Logic and Logical Philosophy
Volume 23 (2014), 301–328
DOI: 10.12775/LLP.2013.026

Paolo Maffezioli[*]
Alberto Naibo

PROOF THEORY OF EPISTEMIC LOGIC OF PROGRAMS

Abstract. A combination of epistemic logic and dynamic logic of programs is presented. Although rich enough to formalize some simple game-theoretic scenarios, its axiomatization is problematic as it leads to the paradoxical conclusion that agents are omniscient. A cut-free labelled Gentzen-style proof system is then introduced where knowledge and action, as well as their combinations, are formulated as rules of inference, rather than axioms. This provides a logical framework for reasoning about games in a modular and systematic way, and to give a step-by-step reconstruction of agents omniscience. In particular, its semantic assumptions are made explicit and a possible solution can be found in weakening the properties of the knowledge operator.

Keywords: epistemic logic; dynamic propositional logic; structural proof theory; labelled sequent calculus; epistemic paradox

1. Introduction

Since the pioneering work of Aumann [1], it has been widely acknowledged that many solution concepts in game theory strictly depend on the assumptions about what players know of the game or about other players' knowledge. In this approach knowledge is defined set-theoretically by Aumann structures, i.e. partitions over a set of informational states. Aumann structures naturally find a logical counterpart in the semantics of modal logic **S5** where knowledge is represented by an equivalence

* Research partially supported by the FWF START Y544-N23 research grant.

Received January 2, 2013. *Revised* July 6, 2013. *Published online* September 12, 2013

relation over the set of conceivable alternatives. However, both Aumann and Kripke structures have several important limitations in the analysis of games since they deal with knowledge in a static way and it is not that obvious to express how agents information can change in consequence of the moves they may play. Nor it is possible to explain why their choices are bounded by their reasoning abilities. Even the most simple game-theoretic situation shows that knowledge and actions are not fixed from the beginning but may evolve and change as the game progresses. It is natural to think that what agents do depends on what they know, and on the other hand, that what they do may change their knowledge. Thus, knowledge and actions are mutually dependent notions involved into a dynamic process. If the notion of rational agent has to be interpreted dynamically, it is then clear that any attempt to formalize rationality must be oriented toward logical frameworks in which knowledge and action may interact.

A possible candidate for studying in an abstract and general way the dynamics of knowledge and action in games is propositional dynamic logic, **PDL** (see [13] for a comprehensive survey). Originally conceived as the logic of computer programming, **PDL** is a multi-modal logic where the modalities are not simply a collection of propositional operators defined over a generic set of indexes, but are inductively generated over a set of atomic programs or, in our terminology, basic actions. In general, a program is a linguistic entity (written in some programming language) that describe a procedure for computing an output for a given input; in particular, in **PDL** a program α is a term denoting an algorithm. Algorithms, as well as the actions that agents perform, are assumed to be mechanically — i.e. step-by-step — executable, and they are supposed to return an output after some finite number of steps. Therefore, programs in **PDL** may be equivalently thought as actions of players in games.

In this sense, the expression $[\alpha]A$ means that by executing α, necessarily A holds, that is, the action α must bring about A. In a "dual" way, the formula $\langle\alpha\rangle A$ is interpreted as: by executing α, possibly A holds, i.e. the action α may bring about A. This interpretation makes clear that in **PDL**, differently from standard classical propositional logic, the truth-values of formulas are not intended as determined from the beginning, and fixed once and for all by a given reality existing independently from human agents. On the contrary, formulas are supposed to acquire a specific truth-value only with respect to the execution of some specific human actions. In this sense, the conception of truth standing behind

PDL becomes very much close to the non-transcendent conception typical of intuitionistic logic or, more generally, of the so-called verificationist positions (see for example [17]).

However, in order to faithfully formalize game-theoretic scenarios, we need to consider a further type of change and allow programs to modify not only truth values but also agents' knowledge. Technically, this means to combine the language of **PDL** with the language of standard epistemic logic, **EL** (see [15]), for reasoning about knowledge. **EL** is a multi-modal logic with a collection of modal operators K_i for each agent i of a given finite set $\mathcal{A}$. Intuitively, the formula $\mathsf{K}_i A$ means that the agent i knows that A. Usually, it is assumed as a minimal requirement that agents have knowledge only of true fact, a false information being at most be believed, but not properly known. Thus, we must assume the principle of factivity of knowledge, that is, $\mathsf{K}_i A \supset A$. The other principles usually given in order to characterize the knowledge operator are more controversial and attracted the attention since they have been introduced. The first one requires agents to be aware of their knowledge, i.e. $\mathsf{K}_i A \supset \mathsf{K}_i \mathsf{K}_i A$ (positive introspection), whereas the other one presupposes also the awareness about their ignorance, $\neg\mathsf{K}_i A \supset \mathsf{K}_i \neg\mathsf{K}_i A$ (negative introspection). We shall assume the three principles throughout and discuss later the possibility of leaving some of them out.

1.1. PDL without iteration

It is worth noting the in standard presentations of **PDL**, the Kleene star operation of iteration (*) is usually taken as a primitive operator, whereas it is not taken into account here. Intuitively, a program α^* corresponds to some finite number of successive applications of α.[1] The reason why we will not deal with it in our presentation is that iterating a program means to be able to (re)use it many different times, and this is far from being trivial insofar as it presupposes to possess a manner of handling computational resources. Thus, in absence of a previously defined operation for controlling the access to resources, the operation of iteration remains only ideally applicable, but not concretely, as we expect instead to be the case in a situation of interaction in which the actions of each agent can be performed only if the actions of the other

[1] It is worth noting that the operation of iteration concerns the construction of programs, and not their execution as a hasty reflection may suggest.

agents have been actually brought to conclusion. A toy example for better understanding this point is given by the analysis of those programs representing natural numbers written in an abstract functional programming language, specifically λ-calculus (cf. the notion of Church numeral given in [26, pp. 20, 282]). By the Curry-Howard correspondence, the program corresponding to the natural number 2 is given by the following proof (written in a sequent calculus version of Girard's *System F*):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\beta : X \to \beta : X \qquad \dfrac{\beta : X \to \beta : X \quad \beta : X \to \beta : X}{\alpha : X \supset X, \beta : X \to \alpha\beta : X}\, {\scriptstyle L\supset}}{\alpha : X \supset X, \alpha : X \supset X, \beta : X \to \alpha(\alpha\,\beta) : X \supset X}\, {\scriptstyle L\supset}}{\alpha : X \supset X, \alpha : X \supset X \to \lambda\beta.\alpha(\alpha\,\beta) : X \supset X}\, {\scriptstyle R\supset}}{\alpha : X \supset X \to \lambda\beta.\alpha(\alpha\,\beta) : X \supset X}\, \mathrm{Ctr}}{\to \lambda\alpha\lambda\beta.\alpha(\alpha\,\beta) : (X \supset X) \supset (X \supset X)}\, {\scriptstyle R\supset}}{\to \Lambda X \lambda\alpha\lambda\beta.\alpha(\alpha\,\beta) : \forall X((X \supset X) \supset (X \supset X))}\, {\scriptstyle R\forall^2}$$

It is quite clear from this example that the iteration of α corresponds to the application of the contraction rule on its formula specification. In this case the appeal to contraction is not too much problematic, as it is applied only once, but in principle it could be possible to apply it any arbitrary finite number of times, and thus to contract any arbitrary finite number of formulas specification of α, potentially generating the whole series of natural numbers. However, as it is known from the linear logic resource interpretation (see [27, pp. 1–2]), such an unbound use of contraction corresponds to conceive the set of resources as perennially exploitable. This means to open the space for potentially infinitary considerations, which seem in fact to be alien from an agent-based perspective, where agents are human beings acting in concrete situations and therefore subject to fixed finitary constraints.[2] Consequently, reasoning about games is not like reasoning about pure logic or mathematics: although some kinds of abstraction and idealization are needed in order to

[2] As it concerns the specific problem of natural numbers, the lesson of linear logic is that the second order categorical definition of natural numbers does not go beyond human epistemic capacities because of an impredicative account of second order quantifiers, but already because at the propositional level some unbounded, and hence potentially unfeasible, operations are allowed, viz. contraction. On the other hand, examples of intrinsically epistemic-aware accounts of natural numbers are given by those logical systems which, with respect to cut elimination, correspond to classes of bounded computational complexity, e.g. Bounded Linear Logic, Light Linear Logic, Soft Linear Logic.

formalize the situation we want to study, the resulting model should still represent something that could effectively occur in real-life situations. Indeed, the presence of multiple players means that the set of actions which are in principle playable by each player is *de facto* bound by the actions already performed by the other players.

1.2. Our proposal

In this paper, we focus on a specific combination of the standard epistemic logic (**EL**) for reasoning about knowledge, and the dynamic logic of programs for reasoning about actions (**PDL**). The same combination has been axiomatically investigated in [25] and [7]. These works are mostly concerned with the following principles connecting knowledge and action:

$[\alpha]\mathsf{K}_i A \supset \mathsf{K}_i[\alpha]A$ (No Learning)

$\mathsf{K}_i[\alpha]A \supset [\alpha]\mathsf{K}_i A$ (Perfect Recall)

$\langle\alpha\rangle\mathsf{K}_i A \supset \mathsf{K}_i\langle\alpha\rangle A$ (Reasoning Ability)

According to the No Learning principle (**NL**) if an agent comes to know that A after any kind of execution of α, then the agent already knows that α brings about A. In other words, agents know in advance the consequences of those actions that necessarily produce a knowledge. Perfect Recall (**PR**) claims that agents never forget information once it is acquired. The third condition, called the Church-Rosser axiom in [25], is more evocative known as Reasoning Ability (**RA**) from [7]. It says that if an agent comes to know that A after some kind of execution of α, then the agent already knows that α could represent a way to obtain A. In other terms, agents know in advance the consequences of those actions that possibly produce a knowledge.

One can consider these principles as axiom schemes to be added to the standard axiomatization of **PDL** and **EL**. We shall refer to such axiomatization as **EPDL**. Although the three previous principles reflect intuitive properties of the combination of knowledge and actions, [25] showed that their acceptance is in fact far from being neutral: when they come into play in connection with **EL** and **PDL**, they bring to some unexpected, if not even puzzling, results of non-conservativity of **EPDL** over **EL**.

We shall introduce a cut-free Gentzen-style proof system for **EPDL** where the notions of knowledge and action, as well as their combinations, are formulated as rules of inference, rather than axioms. Our general aim is to design a logical framework for reasoning about games in a modular and systematic way, and to provide a reconstruction which makes explicit the semantic assumptions leading to the non-conservativity results. Finally, we suggest that a possible solution can be found in weakening the properties of the knowledge operator.

2. Language and Formal Semantics

The language $\mathcal{L}$ of Epistemic Logic of Programs (**EPDL**) is a combination of the language of Epistemic Logic (**EL**) and the iteration-free fragment of the language of Propositional Dynamic Logic (**PDL**). $\mathcal{L}$ contains a countable set $\mathcal{V} = \{P_1, P_2 \ldots\}$ of atomic formulas, a zero-place operator $\bot$, and a countable set of basic programs (interpreted intuitively as actions) $\mathcal{P} = \{\pi_1, \pi_2 \ldots\}$. Programs α are inductively defined from $\mathcal{P}$ using the operations of ; (sequential composition), $\cup$ (alternation) and ? (test), whereas formulas are built from $\mathcal{V}$ and $\bot$ by propositional connectives $\wedge$ (conjunction),$\vee$ (disjunction), $\supset$ (implication), and modal operators K_i (for each agent i from a given set $\mathcal{A}$) and $[\alpha]$ (for each program α). More precisely, if $P \in \mathcal{V}$, $\pi \in \mathcal{P}$, and $i \in \mathcal{A}$, the language of **EPDL** is inductively defined (in Backus-Naur form) as follows:

$$A ::= P \mid \bot \mid A \wedge A \mid A \vee A \mid A \supset A \mid \mathsf{K}_i A \mid [\alpha]A$$
$$\alpha ::= \pi \mid \alpha;\alpha \mid \alpha \cup \alpha \mid A?$$

As usual, $\neg A$ (negation), $A \equiv B$ (equivalence) and $\langle\alpha\rangle A$ are abbreviations for $A \supset \bot$, $(A \supset B) \wedge (B \supset A)$ and $\neg[\alpha]\neg A$, respectively. To execute the composition $\alpha;\beta$ of two programs α and β means to execute α and then β. This corresponds to assuming the axiom scheme $[\alpha;\beta]A \equiv [\alpha][\beta]A$ (**Comp**). An execution of $\alpha \cup \beta$ is either an execution of α or of β (where the choice is non-deterministic), and it is characterized by the axiom scheme $[\alpha \cup \beta]A \equiv [\alpha]A \wedge [\beta]A$ (**Alt**). Finally, the test of a formula A, namely $A?$, corresponds to check whether A is true. The axiom scheme characterizing its behavior is given by $[A?]B \equiv (A \supset B)$ (**Test**). This axiom is particularly interesting because it shows that in **PDL** it is possible to assign an operational interpretation to the usual classical implication in a manner similar to what the BHK interpretation does for the intuitionistic implication.[3]

Formulas of $\mathcal{L}$ are evaluated in relational models — i.e. models based on Kripke frames — combining the semantic features of epistemic and dynamic models.

DEFINITION. Let P be an atomic formula, π a basic action and i an agent. A *model* is a structure $\mathfrak{M} = \langle X, \{R_i\}_{i\in\mathcal{A}}, \{R_\pi\}_{\pi\in\mathcal{P}}, \Vdash\rangle$ where X is non-empty set of states; each R_i and R_π is a binary relation on X; and $\Vdash$ is a binary relation on $X \times \mathcal{V}$. As usual, $x \Vdash P$ means that P is true at x.

The forcing relation $\Vdash$ is given on atoms and it is inductively defined for arbitrary formulas by the following set of valuation clauses:

$x \Vdash \bot$		for no x
$x \Vdash A \wedge B$	iff	$x \Vdash A$ and $x \Vdash B$
$x \Vdash A \vee B$	iff	$x \Vdash A$ or $x \Vdash B$
$x \Vdash A \supset B$	iff	$x \Vdash A$ implies $x \Vdash B$
$x \Vdash \mathsf{K}_i A$	iff	for all y, xR_iy implies $y \Vdash A$
$x \Vdash [\alpha]A$	iff	for all y, $xR_\alpha y$ implies $y \Vdash A$

The latter clause requires R_α, for an arbitrary action α, to be defined. We shall assume the standard intended relational interpretation of programs ([10, p. 110]) and thus call a *standard model* a model satisfying the following conditions:

Act_1	$\forall x\forall y(xR_{\alpha;\beta}y \equiv \exists z(xR_\alpha z \wedge zR_\beta y))$
Act_2	$\forall x\forall y(xR_{\alpha\cup\beta}y \equiv xR_\alpha y \vee xR_\beta y)$
Act_3	$\forall x\forall y(xR_{A?}y \equiv x \Vdash A \wedge x = y)$

The latter condition makes appeal to identity between states. Since in predicate logic identity is assumed to be reflexive — i.e. $x = x$ — and to satisfy the replacement scheme — i.e. $x = y \wedge P(x) \supset P(y)$, for any arbitrary atomic P — we need to impose on standard models the two extra conditions:

Id_1 $\quad \forall x(x = x)$

[3] This result is not astonishing if we think of the usual interpretation of intuitionistic logic into modal logic (cf. in particular [11]), where implication corresponds to a $\Box$ operator.

Id$_2$ $\forall x \forall y(x = y \wedge x \Vdash A \supset y \Vdash A)$

where $P(x)$ has been replaced by $x \Vdash A$, for any arbitrary A.

Finally, notice that the aforementioned principles of factivity, positive introspection, and negative introspection are valid when each R_i is assumed to be an equivalence relation, i.e.

Kn$_1$ $\forall x(xR_ix)$
Kn$_2$ $\forall x \forall y(xR_iy \supset yR_ix)$
Kn$_3$ $\forall x \forall y \forall z(xR_iy \wedge yR_iz \supset xR_iz)$

We call a *standard epistemic model* a standard model where each R_i satisfies the latter three conditions.

3. Proof Theory

Labelled systems [19] are a variant of **G3** sequent calculi (see [28, p. 77]), specifically designed for modal and non-classical logics. In labelled systems formulas do not occur in isolation but always together with the possible state at which they are true. Thus, labelling is the syntactical counterpart of forcing and we shall use the notation $x : A$ to indicate that A is labelled by x, whenever A is a formula and x a possible state. Accessibility relations like xRy may also occur as they allow to handle the transition from one state to another. A labelled sequent $\Gamma \to \Delta$ is then defined as a multiset (list without order) of labelled formulas $x : A$ and relational atoms xRy separated by the symbol $\to$ that represents the derivability relation in the object language. Once the semantics is made explicit part of the syntax, rules for propositional connectives and modalities are directly obtained from the definition of forcing. In particular, since each valuation clause establishes the sufficient and necessary conditions for a formula to be valid, it is possible to formulate rules for introducing that formula in the right-hand side (succedent) and left-hand side (antecedent) of $\to$, respectively. More precisely, the conversion of valuation clauses into inference rules is obtained by converting if-directions into right rules and only-if-directions into left rules. For example, from the sufficient condition for a formula K_iA to be forced at x, we found a rule for introducing the labelled formula $x : \mathsf{K}_iA$ in the succedent

$$\frac{xR_iy, \Gamma \to \Delta, y : A}{\Gamma \to \Delta, x : \mathsf{K}_iA}\ {\scriptstyle R\mathsf{K}}$$

The role of universal quantifiers in the definition is reflected in the rule by the variable condition that y does not appear in the conclusion. Symmetrically, the necessary condition for $\mathsf{K}_i A$ to be forced in x gives the corresponding rule introducing $x : \mathsf{K}_i A$ in the antecedent

$$\frac{y : A, x : \mathsf{K}_i A, xR_i y, \Gamma \to \Delta}{x : \mathsf{K}_i A, xR_i y, \Gamma \to \Delta}\ L\mathsf{K}$$

Analogously to the **G3** rule for universal quantifier in predicate logic the principal formulas $x : \mathsf{K}_i A$ and $xR_i y$ are repeated into the premise. The rules for formulas as $[\alpha]A$ are similar:

$$\frac{y : A, x : [\alpha]A, xR_\alpha y, \Gamma \to \Delta}{x : [\alpha]A, xR_\alpha y, \Gamma \to \Delta}\ L[\,] \qquad \frac{xR_\alpha y, \Gamma \to \Delta, y : A}{\Gamma \to \Delta, x : [\alpha]A}\ R[\,]$$

Notice that differently from modal rules, classical propositional rules leave labels unchanged during their application. Moreover, initial sequents are restricted to atomic formulas. The following set of rules is thus obtained:

$$x : P, \Gamma \to \Delta, x : P \qquad \frac{}{x : \bot, \Gamma \to \Delta}\ L\bot$$

$$\frac{x : A, x : B, \Gamma \to \Delta}{x : A \wedge B, \Gamma \to \Delta}\ L\wedge \qquad \frac{\Gamma \to \Delta, x : A \quad \Gamma \to \Delta, x : B}{\Gamma \to \Delta, x : A \wedge B}\ R\wedge$$

$$\frac{x : A, \Gamma \to \Delta \quad x : B, \Gamma \to \Delta}{x : A \vee B, \Gamma \to \Delta}\ L\vee \qquad \frac{\Gamma \to \Delta, x : A, x : B}{\Gamma \to \Delta, x : A \vee B}\ R\vee$$

$$\frac{\Gamma \to \Delta, x : A \quad x : B, \Gamma \to \Delta}{x : A \supset B, \Gamma \to \Delta}\ L\supset \qquad \frac{x : A, \Gamma \to \Delta, x : B}{\Gamma \to \Delta, x : A \supset B}\ R\supset$$

Finally, we need to consider another group of rules, called structural rules, not introducing any logical operator. Weakening (Wk) corresponds to assuming that the set of derivable sequents is closed under the addition of formulas either in the antecedent or in the consequent of each of its elements. Dually, contraction (Ctr) allows two occurrences of the same formula to be reduced to one:[4]

[4] Notice that contraction in this case is defined either on formulas labelled with possible states or on relational atoms, but not on programs decorated with their formulas specification. Hence, the presence of contraction is not in conflict with what we said in §1.1 about the relationship between iteration and contraction.

$$\frac{\Gamma \to \Delta}{x : A, \Gamma \to \Delta}\ \text{Wk} \qquad \frac{\Gamma \to \Delta}{\Gamma \to \Delta, x : A}\ \text{Wk} \qquad \frac{\Gamma \to \Delta}{xRy, \Gamma \to \Delta}\ \text{Wk}$$

$$\frac{x : A, x : A, \Gamma \to \Delta}{x : A, \Gamma \to \Delta}\ \text{Ctr} \qquad \frac{\Gamma \to \Delta, x : A, x : A}{\Gamma \to \Delta, x : A}\ \text{Ctr} \qquad \frac{xRy, xRy, \Gamma \to \Delta}{xRy, \Gamma \to \Delta}\ \text{Ctr}$$

The last rule to be considered is the rule of cut, which intuitively corresponds to the operation of composition of proofs, that is, to the use of lemmas for constructing new proofs:

$$\frac{\Gamma \to \Delta, x : A \quad x : A, \Gamma' \to \Delta'}{\Gamma, \Gamma' \to \Delta', \Delta}\ \text{CUT}$$

Although structural rules are extremely useful in the process of discovering new proofs by top-down reasoning starting from initial sequents, nonetheless it can be proved that they are dispensable rules: every sequent derivable by applying structural rules is already derivable without any applications of them. In other words, structural rules are *admissible* rules. Moreover, weakening and contraction are also height-preserving admissible (hp-admissible for short), i.e. when applied they do not increase the height of the derivation (defined as the length of its longest branch; for details see [28, Def. 1.1.9, p. 9]).

Theorem 3.1. *The following structural properties are satisfied:*

1. *Arbitrary initial sequents are derivable;*
2. *Substitution of labels is hp-admissible;*
3. *All the rules are hp-invertible;*
4. *Weakening is hp-admissible;*
5. *Contraction is hp-admissible;*
6. *Cut is admissible.*

Proof. To prove 1. we need to show that sequents of the form $x : A, \Gamma \to \Delta, x : A$ with an arbitrary formula $x : A$ are derivable. By induction on $x : A$, we deal with the case of A is of the form $[\alpha]B$, the other ones being analogous. By the inductive hypothesis $y : B, xR_\alpha y, x : [\alpha]B \to y : B$ is derivable since $y : B$ is a subformula of $x : [\alpha]B$ for all y. Then apply $L[\ ]$ and obtain $xR_\alpha y, x : [\alpha]B \to y : B$. Since there is no other occurrence of y, the rule $R[\ ]$ is applicable and the conclusion $x : [\alpha]B \to x : [\alpha]B$ obtains. All the other claims are proved by induction on the height of the derivation and follow the same pattern of [19]. Hp-admissibility of substitution (2.) consists into showing that

$\Gamma[y/x] \to \Delta[y/x]$ are derivable whenever $\Gamma \to \Delta$ is, where $\Gamma[y/x]$ and $\Delta[y/x]$ are obtained from Γ and Δ by replacing every occurrence of the label x by y. Hp-invertibility (3.) reduces to prove the admissibility of the rules obtained by inverting the conclusion with the premise(s). For example, in the case of $R[\,]$ we need to show that for all y, the following rule is admissible:

$$\frac{\Gamma \to \Delta, x : [\alpha]A}{xR_\alpha y, \Gamma \to \Delta, y : A}$$

Inductively on the height of the premise, it is easy to see that if it is initial, also the conclusion is initial. Suppose the premise has been derived by some rule R, say $R[\,]$ with a principal formula $z : [\beta]B$ different from $x : [\alpha]A$; suppose also that is y does not occur in the conclusion of $R[\,]$. We apply 2. and replace y by a new w so to obtain from the premise of $R[\,]$ the sequent $zR_\beta w, \Gamma \to \Delta, x : [\alpha]A, w : B$, then by the inductive hypothesis we get $zR_\beta w, xR_\alpha y, \Gamma \to \Delta, y : A, w : B$, and finally by $R[\,]$ we conclude $xR_\alpha y, \Gamma \to \Delta, y : A, z : [\beta]B$. All the other case are proved similarly. The proof of hp-admissibility of weakening (4.) is not problematic because in general the rules of **G3** systems have been designed exactly for obtaining it; for the pattern of demonstration see [19]. Hp-admissibility of contraction (5.) is also proved by induction, exploiting 3. In the case of two occurrences of $x : [\alpha]A$, one of them derived by an application of $R[\,]$, we need to show that

$$\frac{xR_\alpha y, \Gamma \to \Delta, y : A, x : [\alpha]A}{\Gamma \to \Delta, x : [\alpha]A, x : [\alpha]A}\ {\scriptstyle R[\,]}$$

is admissible. By applying 3. on the premise of $R[\,]$ we obtain $xR_\alpha y, xR_\alpha y, \Gamma \to \Delta, y : A, y : A$, from which by the inductive hypothesis and another application of $R[\,]$ we immediately conclude $\Gamma \to \Delta, x : [\alpha]A$. Cut admissibility (6.) is proved by induction on the structure of the cut formula with sub induction on the sum of the heights of the derivations of the premises of cut. The proof is to a large extent similar to the cut elimination proofs in [21] so we shall consider only the case in which the cut formula is $x : [\alpha]A$ and it is principal in both premises. A derivation of the form

$$\dfrac{\dfrac{xR_\alpha y, \Gamma \to \Delta, y : A}{\Gamma \to \Delta, x : [\alpha]A}\ {\scriptstyle R[\,]} \qquad \dfrac{z : A, xR_\alpha z, x : [\alpha]A, \Gamma' \to \Delta'}{xR_\alpha z, x : [\alpha]A, \Gamma' \to \Delta'}\ {\scriptstyle L[\,]}}{xR_\alpha z, \Gamma, \Gamma' \to \Delta', \Delta}\ {\scriptstyle \text{CUT}}$$

is converted into one with two cuts, one is on a smaller formula, the other is on a shorter derivation. Notice that the left most premise is obtained from $xR_\alpha y, \Gamma \to \Delta, y : A$ by applying 2.

$$\dfrac{\dfrac{xR_\alpha z, \Gamma \to \Delta, z : A \qquad \dfrac{\Gamma \to \Delta, x : [\alpha]A \quad z : A, xR_\alpha z, x : [\alpha]A, \Gamma' \to \Delta'}{z : A, xR_\alpha z, \Gamma, \Gamma, \Gamma' \to \Delta', \Delta, \Delta}\text{CUT}}{xR_\alpha y, \Gamma, \Gamma, \Gamma' \to \Delta', \Delta, \Delta}\text{CUT}}{xR_\alpha y, \Gamma, \Gamma' \to \Delta', \Delta}\text{Ctr} \qquad \dashv$$

The admissibility results stated in Theorem 3.1 allow to build derivations in a systematic way: if a sequent is derivable, then its derivation can be effectively found starting root-first from it and reconstructing its derivation tree by successively applying backwards the instances of those rules the principal formulas of which coincide with one of the formulas present in the sequent under analysis.

It is not difficult to see that the system consisting of rules considered so far characterizes the formulas valid in all relational models. In particular, all classical propositional tautologies and the normality axioms for K and $[\ ]$ — i.e. the sequents $\to x : \mathsf{K}_i(A \supset B) \supset (\mathsf{K}_iA \supset \mathsf{K}_iB)$ and $\to x : [\alpha](A \supset B) \supset ([\alpha]A \supset [\alpha]B)$ — are derivable using the given rules. Still, these rules are not enough. Factivity, positive and negative introspection, and all the formulas involving the operations for building programs cannot be derived. Since these principles of knowledge and actions are valid in all standard epistemic models, in order to render them derivable we need to internalize in the calculus the conditions Kn_1, Kn_2 and Kn_3. We may think of thse condition as mathematical axioms of some axiomatic theory. The question is then how to add proper axioms to the calculus. There are several ways for doing it ([20], § 6.3). Consider the case of R_i. First, we could allow derivations to start with "basic mathematical sequents" ([8, §1.4]) corresponding directly to the properties of each R_i to be reflexive, transitive, and symmetric:

$$\to xR_ix \qquad xR_iy, yR_iz \to xR_iz \qquad xR_iy \to yR_ix$$

The problem is that this kind of initial sequents may be composed together via instances of the cut rule that cannot be proved to be admissible (for a general account of this problem see [9, p. 125] and [28, p. 127]). For example, in order to derive the Euclideanness property an essential and non-eliminable use of the cut rule is needed:

$$\dfrac{xR_iy \to yR_ix \quad yR_ix, xR_iz \to yR_iz}{xR_iy, xR_iz \to yR_iz}\text{CUT}$$

To retrieve cut admissibility we need to consider an alternative way to internalize in the calculus the conditions characterizing standard epistemic models. In general, the idea is to abandon the traditional conception of axioms as borders and starting points of derivations, and to integrate them inside the inferential structure, and to make them part of the body of derivations. In particular, in [21] a general method of adding axioms to sequent calculus in the form of extra-logical inference rules while preserving cut elimination is presented. The method covers specific mathematical theories (apartness, order and lattice theories, affine and projective geometry) and, besides, it successfully applies to modal and non-classical logics, as it is shown in [19]. We start from the classical multi-succedent sequent calculus **G3c** (see [28, p. 77]) and use the existence of conjunctive normal form in classical logic: every quantifier-free formula is equivalent to some formula in conjunctive normal form, that is, to a conjunction of disjunctions of atomic formulas or negation of atomic formulas. Each conjunct is a formula of the form $\neg P_1 \vee \cdots \vee \neg P_m \vee Q_1 \vee \cdots \vee Q_n$ which is classically equivalent to the implication

$$P_1 \wedge \cdots \wedge P_m \supset Q_1 \vee \cdots \vee Q_n$$

Special cases are with $m = 0$, where it reduces to $Q_1 \vee \ldots \vee Q_n$, and with $n = 0$ where it is $\neg(P_1 \wedge \ldots \wedge P_m)$. The universal closure of any such implication is called a regular formula. Regular formulas are then converted into deductively equivalent left rules of the form

$$\frac{Q_1, \overline{P}, \Gamma \to \Delta \quad \ldots \quad Q_n, \overline{P}, \Gamma \to \Delta}{\overline{P}, \Gamma \to \Delta}\ \mathit{Reg}$$

where the multiset $P_1, \ldots, P_m$ has been abbreviated in $\overline{P}$ and repeated in the premises in order to make contraction admissible.

Rules following *Reg* preserve cut elimination in the sense that the new cuts created by means of *Reg* rules can always be reduced to already known types of cuts, for which we posses a reduction strategy. The fundamental aspect is that by transforming axioms into only one kind of rules, i.e. left rules, it is not possible to create cuts between the principal formulas of two *Reg* rules. Looking at our example, it is easy to see that the conditions of reflexivity, transitivity, and symmetry of R_i are instances of regular formulas and thus they can be converted into *Reg* rules:

$$\frac{xR_ix, \Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta}\ \mathit{Ref}_i \qquad \frac{xR_iz, xR_iy, yR_iz, \Gamma \rightarrow \Delta}{xR_iy, yR_iz, \Gamma \rightarrow \Delta}\ \mathit{Trans}_i$$

$$\frac{yR_ix, xR_iy, \Gamma \rightarrow \Delta}{xR_iy, \Gamma \rightarrow \Delta}\ \mathit{Sym}_i$$

When the basic mathematical sequents are replaced by the corresponding rules of inference then the Euclideanness of R_i has a cut-free derivation:

$$\dfrac{\dfrac{yR_iz, yR_ix, xR_iy, xR_iz \rightarrow yR_iz}{yR_ix, xR_iy, xR_iz \rightarrow yR_iz}\ \mathit{Trans}_i}{xR_iy, xR_iz \rightarrow yR_iz}\ \mathit{Sym}_i$$

The derivation above should not suggest that we need initial sequents of the form $xR_iy, \Gamma \rightarrow \Delta, xR_iy$ for deriving formulas of the language of **EPDL**. The addition of these initial sequents is needed only for the derivation of the properties of the accessibility relation.

All the properties characterizing knowledge operators K_i (factivity, positive introspection, and negative introspection) become derivable when our system of rules is extended with the rules Ref_i, Trans_i, and Sym_i. For example, in the case of factivity we have:

$$\dfrac{\dfrac{\dfrac{x : A, xR_ix, x : \mathsf{K}_iA \rightarrow x : A}{xR_ix, x : \mathsf{K}_iA \rightarrow x : A}\ L\mathsf{K}}{x : \mathsf{K}_iA \rightarrow x : A}\ \mathit{Ref}_i}{\rightarrow x : \mathsf{K}_iA \supset A}\ R\supset$$

By the same method, it is possible to convert into rules also existential axioms, or, more generally, axioms of the form of geometric implications (in the sense of category theory). These are the universal closures of implications $A \supset B$ in which A and B do not contain implications or universal quantifiers. Geometric implications can be turned in a useful normal form that consists of conjunctions of formulas

$$\forall \overline{x}(P_1 \wedge \cdots \wedge P_m \supset \exists \overline{y}_1 M_1 \vee \cdots \vee \exists \overline{y}_n M_n)$$

where each P_i is an atomic formula, each M_j a conjunction of a list of atomic formulas $\overline{Q}_j$, and none of the variables in the vectors $\overline{y}_j$ are free in P_i. In turn, each of these formulas can be turned into an inference rule of the following form:

$$\dfrac{\overline{Q}_1(\overline{z}_1/\overline{y}_1), \overline{P}, \Gamma \to \Delta \quad \ldots \quad \overline{Q}_n(\overline{z}_n/\overline{y}_n), \overline{P}, \Gamma \to \Delta}{\overline{P}, \Gamma \to \Delta}\ {\scriptstyle \mathit{Geom}}$$

The variables $\overline{y}_i$ are called the replaced variables of the schema, and the variables $\overline{z}_i$ the proper variables, or *eigenvariables*. In what follows, we shall consider for ease of notation the case in which the vectors of variables $\overline{y}_i$ consist of a single variable. The geometric rule schema is subject to the condition that the *eigenvariables* must not be free in $\overline{P}, \Gamma, \Delta$. As in the case of *Reg* rules, cut elimination still holds in presence of *Geom* rules; a detailed proof can be found in [18].

An example of geometric semantic condition we need to deal with is the condition Act_1 corresponding to the sequential composition of programs. In fact, the left-to-right direction of Act_1 directly follows the geometric scheme, while the right-to-left direction is logically equivalent to the formula $\forall x \forall y \forall z((xR_\alpha z \wedge zR_\beta y) \supset xR_{\alpha;\beta}y)$ which follows the regular scheme. The two directions can thus be converted into the two following rules:

$$\dfrac{xR_\alpha z, zR_\beta y, xR_{\alpha;\beta}y, \Gamma \to \Delta}{xR_{\alpha;\beta}y, \Gamma \to \Delta}\ {\scriptstyle ;1} \qquad \dfrac{xR_{\alpha;\beta}y, xR_\alpha z, zR_\beta y, \Gamma \to \Delta}{xR_\alpha z, zR_\beta y, \Gamma \to \Delta}\ {\scriptstyle ;2}$$

As all the rules following *Geom*, also *;1* must satisfy the variable condition that z does not occur in the conclusion. The axiom $[\alpha;\beta]B \equiv [\alpha][\beta]B$ gets derived as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{z : A, xR_{\alpha;\beta}z, xR_\alpha y, yR_\beta z, x : [\alpha;\beta]A \to z : A}{xR_{\alpha;\beta}z, xR_\alpha y, yR_\beta z, x : [\alpha;\beta]A \to z : A}\ {\scriptstyle L[\,]}}{xR_\alpha y, yR_\beta z, x : [\alpha;\beta]A \to z : A}\ {\scriptstyle ;2}}{xR_\alpha y, x : [\alpha;\beta]A \to y : [\beta]A}\ {\scriptstyle R[\,]}}{x : [\alpha;\beta]A \to x : [\alpha][\beta]A}\ {\scriptstyle R[\,]}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{y : A, z : [\beta]A, xR_\alpha z, zR_\beta y, xR_{\alpha;\beta}y, x : [\alpha][\beta]A \to y : A}{z : [\beta]B, xR_\alpha z, zR_\beta y, xR_{\alpha;\beta}y, x : [\alpha][\beta]A \to y : A}\ {\scriptstyle L[\,]}}{xR_\alpha z, zR_\beta y, xR_{\alpha;\beta}y, x : [\alpha][\beta]A \to y : A}\ {\scriptstyle L[\,]}}{xR_{\alpha;\beta}y, x : [\alpha][\beta]A \to y : A}\ {\scriptstyle ;1}}{x : [\alpha][\beta]A \to x : [\alpha;\beta]A}\ {\scriptstyle R[\,]}$$

Non-deterministic alternation between programs is semantically expressed by condition Act_2. The left-to-right direction of Act_2 directly follows the regular scheme, while the right-to-left direction is logically equivalent to the formula

$$\forall x \forall y (xR_\alpha y \supset xR_{\alpha\cup\beta} y) \wedge \forall x \forall y (xR_\beta y \supset xR_{\alpha\cup\beta} y)$$

which corresponds to the conjunction of two regular formulas. Hence, the following three inference rules are obtained:

$$\frac{xR_\alpha y, xR_{\alpha\cup\beta} y, \Gamma \to \Delta \quad xR_\beta y, xR_{\alpha\cup\beta} y, \Gamma \to \Delta}{xR_{\alpha\cup\beta} y, \Gamma \to \Delta}\ {\scriptstyle \cup \mathit{1}}$$

$$\frac{xR_{\alpha\cup\beta} y, xR_\alpha y, \Gamma \to \Delta}{xR_\alpha y, \Gamma \to \Delta}\ {\scriptstyle \cup \mathit{2}'} \qquad \frac{xR_{\alpha\cup\beta} y, xR_\beta y, \Gamma \to \Delta}{xR_\beta y, \Gamma \to \Delta}\ {\scriptstyle \cup \mathit{2}''}$$

Consequently, $[\alpha \cup \beta]B \equiv [\alpha]B \wedge [\beta]B$ can be derived using these rules.

$$\dfrac{\dfrac{\dfrac{\dfrac{y : A, xR_{\alpha\cup\beta} y, xR_\alpha y, x : [\alpha \cup \beta]A \to y : A}{xR_{\alpha\cup\beta} y, xR_\alpha y, x : [\alpha \cup \beta]A \to y : A}\ {\scriptstyle L[\,]}}{xR_\alpha y, x : [\alpha \cup \beta]A \to y : A}\ {\scriptstyle \cup \mathit{2}'}}{x : [\alpha \cup \beta]A \to x : [\alpha]A}\ {\scriptstyle R[\,]} \quad \dfrac{\dfrac{\dfrac{y : A, xR_{\alpha\cup\beta} y, xR_\beta y, x : [\alpha \cup \beta]A \to y : A}{xR_{\alpha\cup\beta} y, xR_\beta y, x : [\alpha \cup \beta]A \to y : A}\ {\scriptstyle L[\,]}}{xR_\beta y, x : [\alpha \cup \beta]A \to y : A}\ {\scriptstyle \cup \mathit{2}''}}{x : [\alpha \cup \beta]A \to x : [\beta]A}\ {\scriptstyle R[\,]}}{x : [\alpha \cup \beta]A \to x : [\alpha]A \wedge [\beta]A}\ {\scriptstyle R\wedge}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{y : A, xR_\alpha y, xR_{\alpha\cup\beta} y, x : [\alpha]A, x : [\beta]A \to y : A}{xR_\alpha y, xR_{\alpha\cup\beta} y, x : [\alpha]A, x : [\beta]A \to y : A}\ {\scriptstyle L[\,]} \quad \dfrac{y : A, xR_\beta y, xR_{\alpha\cup\beta} y, x : [\alpha]A, x : [\beta]A \to y : A}{xR_\beta y, xR_{\alpha\cup\beta} y, x : [\alpha]A, x : [\beta]A \to y : A}\ {\scriptstyle L[\,]}}{xR_{\alpha\cup\beta} y, x : [\alpha]A, x : [\beta]A \to y : A}\ {\scriptstyle \cup_{\mathit{1}}}}{x : [\alpha]A, x : [\beta]A \to x : [\alpha \cup \beta]A}\ {\scriptstyle R[\,]}}{x : [\alpha]A \wedge [\beta]A \to x : [\alpha \cup \beta]A}\ {\scriptstyle L\wedge}$$

Before introducing the last set of rules corresponding to the semantic condition on the test operator, we need to present the rules corresponding to the conditions for identity Id_1 and Id_2:

$$\frac{x = x, \Gamma \to \Delta}{\Gamma \to \Delta}\ {\scriptstyle \mathit{Ref}_=} \qquad \frac{y : A, x : A, x = y, \Gamma \to \Delta}{x : A, x = y, \Gamma \to \Delta}\ {\scriptstyle \mathit{Rep}_=}$$

Finally, the condition Act_3 can be shown to be equivalent to the conjunction of three regular formulas and then converted into the *Reg* rules:

$$\frac{x : A, xR_{A?} y, \Gamma \to \Delta}{xR_{A?} y, \Gamma \to \Delta}\ {\scriptstyle ?\mathit{1}'} \qquad \frac{x = y, xR_{A?} y, \Gamma \to \Delta}{xR_{A?} y, \Gamma \to \Delta}\ {\scriptstyle ?\mathit{1}''}$$

$$\dfrac{xR_{A?}y, x : A, x = y, \Gamma \to \Delta}{x : A, x = y, \Gamma \to \Delta}\ ?2$$

The corresponding axiom $[A?]B \equiv A \supset B$ is derivable as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{x : B, xR_{A?}x, x = x, x : [A?]B, x : A \to x : B}{xR_{A?}x, x = x, x : [A?]B, x : A \to x : B}\ L[\,]}{x = x, x : [A?]B, x : A \to x : B}\ ?2}{x : [A?]B, x : A \to x : B}\ \mathit{Ref}_=}{x : [A?]B \to x : A \supset B}\ R\supset$$

$$\dfrac{\dfrac{\dfrac{x : A, xR_{A?}y \to y : B, x : A \qquad \dfrac{\dfrac{y : B, x = y, x : A, xR_{A?}y, x : B \to y : B}{x = y, x : A, xR_{A?}y, x : B \to y : B}\ \mathit{Rep}_=}{x : A, xR_{A?}y, x : B \to y : B}\ ?1''}{x : A, xR_{A?}y, x : A \supset B \to y : B}\ L\supset}{xR_{A?}y, x : A \supset B \to y : B}\ ?1'}{x : A \supset B \to x : [A?]B}\ R[\,]$$

4. No Learning, Perfect Recall and Reasoning Ability

Since the language of **EPDL** is freely generated by the union of the two languages of **EL** and **PDL**, and since the models of **EPDL** are obtained by joining up the the models of **EL** and those of **PDL**, it is then clear that **EPDL** can be considered as the fusion of the two logics **EL** and **PDL**. However, **EPDL** also contains mixed formulas where epistemic and action modalities interact. As we have already seen, a quite natural reading can be assigned to them and their peculiarity is to contribute to the formalization of some simple game-theoretic scenarios. In this section we are concerned with the problem of finding cut-elimination preserving rules which correspond to the three principles under consideration. Since they express the connection between the two groups of modalities of **EPDL** it is clear that this connection is reflected also at the semantic level. In [25] the frame conditions characterizing the principles **NL**, **PR** and **RA** are found. They are summarized in the table below.

NL	$[\alpha]\mathsf{K}_i A \supset \mathsf{K}_i[\alpha]A$	$\forall x\forall y\forall z(xR_i y \wedge yR_\alpha z \supset \exists w(xR_\alpha w \wedge wR_i z))$
PR	$\mathsf{K}_i[\alpha]A \supset [\alpha]\mathsf{K}_i A$	$\forall x\forall y\forall z(xR_\alpha y \wedge yR_i z \supset \exists w(xR_i w \wedge wR_\alpha z))$
RA	$\langle\alpha\rangle\mathsf{K}_i A \supset \mathsf{K}_i\langle\alpha\rangle A$	$\forall x\forall y\forall z(xR_\alpha y \wedge xR_i z \supset \exists w(yR_i w \wedge zR_\alpha w))$

These conditions state that the following diagrams can be completed (the completing arrows are the dotted ones):

NL **PR** **RA**

$$\begin{tikzcd} x \arrow[r, dashed, "R_\alpha"] \arrow[d, "R_i"'] & w \arrow[d, dashed, "R_i"] \\ y \arrow[r, "R_\alpha"'] & z \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, "R_\alpha"] \arrow[d, dashed, "R_i"'] & y \arrow[d, "R_i"] \\ w \arrow[r, dashed, "R_\alpha"'] & z \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, "R_\alpha"] \arrow[d, "R_i"'] & y \arrow[d, dashed, "R_i"] \\ z \arrow[r, dashed, "R_\alpha"'] & w \end{tikzcd}$$

Notice that the three conditions coincide with geometric implications and therefore it is possible to make them corresponding to rules of the form

$$\frac{xR_\alpha w, wR_i z, xR_i y, yR_\alpha z, \Gamma \to \Delta}{xR_i y, yR_\alpha z, \Gamma \to \Delta}\,{\scriptstyle NL} \qquad \frac{xR_i w, wR_\alpha z, xR_\alpha y, yR_i z, \Gamma \to \Delta}{xR_\alpha y, yR_i z, \Gamma \to \Delta}\,{\scriptstyle PR}$$

$$\frac{yR_i w, zR_\alpha w, xR_\alpha y, xR_i z, \Gamma \to \Delta}{xR_\alpha y, xR_i z, \Gamma \to \Delta}\,{\scriptstyle RA}$$

All the rules must meet the condition that the label w does not occur in the conclusion. This reflects the role of the existential quantifier in the corresponding semantic condition. The axioms **NL**, **PR** and **RA** from the Hilbert-style axiomatization can now be derived by applying the new rules:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{z : A, w : \mathsf{K}_i A, xR_\alpha w, wR_i z, xR_i y, yR_\alpha z, x : [\alpha]\mathsf{K}_i A \to z : A}{w : \mathsf{K}_i A, xR_\alpha w, wR_i z, xR_i y, yR_\alpha z, x : [\alpha]\mathsf{K}_i A \to z : A}\,{\scriptstyle L\mathsf{K}}}{xR_\alpha w, wR_i z, xR_i y, yR_\alpha z, x : [\alpha]\mathsf{K}_i A \to z : A}\,{\scriptstyle L[\,]}}{xR_i y, yR_\alpha z, x : [\alpha]\mathsf{K}_i A \to z : A}\,{\scriptstyle NL}}{xR_i y, x : [\alpha]\mathsf{K}_i A \to y : [\alpha]A}\,{\scriptstyle R[\,]}}{x : [\alpha]\mathsf{K}_i A \to x : \mathsf{K}_i[\alpha]A}\,{\scriptstyle R\mathsf{K}}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{z : A, w : [\alpha]A, xR_i w, wR_\alpha z, xR_\alpha y, yR_i z, x : \mathsf{K}_i[\alpha]A \to z : A}{w : [\alpha]A, xR_i w, wR_\alpha z, xR_\alpha y, yR_i z, x : \mathsf{K}_i[\alpha]A \to z : A}\,{\scriptstyle L[\,]}}{xR_i w, wR_\alpha z, xR_\alpha y, yR_i z, x : \mathsf{K}_i[\alpha]A \to z : A}\,{\scriptstyle L\mathsf{K}}}{xR_\alpha y, yR_i z, x : \mathsf{K}_i[\alpha]A \to z : A}\,{\scriptstyle PR}}{xR_\alpha y, x : \mathsf{K}_i[\alpha]A \to y : \mathsf{K}_i A}\,{\scriptstyle R\mathsf{K}}}{x : \mathsf{K}_i[\alpha]A \to x : [\alpha]\mathsf{K}_i A}\,{\scriptstyle R[\,]}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{w : A, yR_iw, zR_\alpha w, xR_\alpha y, xR_iz, y : \mathsf{K}_iA \to z : \langle\alpha\rangle A, w : A}{w : A, yR_iw, zR_\alpha w, xR_\alpha y, xR_iz, y : \mathsf{K}_iA \to z : \langle\alpha\rangle A}\,R\langle\,\rangle}{yR_iw, zR_\alpha w, xR_\alpha y, xR_iz, y : \mathsf{K}_iA \to z : \langle\alpha\rangle A}\,L\mathsf{K}}{xR_\alpha y, xR_iz, y : \mathsf{K}_iA \to z : \langle\alpha\rangle A}\,RA}{xR_\alpha y, y : \mathsf{K}_iA \to x : \mathsf{K}_i\langle\alpha\rangle A}\,R\mathsf{K}}{x : \langle\alpha\rangle\mathsf{K}_iA \to x : \mathsf{K}_i\langle\alpha\rangle A}\,L\langle\,\rangle$$

The system obtained by collecting all the rules presented in the last two sections (§§ 3, 4) will be called **G3EPDL**. This system is provably equivalent to the axiomatic system **EPDL**.

5. A paradoxical situation

Let us now temporarily leave our sequent calculus presentation of **EPDL** and get back to the standard axiomatic account. As we mentioned in the Introduction, a main result of [25, Cor. 24, p. 122] is to have shown that **EPDL** is a non-conservative extension of **EL**. In particular, the principle

$$A \supset \mathsf{K}_iA \quad \textbf{OP}$$

which belongs to the language of **EL**, is derivable in **EPDL**, but not in **EL**. In a nutshell,

$$\vdash_{\mathbf{EPDL}} A \supset \mathsf{K}_iA \qquad \text{but} \qquad \nvdash_{\mathbf{EL}} A \supset \mathsf{K}_iA$$

More precisely, what is shown in [25] is that **OP** is derivable simply by combining **EL**, **PDL** and just one of the "mixing" principles **NL**, **PR** or **RA**, i.e.

$$\vdash_{\mathbf{EL+PDL+NL}} A \supset \mathsf{K}_iA \quad \text{or} \quad \vdash_{\mathbf{EL+PDL+PR}} A \supset \mathsf{K}_iA \quad \text{or}$$
$$\vdash_{\mathbf{EL+PDL+RA}} A \supset \mathsf{K}_iA$$

For example, in the case of **NL** the derivation is the following:

1.	$[\alpha]\mathsf{K}_iA \supset \mathsf{K}_i[\alpha]A$	**NL**
2.	$[\neg A?]\mathsf{K}_iA \supset \mathsf{K}_i[\neg A?]A$	Instantiation of 1.
3.	$[A?]B \equiv A \supset B$	**Test**
4.	$[\neg A?]\mathsf{K}_iA \equiv \neg A \supset \mathsf{K}_iA$	Instantiation of 3.
5.	$(\neg A \supset \mathsf{K}_iA) \supset (\mathsf{K}_i[\neg A?]A)$	Transitivity on 4., 2.
6.	$\mathsf{K}_i[\neg A?]A \equiv \mathsf{K}_i(\neg A \supset A)$	Necessitation on 3.

7.	$(\neg A \supset \mathsf{K}_i A) \supset \mathsf{K}_i(\neg A \supset A)$	Transitivity on 5., 6.
8.	$(\neg A \supset \mathsf{K}_i A) \supset \mathsf{K}_i A$	Logical Equivalences from 7.
9.	$\neg A \vee \mathsf{K}_i A$	Logical Tautologies from 8.
10.	$A \supset \mathsf{K}_i A$	Logical Tautologies from 9.

What makes this result extremely puzzling is that **OP** is nothing else but the formal characterization of the omniscience principle. This seems clearly to undermine the possibility of using **EPDL** as a formal framework for studying the knowledge/actions interaction, because if each agent already knows everything that is true, what then would be added by her interaction with the other agents? In other terms, the idea is that if **OP** holds, then either there would be no quest for knowledge at all or the quest would be purely solipsistic. In either case the conclusion would be highly counterintuitive.

In fact, the situation is even more dramatic. In presence of the factivity of knowledge we have that truth and knowledge collapse, i.e.

$$\vdash_{\mathbf{EPDL}} A \equiv \mathsf{K}_i A$$

We are then in a situation analogue to the one of the Church-Fitch paradox of knowability (see [24] and the bibliography therein). Not only in both cases truth and knowledge become two indiscernible concepts, but also their identification seems to be caused by the acceptance of formal presentations of some principles characterizing verificationist theories of truth and meaning. In the case of the Church-Fitch paradox it is the knowability principle (or principle K, see [4, p. 99]) to be involved, while in this case it is **PR** that can be seen as the formal version of a sort of manifestability principle, or better, of a principle concerning the non-transcendence of semantical concepts.[5] Indeed, **PR** may be read as: if an agent i knows that any occurrence of α represents a justification for the assertion of A, then when α occurs i is able to recognize it as such and thus to know that A. In other terms, **PR** states that the justifications for the assertion of a proposition A are epistemically transparent to us. On the other hand, principles **NL** and **RA** formalize a

[5] With the expression 'semantical concepts' we want to denote those concepts on which a theory of meaning or a theory of truth are based on. For example, the semantical key concept of certain verificationist theories of truth is the concept of proof. On the contrary, when we speak of textquoteleft formal semantics' we are just speaking of a mathematical framework in which to express the formal properties that a semantical concept should possess. Thus, in this case, we do not need to indicate which this concept is, but simply to describe its abstract and general properties.

more general and basic condition, not necessarily quintessential of the verificationist account, but also common to other theories of meaning and truth. The idea behind these two principles is that the agents who participate successfully to a linguistic exchange[6] are supposed to know what is a (necessary or possible) justification for the assertion of a certain proposition. In other words, we are assuming that those agents who successfully take part to a common and social enterprise, like linguistic communication or knowledge acquisition, they do not it by chance, but because they already know what counts as a semantical key concept for the situation in which they are acting.

5.1. Possible solutions

Similarly to what happens with the Church-Fitch paradox, the standard solutions proposed to block this new epistemic paradox are essentially of two kinds (see [24]). The first one is a completely syntactical solution as it consists in the restriction on the possible instances of the axiom schemes **NL**, **PR**, **RA** ([25, p. 124]). In a nutshell, the idea is that if A is a formula occurring in one of these axioms, then the place-holders for programs occurring in the same axiom can be instantiated neither with the test for A nor with the test for $\neg A$. In this manner the step 2. of the derivation of $A \supset \mathsf{K}_i A$ above is prevented. The problem of this type of solution is that, on the one hand, it represents a too much *ad hoc* solution. On the other hand, actions and formulas become heterogeneous with respect to the operation of instantiation which is uniform when applied to formulas, but is not when applied to actions.

The second solution is more conceptual, and has the effect of entailing some changes in the formal semantics of **EPDL**. The basic idea is to drop the standard test operator ? and replace it with a new *informational test* operator ¿ (cf. [25, p. 124 *et seqq.*]) such that

$$A¿ \quad \text{means} \quad \textit{test if } A \textit{ is known}$$

In this manner the operation of testing is no more linked directly to the truth value of a formula, but to the epistemic access to it. To test a formula does not mean to test if it is true, but rather if it is known. Semantically, we replace the accessibility relation $R_{A?}$ with a new accessibility relation $R_{A¿}$ for which it holds that for an arbitrary agent i,

[6] Notice that since the Wittgenstein of the *Philosophical Investigations* it is quite standard to conceive linguistic exchanges as particular kind of games (see [29]).

Act$_{3'}$ $\quad \forall x \forall y (x R_{A¿} y \equiv x R_i y \wedge y \Vdash \mathsf{K}_i A)$

This frame property is sufficient in order to validate the following principle, which characterize the behavior of ¿ with respect to a generic agent i

$$[A¿]B \equiv \mathsf{K}_i(\mathsf{K}_i A \supset B) \quad \textbf{Inf Test}$$

It is worth noting that the definition of Act$_{3'}$ does not involve any reference to identity between possible states. Therefore, if ? is replaced by ¿ we do not need the rules $Ref_=$ and $Rep_=$.

The problem of this solution is that it has a too much global character: it works for the whole system **EPDL** without allowing to understand if the cause of the paradox could be circumscribed to the axioms **NL**, **PR**, and **RA**. In particular, there is no analysis of the responsibility that each of these axioms has in the derivation of the paradox, when used in conjunction with the standard test operator. Without such analysis the combination of **EL** and **PDL** is put into question, while a much more parsimonious solution would have been to operate only the revisions suggested by the analysis of the "mixing" axioms, because they are the last axioms to have been added and thus it is reasonable to think of them as the most changeable part of the theory. However, a second problematic aspect of this solution is that by rejecting the standard test operator we have also to abandon the possibility of providing a verificationist account of implication inside a classical logic framework, which represents indeed one of the most debated topics of verificationism nowadays (for a general exposition of the problem see [5, pp. 291–300]).

5.2. A different kind of analysis

It is not difficult to see that the paradox is derivable also in our system **G3EPDL**. However, the use of inference rules instead of axiom schemes guarantees a systematic and modular analysis of the collapse of truth and knowledge. More precisely, thanks to the properties stated in Theorem 3.1, it is possible to operate a proof-search on the sequent $\rightarrow x : A \supset \mathsf{K}_i A$, and since the rules NL, PR, and RA are independent from each other, it is possible to investigate fragments of **G3EPDL** containing only one of them at a time; in this way we can render the analysis of the derivation of **OP** a modular one, focusing just on one of the "mixing" axioms. Moreover, since the axioms of **EPDL** have been transformed into inference rules acting on their corresponding frame conditions, the proof-search becomes a formal and systematic method for extracting the semantic conditions standing behind any given derivable formula of **EPDL**.

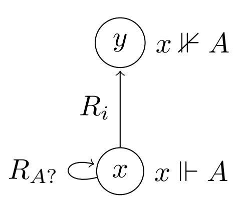
Let us focus now on the principle **NL**, i.e. let us consider only the system **G3EPDL** $\setminus \{PR, RA\}$. By root-first proof search we can find the semantic conditions sufficient for deriving **OP**.

$$
\begin{prooftree}
\AxiomC{$y : A, yR_{A?}w, wR_i x, yR_i x, xR_{A?}x, x = x, xR_i y, x : A \to y : A$}
\RightLabel{$?1'$}
\UnaryInfC{$yR_{A?}w, wR_i x, yR_i x, xR_{A?}x, x = x, xR_i y, x : A \to y : A$}
\RightLabel{NL}
\UnaryInfC{$yR_i x, xR_{A?}x, x = x, xR_i y, x : A \to y : A$}
\RightLabel{Sym_i}
\UnaryInfC{$xR_{A?}x, x = x, xR_i y, x : A \to y : A$}
\RightLabel{$?2$}
\UnaryInfC{$x = x, xR_i y, x : A \to y : A$}
\RightLabel{$Ref_=$}
\UnaryInfC{$xR_i y, x : A \to y : A$}
\RightLabel{$R\mathsf{K}$}
\UnaryInfC{$x : A \to x : \mathsf{K}_i A$}
\RightLabel{$R\supset$}
\UnaryInfC{$\to x : A \supset \mathsf{K}_i A$}
\end{prooftree}
$$

The peculiarity of this derivation is that it makes appeal to the rule Sym_i, while the use of the properties of the knowledge operator remained hidden in the Hilbert-style axiomatic derivation.

5.3. An alternative solution

From the previous derivation we can simply infer that Sym_i is a sufficient condition for the derivation of **OP** with respect to the system **G3EPDL** $\setminus \{PR, RA\}$: if Sym_i is assumed, then **OP** can be derived. The derivation itself gives no information about whether Sym_i is also a necessary condition for **OP**. Nevertheless, by pruning the tree just before the application of Sym_i a countermodel for **OP** can be extracted (for ease of representation, the R_i-reflexivity relations on x and y are omitted):



In this model $x \nVdash A \supset \mathsf{K}_i A$, because $x \Vdash A$, but $x \nVdash \mathsf{K}_i A$. This shows that Sym_i is also a necessary condition for **OP**: without Sym_i, **OP** becomes invalid.

The previous result shows that the standard test operator and **NL** are not problematic *per se*, but become so when epistemic modalities correspond to **S5** modalities. From a conceptual point of view, this means that when knowledge and actions interact, the characterization of knowledge by using the principles of **S5** could be too strong. In other terms, if knowledge is studied as a dynamic notion and not a static one, then it could be necessary to formalize it in a more permissive way. In particular, the solution that naturally stems from the previous result consists in weakening the properties of the knowledge operator and leave out the symmetry of R_i. Without symmetric accessibility relations the Euclideanness of R_i does not follows anymore. And since Euclideanness correspond to the principle of negative introspection (see [6, p. 63]) we should abandon this principle. This idea seems to be confirmed by the following informal argument meant to show that this principle is not valid when knowledge is dynamic: the fact the we do not know that A is not sufficient to entail that we know that we do not know that A, since it could be the case that in the future a certain action could bring about A and therefore put us in the condition to eventually know that A.

6. Conclusions and related works

Although in this paper we focused on the standard **PDL** (without iteration) to formalize the reasoning about actions in games, there are variants of **PDL** that have been specifically introduced to this puropose. Among them, Game Logic [22] is a generalization of **PDL** that formalize the resoning about determined 2-palyer games. It should be also noticed that an alternative proof-theoretic approach to **EPDL** is presented in [23] where a Tait-style sequent calculus is introduced and the proof-search procedure is shown to be decidable. However, a restricted version of the cut rule is considered, whereas in this paper we work with cut rule in its full generality. Moreover, in [23] axiom schemes are transformed into inference rules following a method similar to the fold/unfold pattern described in [3]: these rules act directly on formulas and not on their semantic conditions. The drawback is that proof-search procedure cannot be used for systematically find the semantic conditions of derivable sequents of **EPDL**. Other works are concerned with either the epistemic part or the program part of **EPDL**. Within the tradition of labelled systems [12] presents a sequent calculus for multi-agent epistemic logic with the operator of distributed knowledge. The rules for K_i and the relational rules for R_i of our system **G3EPDL** are the same rules of the system presented in [12]. As it concerns propositional dynamic logic several approaches have been recently proposed. In [14] a tree-hypersequent calculus for the full language of **PDL** is introduced. Although the system satisfies the structural properties, the rules that correspond to the operator * has an infinite number of premises. A system for the iteration-free fragment of **EPDL** is presented in [2].

In this paper we have investigated the connection between the formal notions of knowledge and action from a proof-theoretic perspective. Such a connection can be formulated within the language of **EPDL** and is known to be problematic as it leads to the collapse of knowledge and truth and consequently to the omniscience of agents. The phenomenon is clearly related to the paradox of knowability. In either case from intuitively valid principles combining two kinds of modal operators it is possible to conclude a highly counterintuitive formula where one of the two modality disappeared. As in the Church-Fitch paradox of knowability, also in **EPDL** the collapse of knowledge and truth is presented axiomatically. Following the strategy adopted in [16] we formulate these principles as rules of inference, rather than axioms, and provide an analysis of the derivation of omniscience where every step is made explicit. In particular, our analysis allows to recognize that the assumptions on knowledge operator, although neglected in the axiomatic derivation, play an important role. In particular, it becomes clear where exactly in the proof of omniscience from **NL** the assumptions about knowledge as an **S5** modality are needed. Thus, an alternative reading of the result — and consequently an alternative solution — is made possible by our analysis: that knowledge, as **S5** operator, is somehow inadequate if we aim at formalizing a dynamic notion of knowledge. The presence of principles connecting knowledge and action forces us to revise our notion of knowledge. If the properties of knowledge operator are weakened, the price one has to pay is to loose the correspondence between Kripke frames and Aumann structures. Since in Aumann structures knowledge is represented by a partition on a set of state, we need to consider the equivalence class of the accessibility relations in the corresponding Kripke frames. This is possible only when the R_i's are equivalence relations. Nevertheless, the set-theoretic approach by Aumann was specifically designed for dealing with common knowledge. Since we do not take into consideration this operator here, the loss of the correspondence with Aumann structures is not conceptually problematic.

As it concerns the future developments of our work, a first idea could be to "inferentialize" the information test of [25]. Notice that the semantics of $R_¿$ follows the regular scheme and therefore the method of conversion of axioms into rules applies. Thus, we can provide a proof-theoretic analysis of the solution proposed in [25].

Another possible direction is to consider other alternative logics underlying **EPDL**. As we have mentioned above, the study of game-like situations reflects a verificationist account of truth and knowledge. Since the logical framework usually accepted by this kind of account is intuitionistic logic, it could be interesting to study an intuitionistic version of **EDPL**. In particular, it seems to us that intuitionistic **EPDL** could shed some lights about the derivability of omniscience using **PR** and **RA**.

References

[1] Aumann, R., "Agreeing to disagree", *Annals of Statistics*, 4 (1976): 1236–1239. DOI: 10.1214/aos/1176343654

[2] Bull, R. A., "Cut elimination for propositional dynamic logic without $*$", *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 38 (1992): 85–100. DOI: 10.1002/malq.19920380107

[3] Dowek, G., "About folding-unfolding cuts and cuts modulo", *Journal of Logic and Computation*, 11 (2001): 419–429. DOI: 10.1093/logcom/11.3.419

[4] Dummett, M., "What is a theory of meaning? (II)", pages 67–137 in *Truth and Meaning. Essays in semantics*, G. Evans and J. McDowell (eds.), Oxford: Clarendon Press, 1976. DOI: 10.1093/acprof:oso/9780199207275.003.0005

[5] Dummett, M., *The Logical Basis of Metaphysics*, London: Duckworth, 1991.

[6] Fagin, R., J. Y. Halpern, Y. Moses, and M. Y. Vardi, *Reasoning about Knowledge*, Cambridge (USA): MIT Press, 1995.

[7] Fitting, M., "Reasoning about games", *Studia Logica*, 99 (2011): 143–169. DOI: 10.1007/s11225-011-9358-7

[8] Gentzen, G., "New version of the consistency proof for elementary number theory", pages 252–286 in *The Collected Papers of Gerhard Gentzen*, M. Szabo (ed.), Amsterdam: North-Holland, 1969.

[9] Girard, J.-Y., *Proof Theory and Logical Complexity* (Vol. 1), Naples: Bibliopolis, 1987.

[10] Goldblatt, R., *Logic of Time and Computation* (2nd ed.), Stanford: CSLI Publications, 1992.

[11] Gödel, K., "An interpretation of the intuitionistic propositional calculus", pages 296–302 in *Collected Works* (Vol. 3), S. Feferman *et al.* (eds.), Oxford: Oxford University Press, 2001.

[12] Hakli, R., and S. Negri, "Proof theory for distributed knowledge", pages 100–116 in *Computational Logic in Multi-Agent Systems 8th International Workshop, CLIMA VIII, Porto, Portugal, September 10-11, 2007*, F. Sadri and K. Satoh (eds.), Lecture Notes in Artificial Intelligence (Vol. 5405), Berlin-Heidelberg: Springer, 2008. DOI: 10.1007/978-3-540-88833-8_6

[13] Harel, D., D. Kozen, and J. Tiuryn, *Dynamic Logic*, Cambridge, MA: MIT Press, 2000.

[14] Hill, B., and F. Poggiolesi, "A contraction-free and cut-free sequent calculus for propositional dynamic logic", *Studia Logica*, 94 (2010): 47–72. DOI: 10.1007/s11225-010-9224-z

[15] Hintikka, J., *Knowledge and Belief. An introduction to the logic of the two notions*, Ithaca & London: Cornell University Press, 1962.

[16] Maffezioli, P., A. Naibo, and S. Negri, "The Church-Fitch knowability paradox in the light of structural proof theory", *Synthese* (2012). DOI: 10.1007/s11229-012-0061-7

[17] Martin-Löf, P., "Verificationism then and now", pages 187–196 in *The Foundational Debate. Complexity and constructivity in mathematics and physics*, W. DePauli-Schimanovich *et al.* (eds.), Dordrecht: Kluwer, 1995. DOI: 10.1007/978-94-017-3327-4_14

[18] Negri, S., "Contraction-free sequent calculi for geometric theories, with an application to Barr's theorem", *Archive for Mathematical Logic*, 42 (2003): 389–401. DOI: 10.1007/s001530100124

[19] Negri, S., "Proof analysis in modal logic", *Journal of Philosophical Logic*, 34 (2005): 507–544. DOI: 10.1007/s10992-005-2267-3

[20] Negri, S., and J. von Plato, *Structural Proof Theory*, Cambridge: Cambridge University Press, 2001.

[21] Negri, S., and L. von Plato, *Proof Analysis. A contribution to Hilbert's last problem*, Cambridge: Cambridge University Press, 2011.

[22] Pauly, M., and R. Parikh, "Game Logic: An overview", *Studia Logica*, 75 (2003): 165–182.

[23] Sakalauskaitė, J., "A sequent calculus for propositional dynamic logic for agents with interactions", *Lithuanian Mathematical Journal*, 45 (2005): 217–224. DOI: 10.1007/s10986-005-0025-4

[24] Salerno, J. (ed.), *New Essays on the Knowability Paradox*, Oxford: Oxford University Press, (2009).

[25] Schmidt, R., and D. Tishkovsky, "On combination of propositional dynamic logic and doxastic modal logics", *Journal of Logic, Language and Information*, 17 (2008): 109–129. DOI: 10.1007/s10849-007-9041-6
[26] Sørensen, M. H., and P. Urzyczyn, *Lectures on the Curry-Howard Isomorphism*, Amsterdam: North-Holland, 2006.
[27] Troelstra, A. S., *Lectures on Linear Logic*, Stanford: CSLI Publications, 1992.
[28] Troelstra, A. S., and H. Schwichtenberg, *Basic Proof Theory* (2nd ed.), Cambridge: Cambridge University Press, 2000.
[29] Wittgenstein, L., *Philosophical Investigations*, Oxford: Blackwell, 1953.

PAOLO MAFFEZIOLI
Faculty of Philosophy
University of Groningen
Oude Boteringestraat 52
9712 GL Groningen, the Netherlands
P.Maffezioli@rug.nl

ALBERTO NAIBO
Department of Philosophy
University of Paris 1 Panthéon-Sorbonne
17, rue de la Sorbonne
75005 Paris, France
alberto.naibo@univ-paris1.fr