D_{f(w_i)g(t_j)} \subseteq D_{f(w_i)g(t_i)}$. Therefore, $v(a) \in D_{\omega_i\tau_{[i]}} = v_{\omega_i\tau_{[i]}}(\mathrm{E})$. It follows that $\mathrm{E}a$ is true in $f(w_i)$ at $g(t_i)$ in $\mathcal{M}$, and we can take $\mathcal{M}'$ to be $\mathcal{M}$.

The other cases are proved similarly. ⊣

THEOREM 24 (Completeness with Barcan Rules). *VS + any subset of the domain-inclusion (Barcan rules) is (strongly) complete with respect to its semantics.*

PROOF. In the relevant Completeness Theorem, we have to check that the induced model is of the appropriate kind.

(T-ACBF) Suppose that $R\omega_i\omega_j\tau_{[k]}$ and that $o_a \in D_{\omega_i\tau_{[k]}}$. Then $rw_iw_jt_k$ and $\mathrm{E}a, w_it_k$ are on the branch. Since the branch is complete (T-ACBF) has been applied. Accordingly, $\mathrm{E}a, w_jt_k$ is on the branch. Consequently, $o_a \in D_{\omega_j\tau_{[k]}}$.

(T-TBF) Suppose that $\tau_{[i]} < \tau_{[j]}$ and that $o_a \in D_{\omega_i\tau_{[j]}}$. Then $t_i < t_j$ and $\mathrm{E}a, w_it_j$ are on the branch. Since the branch is complete (T-TBF) has been applied. Hence, $\mathrm{E}a, w_i t_i$ is on the branch. It follows that $o_a \in D_{\omega_i \tau_{[i]}}$. The other cases are proved similarly. ⊣

6.2.4. General correctness theorem

Theorem 25 (General Correctness Theorem). *Let S be any of the variable domain tableau systems discussed in this essay (without identity). Then S is (strongly) sound and complete with respect to its semantics.*

Proof. The proof is similar to the proof in the constant domain case. ⊣

6.3. Constant and variable domain logics

6.3.1. General correctness theorem

Theorem 26 (General Correctness Theorem). *Let S be any constant and variable domain tableau system discussed in this essay (without identity). Then S is (strongly) sound and complete with respect to its semantics.*

Proof. Combine the proofs for the constant systems and the variable systems. ⊣

6.4. Systems with necessary identity

So far we have assumed that the identity predicate is not part of our language. In this section we will prove soundness and completeness for systems with necessary identity. In Subsection 6.4.3 we will see what happens when we add descriptors to our language. In the next section we turn to systems with contingent identity.

Adding the identity predicate does nothing to affect the proofs of the Locality and Denotation Lemmas; they still hold.

6.4.1. Soundness, necessary identity

Theorem 27 (Soundness Necessary Identity). *Let S be any system in this essay (without identity). Then S + the rules for necessary identity is strongly sound with respect to its semantics (variable, constant or constant and variable).*

Proof. There are three new cases in the Soundness Lemma.

(T-R=) Trivial.

(T-S=) At this stage the only (non-temporal) constants in our language are rigid. So, let $s = a$ and $t = b$. Since f and g show that the branch is satisfiable in $\mathcal{M}$, $v(a) = v(b)$ and $\langle v(a_1), \ldots, v(a), \ldots, v(a_n)\rangle \in v_{\omega_i\tau_{[j]}}(P)$. Accordingly, $\langle v(a_1), \ldots, v(b), \ldots, v(a_n)\rangle \in v_{\omega_i\tau_{[j]}}(P)$. Consequently, $Pa_1 \ldots b \ldots a_n$ is true in $f(w_i)$ at $g(t_j)$ in $\mathcal{M}$. So, we may take $\mathcal{M}'$ to be $\mathcal{M}$.

(T-N=) Since f and g show that the branch is satisfiable in $\mathcal{M}$, $a = b$ is true in $f(w_i)$ at $g(t_j)$. Accordingly, $v(a) = v(b)$. Hence, $a = b$ is true in $f(w_k)$ at $g(t_l)$, and we may take $\mathcal{M}'$ to be $\mathcal{M}$.

The Soundness Theorem then follows as usual. ⊣

6.4.2. Completeness, necessary identity

DEFINITION 28 (Induced Model). We define the induced model as before, but with the following modification. Define $a \sim b$ to mean that $a = b, w_0t_0$ is on the branch. This is obviously an equivalence relation. Let $[a]$ be the equivalence class of a under $\sim$. $D = \{[a] : a \in C\}$ (or, if $C = \emptyset$, $D = \{o\}$ for an arbitrary o). $v(a) = [a]$, and $\langle [a_1], \ldots, [a_n]\rangle \in v_{\omega_i\tau_{[j]}}(P)$ iff $Pa_1 \ldots a_n, w_it_j$ is on $\mathcal{B}$, given that P is any n-place predicate other than identity. (Due to (T-N=) and (T-S=) this is well defined.) For the variable domain case, $D_{\omega_i\tau_{[j]}} = v_{\omega_i\tau_{[j]}}(\mathrm{E})$.

THEOREM 29 (Completeness Necessary Identity). *Let S be any system in this essay (without identity). Then $S+$ the rules for necessary identity is strongly complete with respect to its semantics (variable, constant or constant and variable).*

PROOF. Here are the modified cases in the Completeness Lemma.

$Pa_1...a_n, w_it_j$ is on $\mathcal{B} \Rightarrow \langle [a_1], ..., [a_n]\rangle \in v_{\omega_i\tau_{[j]}}(P) \Rightarrow \langle v(a_1), ..., v(a_n)\rangle \in v_{\omega_i\tau_{[j]}}(P) \Rightarrow v_{\omega_i\tau_{[j]}}(Pa_1...a_n) = 1$.

$\neg Pa_1...a_n, w_it_j$ is on $\mathcal{B} \Rightarrow Pa_1...a_n, w_it_j$ is not on $\mathcal{B}$ ($\mathcal{B}$ open) $\Rightarrow \langle [a_1], ..., [a_n]\rangle \notin v_{\omega_i\tau_{[j]}}(P) \Rightarrow \langle v(a_1), ..., v(a_n)\rangle \notin v_{\omega_i\tau_{[j]}}(P) \Rightarrow v_{\omega_i\tau_{[j]}}(Pa_1...a_n) = 0$.

$a = b, w_it_j$ is on $\mathcal{B} \Rightarrow a \sim b$ (T-N=) $\Rightarrow [a] = [b] \Rightarrow v(a) = v(b) \Rightarrow v_{\omega_i\tau_{[j]}}(a = b) = 1$.

$\neg a = b, w_it_j$ is on $\mathcal{B} \Rightarrow a = b, w_0t_0$ is not on $\mathcal{B}$ ($\mathcal{B}$ open) $\Rightarrow$ it is not the case that $a \sim b \Rightarrow [a] \neq [b] \Rightarrow v(a) \neq v(b) \Rightarrow v_{\omega_i\tau_{[j]}}(a = b) = 0$.

The proof of the Completeness Theorem then goes through as before. ⊣

6.4.3. Soundness and completeness with descriptors and necessary identity

In this section we show that systems including descriptors are sound and complete. The proofs of the Locality and Denotation Lemmas go through as before. The only modification is that we replace anything of the form $v(t)$ by $v_{\omega_i\tau_{[j]}}(t)$, where t is a non-temporal term. Note that the co-referring constants are rigid in the Denotation Lemma. (Descriptors that co-refer at a world-moment pair do not necessarily co-refer at all world-moment pairs.)

Theorem 30 (Soundness Descriptors). *Let S be any system in this essay* (*with necessary identity*). *Then $S +$ the rule for descriptors is strongly sound with respect to its semantics* (*variable, constant or constant and variable*).

Proof. There is one novel case in the Soundness Lemma. The rest is as in the necessary identity case.

(T-D=) Suppose that f and g show that the branch, $\mathcal{B}$, to which we apply the rule, is satisfiable in $\mathcal{M}$. In world $f(w_i)$ at time $g(t_j)$, α has some denotation, d. Thus, $v(k_d) = v_{f(w_i)g(t_j)}(\alpha)$. Let $\mathcal{M}'$ be the same as $\mathcal{M}$, except that $v'(c) = d$. $v'(c) = d = v(k_d) = v_{f(w_i)g(t_j)}(\alpha)$. Hence, $c = \alpha$ is true in $f(w_i)$ at $g(t_j)$ in $\mathcal{M}'$. And the rest of the branch is satisfiable in $\mathcal{M}'$ as well, by the Locality Lemma. This is shown by f and g, since c does not occur in any formula on $\mathcal{B}$. ⊣

Theorem 31 (Completeness Descriptors). *Let S be any system in this essay* (*with necessary identity*). *Then $S +$ the rule for descriptors is strongly complete with respect to its semantics* (*variable, constant or constant and variable*).

Proof. We want to show that the Completeness Lemma still holds. So, we extend the definition of the induced model to descriptors. For any α, w_i and t_j, on the branch, there is a line of the form $a = \alpha, w_it_j$. Take any one such a (it does not matter which, because of (T-S=)), and let this be $\widehat{\alpha}$. Let $\widehat{b}$ be b itself, for any rigid designator, b. In the induced model, we define $v_{\omega_i\tau_{[j]}}(\alpha)$ to be $[\widehat{\alpha}]$. Here are the necessary modifications in the Completeness Lemma.

$Pt_1...t_n, w_it_j$ is on $\mathcal{B} \Rightarrow P\widehat{t_1}...\widehat{t_n}, w_it_j$ is on $\mathcal{B}$ (T-S=) $\Rightarrow \langle[\widehat{t_1}], ..., [\widehat{t_n}]\rangle \in v_{\omega_i\tau_{[j]}}(P) \Rightarrow \langle v_{\omega_i\tau_{[j]}}(t_1), ..., v_{\omega_i\tau_{[j]}}(t_n)\rangle \in v_{\omega_i\tau_{[j]}}(P) \Rightarrow v_{\omega_i\tau_{[j]}}(Pt_1...t_n) = 1$.

$\neg Pt_1 \dots t_n, w_i t_j$ is on $\mathcal{B} \Rightarrow P\widehat{t_1}...\widehat{t_n}, w_i t_j$ is not on $\mathcal{B}$ ((T-S=), $\mathcal{B}$ open) $\Rightarrow \langle [\widehat{t_1}], ..., [\widehat{t_n}] \rangle \notin v_{\omega_i \tau_{[j]}}(P) \Rightarrow \langle v_{\omega_i \tau_{[j]}}(t_1), ..., v_{\omega_i \tau_{[j]}}(t_n) \rangle \notin v_{\omega_i \tau_{[j]}}(P) \Rightarrow v_{\omega_i \tau_{[j]}}(Pt_1 \dots t_n) = 0$.

$t_1 = t_2, w_i t_j$ is on $\mathcal{B} \Rightarrow \widehat{t_1} = \widehat{t_2}, w_i t_j$ is on $\mathcal{B}$ (T-S=) $\Rightarrow \widehat{t_1} = \widehat{t_2}, w_0 t_0$ is on $\mathcal{B}$ (T-N=) $\Rightarrow \widehat{t_1} \sim \widehat{t_2} \Rightarrow [\widehat{t_1}] = [\widehat{t_2}] \Rightarrow v_{\omega_i \tau_{[j]}}(t_1) = v_{\omega_i \tau_{[j]}}(t_2) \Rightarrow v_{\omega_i \tau_{[j]}}(t_1 = t_2) = 1$.

$\neg t_1 = t_2, w_i t_j$ is on $\mathcal{B} \Rightarrow \widehat{t_1} = \widehat{t_2}, w_i t_j$ is not on $\mathcal{B}$ ((T-S=), $\mathcal{B}$ open) $\Rightarrow \widehat{t_1} = \widehat{t_2}, w_0 t_0$ is not on $\mathcal{B}$ ((T-N=), $\mathcal{B}$ open) $\Rightarrow$ it is not the case that $\widehat{t_1} \sim \widehat{t_2} \Rightarrow [\widehat{t_1}] \neq [\widehat{t_2}] \Rightarrow v_{\omega_i \tau_{[j]}}(t_1) \neq v_{\omega_i \tau_{[j]}}(t_2) \Rightarrow v_{\omega_i \tau_{[j]}}(t_1 = t_2) = 0$.

The Completeness Theorem then follows as usual. ⊣

6.5. Contingent identity logic

In this section we will prove soundness and completeness for systems with contingent identity. In Subsection 6.5.3 we add descriptors to our systems and prove soundness and completeness.

Our models now have one new component, H. With this exception, the Locality and Denotation Lemmas are as in the constant or variable domain cases.

Theorem 32 (Denotation and Locality). *The Denotation and Locality Lemmas hold in contingent identity semantics.*

Proof. The proofs are as in the constant or variable domain cases, except for the atomic formulas.

Locality: $v_{1\omega\tau}(Pa_1...a_n) = 1$ iff $\langle |v_1(a_1)|_{\omega\tau}, ..., |v_1(a_n)|_{\omega\tau} \rangle \in v_{1\omega\tau}(P)$ iff $\langle |v_2(a_1)|_{\omega\tau}, ..., |v_2(a_n)|_{\omega\tau} \rangle \in v_{2\omega\tau}(P)$ iff $v_{2\omega\tau}(Pa_1 \dots a_n) = 1$.

Denotation: $v_{\omega\tau}(Pa_1...a...a_n) = 1$ iff $\langle |v(a_1)|_{\omega\tau}, ..., |v(a)|_{\omega\tau}, ..., |v(a_n)|_{\omega\tau} \rangle \in v_{\omega\tau}(P)$ iff $\langle |v(a_1)|_{\omega\tau}, ..., |v(b)|_{\omega\tau}, ..., |v(a_n)|_{\omega\tau} \rangle \in v_{\omega\tau}(P)$ iff $v_{\omega\tau}(Pa_1...b...a_n) = 1$. ⊣

6.5.1. Soundness, contingent identity

Theorem 33 (Soundness Contingent Identity). *Let S be any system in this essay (without identity). Then $S+$ the rules for contingent identity is strongly sound with respect to its semantics (variable, constant or constant and variable).*

Proof. The proof is as in the necessary identity case with some minor modifications. E.g. here is the step for (T-S=). Since f and g show that the branch is satisfiable in $\mathcal{M}$, $|v(a)|_{\omega_i \tau_{[j]}} = |v(b)|_{\omega_i \tau_{[j]}}$ and

$\langle |v(a_1)|_{\omega_i\tau_{[j]}}, ..., |v(a)|_{\omega_i\tau_{[j]}}, ..., |v(a_n)|_{\omega_i\tau_{[j]}}\rangle \in v_{\omega_i\tau_{[j]}}(P)$. So, $\langle |v(a_1)|_{\omega_i\tau_{[j]}}, ..., |v(b)|_{\omega_i\tau_{[j]}}, ..., |v(a_n)|_{\omega_i\tau_{[j]}}\rangle \in v_{\omega_i\tau_{[j]}}(P)$. Accordingly, $Pa_1...b...a_n$ is true in $f(w_i)$ at $g(t_j)$ in $\mathcal{M}$, and so we may take $\mathcal{M}'$ to be $\mathcal{M}$. ⊣

6.5.2. Completeness, contingent identity

Definition 34 (Induced Model). Given an open complete branch $\mathcal{B}$ of a tableau, the induced model is defined as usual, but with the following modifications. If there are no constants on the branch, $D = \{o\}$, for an arbitrary o; $H = \{h\}$, for an arbitrary h; and for every $\omega \in W$ and $\tau \in T$, $|o|_{\omega\tau} = h$. Otherwise, $D = \{o_a : a \textit{ occurs on } \mathcal{B}\}$ as usual. The objects in D are now functions from $W \times T$ to H. We shall say that $a \sim_{\omega_i\tau_{[j]}} b$ iff $a = b, w_it_j$ occurs on $\mathcal{B}$. $\sim_{\omega_i\tau_{[j]}}$ is an equivalence relation. Let $[a]_{\omega_i\tau_{[j]}}$ be the equivalence class of a under $\sim_{\omega_i\tau_{[j]}}$. $H = \{[a]_{\omega_i\tau_{[j]}} : \textit{for all } a, w_it_j \textit{ on } \mathcal{B}\}$. Let $|o_a|_{\omega_i\tau_{[j]}} = [a]_{\omega_i\tau_{[j]}}$, for $\omega_i \in W$ and $\tau_{[j]} \in T$. For each (non-temporal, rigid) constant, a, $v(a) = o_a$. For each n-place predicate, P, other than identity: $\langle [a_1]_{\omega_i\tau_{[j]}}, ..., [a_n]_{\omega_i\tau_{[j]}}\rangle \in v_{\omega_i\tau_{[j]}}(P)$ iff $Pa_1...a_n, w_it_j$ is on $\mathcal{B}$. Any n-tuple that contains a substratum that is not of the form $[a]_{\omega_i\tau_{[j]}}$ is not in $v_{\omega_i\tau_{[j]}}(P)$. Because of (T-S=), it does not matter which member of an equivalence class we chose. If the model is a variable domain model, $D_{\omega_i\tau_{[j]}} = \{d \in D : |d|_{\omega_i\tau_{[j]}} \in v_{\omega_i\tau_{[j]}}(\mathrm{E})\}$.

Theorem 35 (Completeness Contingent Identity). *Let S be any system in this essay* (*without identity*)*. Then $S+$ the rules for contingent identity is strongly complete with respect to its semantics* (*variable, constant or constant and variable*)*.*

Proof. The proof of the Completeness Lemma is as in the non-identity case, except for the following steps.

$Pa_1 \dots a_n, w_it_j$ is on $\mathcal{B} \Rightarrow \langle [a_1]_{\omega_i\tau_{[j]}}, \dots, [a_n]_{\omega_i\tau_{[j]}}\rangle \in v_{\omega_i\tau_{[j]}}(P) \Rightarrow \langle |o_{a_1}|_{\omega_i\tau_{[j]}}, ..., |o_{a_n}|_{\omega_i\tau_{[j]}}\rangle \in v_{\omega_i\tau_{[j]}}(P) \Rightarrow \langle |v(a_1)|_{\omega_i\tau_{[j]}}, \dots, |v(a_n)|_{\omega_i\tau_{[j]}}\rangle \in v_{\omega_i\tau_{[j]}}(P) \Rightarrow v_{\omega_i\tau_{[j]}}(Pa_1 \dots a_n) = 1$.

$\neg Pa_1 \dots a_n, w_it_j$ is on $\mathcal{B} \Rightarrow Pa_1 \dots a_n, w_it_j$ is not on $\mathcal{B} \Rightarrow \langle [a_1]_{\omega_i\tau_{[j]}}, \dots, [a_n]_{\omega_i\tau_{[j]}}\rangle \notin v_{\omega_i\tau_{[j]}}(P) \Rightarrow \langle |o_{a_1}|_{\omega_i\tau_{[j]}}, \dots, |o_{a_n}|_{\omega_i\tau_{[j]}}\rangle \notin v_{\omega_i\tau_{[j]}}(P) \Rightarrow \langle |v(a_1)|_{\omega_i\tau_{[j]}}, \dots, |v(a_n)|_{\omega_i\tau_{[j]}}\rangle \notin v_{\omega_i\tau_{[j]}}(P) \Rightarrow v_{\omega_i\tau_{[j]}}(Pa_1 \dots a_n) = 0$.

$a = b, w_it_j$ is on $\mathcal{B} \Rightarrow a \sim_{\omega_i\tau_{[j]}} b \Rightarrow [a]_{\omega_i\tau_{[j]}} = [b]_{\omega_i\tau_{[j]}} \Rightarrow |o_a|_{\omega_i\tau_{[j]}} = |o_b|_{\omega_i\tau_{[j]}} \Rightarrow |v(a)|_{\omega_i\tau_{[j]}} = |v(b)|_{\omega_i\tau_{[j]}} \Rightarrow v_{\omega_i\tau_{[j]}}(a = b) = 1$.

$\neg a = b, w_it_j$ is on $\mathcal{B} \Rightarrow a = b, w_it_j$ is not on $\mathcal{B}$, ($\mathcal{B}$ open) $\Rightarrow$ it is not the case that $a \sim_{\omega_i\tau_{[j]}} b \Rightarrow [a]_{\omega_i\tau_{[j]}} \neq [b]_{\omega_i\tau_{[j]}} \Rightarrow |o_a|_{\omega_i\tau_{[j]}} \neq |o_b|_{\omega_i\tau_{[j]}} \Rightarrow |v(a)|_{\omega_i\tau_{[j]}} \neq |v(b)|_{\omega_i\tau_{[j]}} \Rightarrow v_{\omega_i\tau_{[j]}}(a = b) = 0$.

The rest of the Completeness Theorem then follows as usual. ⊣

6.5.3. Soundness and completeness with descriptors and contingent identity

In this section we consider the addition of descriptors to contingent identity systems. The Locality and Denotation Lemmas are established as in the constant or variable domain cases. The proofs of the soundness and completeness theorems are as in the necessary identity case, with some minor modifications.

THEOREM 36 (Soundness Descriptors). *Let S be any system in this essay (with contingent identity). Then $S+$ the rule for descriptors is strongly sound with respect to its semantics (variable, constant or constant and variable).*

PROOF. Suppose that f and g show that the branch $\mathcal{B}$ is satisfiable in $\mathcal{M}$. Here are the new interesting cases.
(T-S=) For the sake of illustration, assume that there is only one occurrence of s. Accordingly, $|v_{f(w_i)g(t_j)}(s)|_{f(w_i)g(t_j)} = |v_{f(w_i)g(t_j)}(t)|_{f(w_i)g(t_j)}$ and $\langle |v_{f(w_i)g(t_j)}(t_1)|_{f(w_i)g(t_j)}, ..., |v_{f(w_i)g(t_j)}(s)|_{f(w_i)g(t_j)}, ..., |v_{f(w_i)g(t_i)}(t_n)|_{f(w_i)g(t_j)} \rangle \in v_{f(w_i)g(t_j)}(P)$. So, $\langle |v_{f(w_i)g(t_j)}(t_1)|_{f(w_i)g(t_j)}, ..., |v_{f(w_i)g(t_j)}(t)|_{f(w_i)g(t_j)}, ..., |v_{f(w_i)g(t_j)}(t_n)|_{f(w_i)g(t_j)} \rangle \in v_{f(w_i)g(t_j)}(P)$. It follows that $Pt_1...t...t_n$ is true in $f(w_i)$ at $g(t_j)$ in $\mathcal{M}$, and we may take $\mathcal{M}'$ to be $\mathcal{M}$.

(T-D=) In $f(w_i)$ at $g(t_j)$, α has some denotation, $d \in D$. Thus $|v(k_d)|_{f(w_i)g(t_j)} = |v_{f(w_i)g(t_j)}(\alpha)|_{f(w_i)g(t_j)}$. Let $\mathcal{M}'$ be the same as $\mathcal{M}$, except that $v(c) = d$. $|v(c)|_{f(w_i)g(t_j)} = |d|_{f(w_i)g(t_j)} = |v(k_d)|_{f(w_i)g(t_j)} = |v_{f(w_i)g(t_j)}(\alpha)|_{f(w_i)g(t_j)}$. Hence, $c = \alpha$ is true in $f(w_i)$ at $g(t_j)$ in $\mathcal{M}'$. Furthermore, f and g show that the rest of the branch is satisfiable in $\mathcal{M}'$, by the Locality Lemma. For c does not occur in any formula on $\mathcal{B}$. ⊣

THEOREM 37 (Completeness Descriptors). *Let S be any system in this essay (with contingent identity). Then $S+$ the rule for descriptors is strongly complete with respect to its semantics (variable, constant or constant and variable).*

PROOF. We modify the definition of an induced model so that it applies to descriptors as well. Then we check that the Completeness Lemma holds. For any descriptor, α, and any w_i and t_j on the branch, there is a line of the form $a = \alpha, w_i t_j$. Let any one such a (it does not matter which, because of (T-S=)) be $\widehat{\alpha}$. If b is a rigid designator, let $\widehat{b}$ be b itself.

Finally: $v_{\omega_i\tau_{[j]}}(\alpha) = o_{\widehat{\alpha}}$. Here are the modified steps in the Completeness Lemma.

$Pt_1...t_n, w_it_j$ is on $\mathcal{B} \Rightarrow P\widehat{t_1}\dots\widehat{t_n}, w_it_j$ is on $\mathcal{B}$ (T-S=) $\Rightarrow \langle[\widehat{t_1}]_{\omega_i\tau_{[j]}}, \dots, [\widehat{t_n}]_{\omega_i\tau_{[j]}}\rangle \in v_{\omega_i\tau_{[j]}}(P) \Rightarrow \langle|o_{\widehat{t_1}}|_{\omega_i\tau_{[j]}}, \dots, |o_{\widehat{t_n}}|_{\omega_i\tau_{[j]}}\rangle \in v_{\omega_i\tau_{[j]}}(P) \Rightarrow \langle|v_{\omega_i\tau_{[j]}}(t_1)|_{\omega_i\tau_{[j]}}, ..., |v_{\omega_i\tau_{[j]}}(t_n)|_{\omega_i\tau_{[j]}}\rangle \in v_{\omega_i\tau_{[j]}}(P) \Rightarrow v_{\omega_i\tau_{[j]}}(Pt_1...t_n) = 1$.

$\neg Pt_1...t_n, w_it_j$ is on $\mathcal{B} \Rightarrow Pt_1...t_n, w_it_j$ is not on $\mathcal{B}$ ($\mathcal{B}$ open) $\Rightarrow P\widehat{t_1}...\widehat{t_n}, w_it_j$ is not on $\mathcal{B}$ ((T-S=), $\mathcal{B}$ open) $\Rightarrow \langle[\widehat{t_1}]_{\omega_i\tau_{[j]}}, \dots, [\widehat{t_n}]_{\omega_i\tau_{[j]}}\rangle \notin v_{\omega_i\tau_{[j]}}(P) \Rightarrow \langle|o_{\widehat{t_1}}|_{\omega_i\tau_{[j]}}, \dots, |o_{\widehat{t_n}}|_{\omega_i\tau_{[j]}}\rangle \notin v_{\omega_i\tau_{[j]}}(P) \Rightarrow \langle|v_{\omega_i\tau_{[j]}}(t_1)|_{\omega_i\tau_{[j]}}, \dots, |v_{\omega_i\tau_{[j]}}(t_n)|_{\omega_i\tau_{[j]}}\rangle \notin v_{\omega_i\tau_{[j]}}(P) \Rightarrow v_{\omega_i\tau_{[j]}}(Pt_1\dots t_n) = 0$.

$t_1 = t_2, w_it_j$ is on $\mathcal{B} \Rightarrow \widehat{t_1} = \widehat{t_2}, w_it_j$ is on $\mathcal{B}$ (T-S=) $\Rightarrow \widehat{t_1} \sim_{\omega_i\tau_{[j]}} \widehat{t_2} \Rightarrow [\widehat{t_1}]_{\omega_i\tau_{[j]}} = [\widehat{t_2}]_{\omega_i\tau_{[j]}} \Rightarrow |o_{\widehat{t_1}}|_{\omega_i\tau_{[j]}} = |o_{\widehat{t_2}}|_{\omega_i\tau_{[j]}} \Rightarrow |v_{\omega_i\tau_{[j]}}(t_1)|_{\omega_i\tau_{[j]}} = |v_{\omega_i\tau_{[j]}}(t_2)|_{\omega_i\tau_{[j]}} \Rightarrow v_{\omega_i\tau_{[j]}}(t_1 = t_2) = 1$.

$\neg t_1 = t_2, w_it_j$ is on $\mathcal{B} \Rightarrow t_1 = t_2, w_it_j$ is not on $\mathcal{B}$ ($\mathcal{B}$ open) $\Rightarrow \widehat{t_1} = \widehat{t_2}, w_it_j$ is not on $\mathcal{B}$, ((T-S=), $\mathcal{B}$ open) $\Rightarrow$ it is not the case that $\widehat{t_1} \sim_{\omega_i\tau_{[j]}} \widehat{t_2} \Rightarrow [\widehat{t_1}]_{\omega_i\tau_{[j]}} \neq [\widehat{t_2}]_{\omega_i\tau_{[j]}} \Rightarrow |o_{\widehat{t_1}}|_{\omega_i\tau_{[j]}} \neq |o_{\widehat{t_2}}|_{\omega_i\tau_{[j]}} \Rightarrow |v_{\omega_i\tau_{[j]}}(t_1)|_{\omega_i\tau_{[j]}} \neq |v_{\omega_i\tau_{[j]}}(t_2)|_{\omega_i\tau_{[j]}} \Rightarrow v_{\omega_i\tau_{[j]}}(t_1 = t_2) = 0$.

The Completeness Theorem now follows in the usual fashion. $\dashv$

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Daniel Rönnedal
Stockholm University
Department of Philosophy
106 91 Stockholm, Sweden
daniel.ronnedal@philosophy.su.se