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DECISION PROCEDURES FOR SOME STRONG HYBRID LOGICS

Abstract. Hybrid logics are extensions of standard modal logics, which significantly increase the expressive power of the latter. Since most of hybrid logics are known to be decidable, decision procedures for them is a widely investigated field of research. So far, several tableau calculi for hybrid logics have been presented in the literature. In this paper we introduce a sound, complete and terminating tableau calculus $\mathcal{T}_{\mathcal{H}(@,\mathsf{E},\mathsf{D},\Diamond^-)}$ for hybrid logics with the satisfaction operators, the universal modality, the difference modality and the inverse modality as well as the corresponding sequent calculus $\mathcal{S}_{\mathcal{H}(@,\mathsf{E},\mathsf{D},\Diamond^-)}$. They not only uniformly cover relatively wide range of various hybrid logics but they are also conceptually simple and enable effective search for a minimal model for a satisfiable formula. The main novelty is the exploitation of the *unrestricted blocking* mechanism introduced as an explicit, sound tableau rule.

Keywords: hybrid logics; modal logics; tableau calculi; sequent calculi; decision procedures; automated reasoning

1. Introduction

Hybrid logics are powerful extensions of modal logics which allow referring to particular states of a model without using meta-language. In order to achieve it, the language of standard modal logics is enriched with the countably infinite set of propositional expressions called *nominals* (we fix the notation $\text{NOM} = \{i, j, k, \dots\}$ to stand for the set of nominals), disjoint from the set of propositional variables PROP. Each nominal is true at exactly one world and therefore can serve both as a

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label and as a formula. Supplying a language with nominals significantly strengthens its expressive power. In the presented paper we also consider further modifications of hybrid logic obtained by adding the so-called satisfaction operators, the universal modality, the difference modality and the inverse modality. The satisfaction operators of the form $@_i$ allow stating that a particular formula holds at a world labelled by i. The universal modality E expresses the fact that there exists a world in a domain, at which a particular formula holds. The difference modality D stands for the fact that a particular formula holds at a world different from the current one. Eventually, the inverse modality allows us "jump back" to a predecessor-world along the accessibility relation.

Some hybrid logics additionally contain a different sort of expressions, the *state variables*, which allow quantifying over worlds, and additional operators like the *down-arrow operator* or the *state quantifiers*. However, these logics are proven to be undecidable (cf. [2]) so, in principle, they cannot be subjected to a terminating tableau-based decision procedure. We therefore confine ourselves only to the foregoing decidable hybrid logic.

In the present paper we introduce a sound, complete and terminating tableau calculus $\mathcal{T}_{\mathcal{H}(@,\mathsf{E},\mathsf{D},\Diamond^-)}$ for hybrid logics with $@$, E, D and $\Diamond^-$ operators. Our approach, unlike that in [12] and [4], is focused on the uniform treatment of all aforementioned logics, conceptual simplicity and minimality of models generated by $\mathcal{T}_{\mathcal{H}(@,\mathsf{E},\mathsf{D},\Diamond^-)}$. Basing on [19], we introduce the *unrestricted blocking* mechanism that satisfies these conditions.

In Section 2 a characterisation of the logic $\mathcal{H}(@,\mathsf{E},\mathsf{D},\Diamond^-)$ is provided. In Section 3 we introduce the tableau calculus $\mathcal{T}_{\mathcal{H}(@,\mathsf{E},\mathsf{D},\Diamond^-)}$ and we describe the decision procedure for $\mathcal{H}(@,\mathsf{E},\mathsf{D},\Diamond^-)$. In Section 4 we prove soundness and completeness of $\mathcal{T}_{\mathcal{H}(@,\mathsf{E},\mathsf{D},\Diamond^-)}$ and Section 5 introduces a sequent calculus $\mathcal{S}_{\mathcal{H}(@,\mathsf{E},\mathsf{D},\Diamond^-)}$ equivalent to $\mathcal{T}_{\mathcal{H}(@,\mathsf{E},\mathsf{D},\Diamond^-)}$. Section 6 provides a closer look at the termination problem for $\mathcal{T}_{\mathcal{H}(@,\mathsf{E},\mathsf{D},\Diamond^-)}$. We conclude the paper in Section 7.

2. Hybrid logic

Syntax. Let $\mathcal{O} \subseteq \{@,\mathsf{E},\mathsf{D},\Diamond^-\}$. By $\mathcal{H}(\mathcal{O})$ we will denote the hybrid logic with operator(s) $\mathcal{O}$.

We recursively define the set FORM of well-formed formulas of the logic $\mathcal{H}(@, \mathsf{E}, \mathsf{D}, \Diamond^-)$ in the following manner:

$$\text{FORM} \ni \varphi := p \mid i \mid \neg\psi \mid \psi \wedge \chi \mid \Diamond\psi \mid @_i\psi \mid \mathsf{E}\psi \mid \mathsf{D}\psi \mid \Diamond^-\psi,$$

where $p \in \text{PROP}$, $i \in \text{NOM}$ and $\psi, \chi \in \text{FORM}$.

Other connectives and operators are defined in a standard way. All E, D and $\Diamond^-$ have dual operators. @ is self-dual. We abbreviate $\neg\mathsf{E}\neg$ as A.

Semantics. A model $\mathfrak{M}$ for hybrid logic $\mathcal{H}(@, \mathsf{E}, \mathsf{D}, \Diamond^-)$ is a triple $\langle W, R, V\rangle$ where:

$W \neq \emptyset$ is called a *domain*,
$R \subseteq W^2$ is called an *accessibility relation*,
$V : \text{PROP} \cup \text{NOM} \longrightarrow \mathcal{P}(W)$ such that for each $i \in \text{NOM}$ $V(i)$ is a singleton set; V is called a *valuation function*.

Relation $\models$ (*forcing*) is defined inductively:

$$\begin{aligned}
\mathfrak{M}, w \models p &\iff w \in V(p), \quad p \in \text{PROP},\\
\mathfrak{M}, w \models i &\iff \{w\} = V(i), \quad i \in \text{NOM},\\
\mathfrak{M}, w \models \neg\varphi &\iff \mathfrak{M}, w \not\models \varphi,\\
\mathfrak{M}, w \models \varphi \wedge \psi &\iff \mathfrak{M}, w \models \varphi \text{ and } \mathfrak{M}, w \models \psi,\\
\mathfrak{M}, w \models \Diamond\varphi &\iff \text{there exists } z \in W \text{ such that } wRz \text{ and } \mathfrak{M}, z \models \varphi,\\
\mathfrak{M}, w \models @_i\varphi &\iff \{z\} = v(i) \text{ and } \mathfrak{M}, z \models \varphi,\\
\mathfrak{M}, w \models \mathsf{E}\varphi &\iff \text{there exists } z \in W \text{ such that } \mathfrak{M}, z \models \varphi,\\
\mathfrak{M}, w \models \mathsf{D}\varphi &\iff \text{there exists } z \in W \text{ such that } z \neq w \text{ and } \mathfrak{M}, z \models \varphi,\\
\mathfrak{M}, w \models \Diamond^-\varphi &\iff \text{there exists } z \in W \text{ such that } zRw \text{ and } \mathfrak{M}, z \models \varphi.
\end{aligned}$$

3. Tableau calculus for the logic $\mathcal{H}(@, \mathsf{E}, \mathsf{D}, \Diamond^-)$

Tableau calculi. Two main types of tableau calculi for hybrid logics are present in the literature, namely the *prefixed* and the *internalised* calculi. The prefixed calculi consist in introducing another sort of expressions, namely *prefixes*. They serve as labels for worlds, which, unlike nominals, are of meta-linguistic provenience. Another type of meta-language expressions occurring in prefixed tableaux are the *accessibility expressions*.

The equality between two prefixes is expressed implicitly by imposing on them the satisfaction of the same nominal. Apparently, prefixed calculi are less complex than internalised calculi. Besides, basic hybrid logic $\mathcal{H}$ is not supplied with sufficient expressive power to internalise its own semantics. It therefore requires the domain expressions occurring in the calculus. The most widely known prefixed tableau calculi for hybrid logics come from Tzakova [21], Bolander and Braüner (who improved Tzakova's calculus to the terminating version) [6], Kaminski and Smolka [14].[1] The tableau calculus for hybrid logics obtained from the synthesised framework from [19] is also subsumed under the prefixed calculi class.

Internalised calculi for hybrid logics take advantage of the high expressive power of these logics which allows encoding the domain expressions within the language. Although internalisation of the logic allows dispensing with certain rules present in prefixed tableau calculi, it also jeopardises termination of the calculus by, e.g., using pure axioms (not including other formulas but nominals) to characterise frame conditions (cf. [4]).

In this section we present an internalised tableau calculus covering hybrid logics with the satisfaction operators, the universal modality, the difference modality and the inverse modality. It resembles Blackburn's calculus from [3] modified by Bolander and Braüner in [6] and by Blackburn and Bolander in [4]. However, certain rules have been added (e.g. the rules for D).

Encoding the domain expressions. In [3] Blackburn made an observation that the language of hybrid logic with @ operators is sufficiently rich to express semantics within itself. As we mentioned in Section 2, there are three types of the domain expressions: satisfaction statements ($\mathfrak{M}, w \models \varphi$), accessibility statements (wRv) and equality statements ($w = v$). Hybrid equivalents of the foregoing expressions are shown below.

$$\begin{array}{ll} \mathcal{H}(\mathfrak{M}, w \models \varphi) = @_{i_w}\varphi & \mathcal{H}(\mathfrak{M}, w \not\models \varphi) = @_{i_w}\neg\varphi \\ \mathcal{H}(R(w,v)) = @_{i_w}\Diamond j_v & \mathcal{H}(\neg R(w,v)) = @_{i_w}\neg\Diamond j_v \\ \mathcal{H}(w = v) = @_{i_w} j_v & \mathcal{H}(w \neq v) = @_{i_x}\neg j_y \end{array}$$

[1] In the case of Kaminski and Smolka's system from [14] one could put in question its prefixed character since it is prefixed by expressions called *nominals* by the authors. However, apart from the terminology there is no trace of internalisation in this calculus. Nominals occurring within formulas are handled in a different fashion than nominals labelling formulas, namely by using the $\{\cdot\}$ operator.

Rules for the connectives:

$$(\neg)\ \frac{i : \neg j}{j : j} \qquad (\neg\neg)\ \frac{i : \neg\neg\varphi}{i : \varphi} \qquad (\wedge)\ \frac{i : \varphi \wedge \psi}{i : \varphi, i : \psi} \qquad (\neg\wedge)\ \frac{i : \neg(\varphi \wedge \psi)}{i : \neg\varphi \mid i : \neg\psi}$$

$$(\Diamond)^*\ \frac{i : \Diamond\varphi}{i : \Diamond j, j : \varphi} \qquad (\neg\Diamond)\ \frac{i : \neg\Diamond\varphi, i : \Diamond j}{j : \neg\varphi}$$

$$(@)\ \frac{i : @_j\varphi}{j : \varphi} \qquad (\neg @)\ \frac{i : \neg @_j\varphi}{j : \neg\varphi}$$

$$(\mathsf{E})^*\ \frac{i : \mathsf{E}\varphi}{j : \varphi} \qquad (\neg\mathsf{E})\ \frac{i : \neg\mathsf{E}\varphi, j : j}{j : \neg\varphi}$$

$$(\mathsf{D})^*\ \frac{i : \mathsf{D}\varphi}{i : \neg j, j : \varphi} \qquad (\neg\mathsf{D})\ \frac{i : \neg\mathsf{D}\varphi, i : \neg j}{j : \neg\varphi}$$

$$(\Diamond^-)^*\ \frac{i : \Diamond^-\varphi}{j : \Diamond i, j : \varphi} \qquad (\neg\Diamond^-)\ \frac{i : \neg\Diamond^-\varphi, j : \Diamond i}{j : \neg\varphi}$$

Equality rules:

$$(\mathsf{ref})\ \frac{i : \varphi}{i : i} \qquad (\mathsf{sub})\ \frac{i : j, i : \varphi}{j : \varphi}$$

Closure rule and unrestricted blocking rule:

$$(\bot)\ \frac{i : \varphi, i : \neg\varphi}{\bot} \qquad (\mathsf{ub})\ \frac{i : i, j : j}{i : j \mid i : \neg j}$$

* Nominals in the conclusions are fresh on the branch.

Figure 1. Rules for the calculus $\mathcal{T}_{\mathcal{H}(@,\mathsf{E},\mathsf{D},\Diamond^-)}$

Both E and D operators allow mimicking @ operators: $@_i\varphi := \mathsf{E}(i \wedge \varphi)$ and $@_i\varphi := (i \wedge \varphi) \vee \mathsf{D}(i \wedge \varphi)$. Therefore, in the calculus we use the notation $i : \varphi$, rather than $@_i\varphi$, to keep its universal character. This *colon* notation will stand for one of the foregoing expressions, depending on a considered logic[2], except for the fact that whenever a logic includes @ operators, $i : \varphi$ means $@_i\varphi$.

Tableau calculus. Figure 1 presents the rules of the tableau calculus $\mathcal{T}_{\mathcal{H}(@,\mathsf{E},\mathsf{D},\Diamond^-)}$ for the logic $\mathcal{H}(@, \mathsf{E}, \mathsf{D}, \Diamond^-)$.

[2] Basically, it means that we choose the representation of the *colon* sign that we have at our disposal in a considered logic. E.g. if we consider the logic $\mathcal{H}(\mathsf{E})$, $i : \varphi$ will stand for $\mathsf{E}(i \wedge \varphi)$

Boolean rules are straightforward and require no additional comments. ($\Diamond$), (E), (D) and ($\Diamond^-$) are rules introducing new labels, which was marked as the side-condition for them. In the case of ($\neg\mathsf{E}$) and ($\neg\mathsf{D}$) the standard side-condition of former occurrence of a label on a branch was replaced by introducing an explicit premiss stating that a particular nominal has appeared as a label on a branch. The rule (ref) is a reflexivity rule that introduces to a branch the explicit information that a nominal occurred as a label within a branch. (sub) expresses the substitutability of two nominals as labels, provided that one of them is labelled by the other. The ($\bot$) rule is self-evident.

The (ub) rule is a special variant of the *analytical cut rule* which was sucessfully applied to reduce a branching factor and, in consequence, the size of proofs in tableau-like calculus KE (cf. [1]). While applying full analytic cut we can divide any branch with respect to a subformula and its negation of any formula already occuring on the branch. (ub) is more restrictive since it is applied only to nominals. Intuitively, if two labels appear on a branch, they either label two distinct worlds or the same world. Thus, (ub) allows comparing any pair of labels that appeared on a branch. As it will turn out before long, this possibility is essential for termination of the whole calculus. The main rationale behind introduction of (ub) is connected with termination problems. It is an explicit rule-realization of unrestricted blocking strategy. However, it appears that introduction of this rule leads also to some other advantages connected with the form of rules for D and reduction of branching factor.

It is worth noticing that both rules for D are nonbranching in contrast to other proposals. In particular, tableau rules defined for D in the context of ordinary modal logic (i.e. with no nominals) are quite complicated and lead to multiple branching dependent on the number of prefixes already occurring on the branch (see [10] or [11]). In hybrid logics the possibility of expressing inequality of nominals simplifies matters greatly and the rule (D) is nonbranching in the effect. But still suitable rule for negated D is branching in most systems (e.g. [17], [13], [14]); we can formulate it as follows:

$$\frac{i : \neg\mathsf{D}\varphi, j : j}{i : j \mid j : \neg\varphi}, \tag{$*$}$$

In [4] Blackburn and Bolander noticed that the ($\neg\mathsf{D}$) rule in the form present in our calculus breaks the completeness of the whole calculus but they did not explain why branching rule does not violate the completeness of the calculus, whereas version without branching conclusions does. In [19] Schmidt and Tishkovsky introduced the (†) condition which determines whether we can decrease the branching factor of a rule by moving some of the conclusions to the premises:

THEOREM 1 ([19]). *Let $\mathcal{T}$ be a tableau calculus. Let β be the rule of the form:*

$$\frac{X_0}{X_1 \mid \cdots \mid X_n}$$

and let $\mathcal{T}^R$ be a refined version of $\mathcal{T}$, including the rule β^R of the form:

$$\frac{X_0, \neg X_1}{X_2 \mid \cdots \mid X_n}.$$

Suppose that $\mathcal{B}^R$ is an arbitrary open and fully-expanded branch in a $\mathcal{T}^R$-tableau. Let $F = \{\varphi_1, \ldots, \varphi_l\}$ be a set of all formulas from $\mathcal{B}^R$ reflected in $\mathfrak{M}(\mathcal{B}^R)$. Then the refined rule β^R is admissible (i.e. the resulting calculus $\mathcal{T}^R$ is still complete) if the following condition is satisfied:

$$\begin{array}{l} \textit{If } X_0(\varphi_{i_1}, \ldots, \varphi_{i_k}) \in \mathcal{B}^R \\ \textit{then } \mathfrak{M}(\mathcal{B}^R) \models X_m(\varphi_{i_1}, \ldots, \varphi_{i_k}), \textit{ for some } m \in \{1, \ldots, n\}, \end{array} \tag{†}$$

where $X_i(\varphi_{j_1}, \ldots, \varphi_{j_l})$ is an instantiation of a formula X_i, involving formulas $\varphi_{j_1}, \ldots, \varphi_{j_l}$.

In most modal and description logics (†) holds for $(\neg\Diamond)$ but, as it turns out, it fails for $(\neg\mathsf{D})$. Usually, to make such refinements possible without breaking completeness, we need the *analytical cut* rule of the form:

$$\frac{i : \varphi, j : j}{j : \varphi \mid j : \neg\varphi},$$

which significantly increases the branching factor of the whole calculus. However, it appears that in the case of $(\neg\mathsf{D})$ rule we only need some restricted form of the analytical cut for nominals:

$$\frac{i : i, j : j}{i : j \mid i : \neg j},$$

which is present in $\mathcal{T}_{\mathcal{H}(@,\mathsf{E},\mathsf{D},\Diamond^-)}$ under the name of (ub). It turns out that thanks to the tool that was introduced for ensuring termination, we

1.	$i : \neg\mathsf{D}\varphi$		(Assumption)
2.	$j : j$		(Assumption)
3.	$i : i$		((ref): 1)
4.	$i : j$	$i : \neg j$	((ub): 2,3)
5.		$j : \neg\varphi$	(($\neg$D): 1,4)

Figure 2. Derivation of the rule ($*$) using the rules ($\neg$D) and (ub)

obtain the refined version of ($\neg$D) rule for free! Taking the rules ($\neg$D) and (ub) as primitive, we can derive the rule ($*$) (see Fig. 2).

Before we provide a proper method of constructing a tableau, we need to introduce several preliminary definitions.

DEFINITION 1. We call a branch of a $\mathcal{T}_{\mathcal{H}(@,\mathsf{E},\mathsf{D},\Diamond^-)}$ tableau *closed* if the closure rule was applied on it. If a branch is not closed, it is open. An open branch is *fully expanded* if no other rules are applicable on it.

DEFINITION 2. Let $\textsc{nom}(\mathcal{B})$ be a set of nominals occurring as labels on a fully expanded branch $\mathcal{B}$ of a $\mathcal{T}_{\mathcal{H}(@,\mathsf{E},\mathsf{D},\Diamond^-)}$ tableau for a given input formula. We introduce the $\approx_{\mathcal{B}}$ relation over $\textsc{nom}(\mathcal{B})$ which we define in the following way:

$$i \approx_{\mathcal{B}} j \text{ iff } i : j \in \mathcal{B}.$$

PROPOSITION 1. *$\approx_{\mathcal{B}}$ is the equivalence relation.*

PROOF. *Reflexivity* is ensured by the (ref) rule. For *symmetry* assume that $i : j$ is on $\mathcal{B}$. By (ref) we obtain $i : i$ and after applying (sub) to these two premises we obtain $j : i$. For *transitivity* suppose that $i : j$ and $j : k$ are on $\mathcal{B}$. By symmetry we have that $j : i$ is also on $\mathcal{B}$. We therefore take $j : i$ and $j : k$ as premises of (sub) and obtain $i : k$. ⊣

DEFINITION 3. A rule of the $\mathcal{T}_{\mathcal{H}(@,\mathsf{E},\mathsf{D},\Diamond^-)}$ is applied *eagerly* in a tableau iff whenever it is applicable, it is applied.

DEFINITION 4. Let $\prec_{\mathcal{B}}$ be an ordering on $\textsc{nom}(\mathcal{B})$ defined as follows:

$$i \prec_{\mathcal{B}} j \text{ iff } i : i \text{ occurred on } \mathcal{B} \text{ earlier than } j : j.$$

Note that $\prec_{\mathcal{B}}$ is well-founded and linear since no rule introduces more than one labelling nominal as a conclusion.

DEFINITION 5. To each $\mathcal{T}_{\mathcal{H}(@,\mathsf{E},\mathsf{D},\Diamond^-)}$ rule we affix the *priority number*. It indicates what the order of application of particular rules should be. The lower the number is, the sooner the rule should be applied. We have: (ref), ($\neg$): 1, (ub), ($\bot$): 2, (sub):3, ($\neg\neg$), ($\wedge$), ($\neg\wedge$): 4, ($\neg\Diamond$), ($\neg$E), ($\neg$D), ($\neg\Diamond^-$): 5, ($\Diamond$), (E), (D), $\Diamond^-$: 6.

Now we are ready to provide the tableau construction algorithm. As usual, we do it inductively.

DEFINITION 6 (Tableau construction algorithm). **Basic step**: For a given input formula φ put $i : \neg\varphi$ at the initial node. i is a nominal not occurring in φ.
Inductive step: Suppose that you performed n steps of a derivation. In the $n + 1$th step apply the rules of $\mathcal{T}_{\mathcal{H}(@,\mathsf{E},\mathsf{D},\Diamond^-)}$ eagerly respecting the priority ordering given in Definition 5 and fulfilling the following conditions:

(c1) if the application of a rule results in formulas all of which are already present on a branch, do not perform this application;
(c2) rules of priority 5 and 6 can only by applied to labels that are the least elements (with respect to $\prec_{\mathcal{B}}$) of the equivalence class (with respect to $\approx_{\mathcal{B}}$);
(c3) the ($\Diamond$) must not be applied to formulas of the form $i : \Diamond j$. We call them the *accessibility formulas*.

If after the $n + 1$th step of derivation:

(a) all tableau branches are closed, stop and return: *theorem*,
(b) there are open branches in a tableau and no further rules are applicable (respecting conditions (c1)–(c3)), stop and return: *non-theorem*;
(c) there are open branches in a tableau and further rules are applicable (respecting conditions (c1)–(c3)), proceed to the $n + 2$th step.

We will explain the way the (ub) rule works more carefully in Section 6.

4. Soundness and Completeness of $\mathcal{T}_{\mathcal{H}(@,\mathsf{E},\mathsf{D},\Diamond^-)}$

In the current section we state and prove soundness and completeness of the foregoing calculus. First, we formulate the following

Definition 7. We call a tableau calculus $\mathcal{T}$ *sound* if and only if for each satisfiable input formula φ each tableau $\mathcal{T}(\varphi)$ is open, i.e., there exists a fully expanded branch on which no closure rule was applied. A tableau calculus is called *complete* if and only if for each unsatisfiable input formula φ there exists a closed tableau, i.e. a tableau where a closure rule was applied on each branch.

For soundness it amounts to proving that particular rules preserve satisfiability. For completeness we take the contrapositive of the condition given in Definition 7 and demonstrate that if there exists an open, fully expanded branch $\mathcal{B}$ in a tableau for φ then there exists a model for φ.

Theorem 2. *$\mathcal{T}_{\mathcal{H}(@,\mathsf{E},\mathsf{D},\Diamond^-)}$ is sound.*

Proof. By easy verification of all the rules. $\dashv$

Suppose that $\mathcal{B}$ is an open, fully expanded branch in a $\mathcal{T}_{\mathcal{H}(@,\mathsf{E},\mathsf{D},\Diamond^-)}$ tableau for φ. We define a model $\mathfrak{M}(\mathcal{B}) = \langle W, R, V\rangle$ derived from $\mathcal{B}$ in the following way:

$W = \{[i]_{\approx_\mathcal{B}} \mid i : i \in \mathcal{B}\}$;
$R = \{([i]_{\approx_\mathcal{B}}, [j]_{\approx_\mathcal{B}}) \mid i : \Diamond j :\in \mathcal{B}\}$;
$V = \{(i, [i]_{\approx_\mathcal{B}}) \mid i : i \in \mathcal{B}\} \cup \{(p, U) \mid p \in \textsc{prop}$, p occurred in $\mathcal{B}$ and $U = \{[i]_{\approx_\mathcal{B}} \mid i : p \in \mathcal{B}\}\}$.

Lemma 1. *Suppose that $\mathcal{B}$ is an open, fully expanded branch in a $\mathcal{T}_{\mathcal{H}(@,\mathsf{E},\mathsf{D},\Diamond^-)}$ tableau for φ. Then if $i : \psi \in \mathcal{B}$ then $\mathfrak{M}(\mathcal{B}), [i]_{\approx_\mathcal{B}} \models \psi$.*

Proof. By induction on the complexity of ψ. Since all cases save $\psi = \mathsf{D}\chi$ and $\psi = \neg\mathsf{D}\chi$ are covered by proofs given in [4] and [6] (in a little more complicated form involving the notion of *urfather* instead of the ordinary equivalence relation $\approx$), we only consider missing cases.

Case: $\psi = \mathsf{D}\chi$. We have $i : \mathsf{D}\chi \in \mathcal{B}$. After applying (D) we obtain $i : \neg j \in \mathcal{B}$ and $j : \chi \in \mathcal{B}$. By the inductive hypothesis we have that $\mathfrak{M}(\mathcal{B}), [j]_{\approx_\mathcal{B}} \models \chi$. It suffices to show that $[i]_{\approx_\mathcal{B}}$ and $[j]_{\approx_\mathcal{B}}$ are distinct. Suppose that they are the same equivalence class. But then, by Def. 2, $i : j \in \mathcal{B}$, which contradicts the fact that $\mathcal{B}$ is open.

Case: $\psi = \neg\mathsf{D}\chi$. We have $i : \neg\mathsf{D}\chi \in \mathcal{B}$. Let $\textsc{nom}(\mathcal{B})$ be the set of labels that appeared on $\mathcal{B}$. Since the (ub) rule is applied eagerly, for each label $j_k \in \textsc{nom}(\mathcal{B})$ either $i : j_k \in \mathcal{B}$ or $i : \neg j_k \in \mathcal{B}$. If for all labels j_k from $\textsc{nom}(\mathcal{B})$ we have $i : j_k \in \mathcal{B}$, it means that $W = \{[i]_{\approx_\mathcal{B}}\}$

and therefore $\mathfrak{M}(\mathcal{B}), [i]_{\approx_\mathcal{B}} \models \neg \mathsf{D}\chi$ trivially holds. Suppose that there exists a label $j_l \in \mathcal{L}$ such that $i : \neg j_l \in \mathcal{B}$. Then, after applying $(\neg\mathsf{D})$ to $i : \neg\mathsf{D}\chi$ and $i : \neg j_l$, we obtain that $j_l : \neg\chi \in \mathcal{B}$. By the inductive hypothesis, $\mathfrak{M}(\mathcal{B}), [j_l]_{\approx_\mathcal{B}} \models \neg\chi$ and $[i]_{\approx_\mathcal{B}} \neq [j_k]_{\approx_\mathcal{B}}$. Since j_l was picked arbitrarily, we obtain the conclusion. ⊣

Theorem 3. *$\mathcal{T}_{\mathcal{H}(@,\mathsf{E},\mathsf{D},\Diamond^-)}$ is complete.*

Proof. By Definition 7 and Lemma 1. ⊣

5. Sequent Calculus $\mathcal{S}_{\mathcal{H}(@,\mathsf{E},\mathsf{D},\Diamond^-)}$

Before we go to termination matters, we develop the sequent calculus corresponding strictly to tableau calculus presented above. We use sequents built from finite sets of formulas which allow to avoid structural rules and to keep strong resemblance to the rules of the tableau calculus (see Fig. 3).

The special rule for unrestricted blocking has the form:

$$(\mathsf{ub})\quad \frac{i : i, j : j, \Gamma \Rightarrow \Delta, i : j \mid i : j, i : i, j : j, \Gamma \Rightarrow \Delta}{i : i, j : j, \Gamma \Rightarrow \Delta}$$

The proof is defined in a standard way as a tree of sequents where each leaf is labelled with an instance of an axiom and all edges are obtained by means of specified rules.

Remark 1. All the rules are in one-to-one correspondence to tableau rules, except two rules for negation. Here instead of one tableau rule $(\neg\neg)$ we have two rules; on the other hand, the effect of tableau rule $(\neg)$ is covered by the more general formulation of the rule (ref). We explicitly introduced $j : j$ into antecedents of schemata of $(\Rightarrow\mathsf{E})$, $(\Rightarrow\mathsf{D})$ and (ub) in order to closely follow the formulation of $(\neg\mathsf{E})$, $(\neg\mathsf{D})$, (ub). Clearly, one can formulate this demand as a side condition for both rules and thus avoid their presence.

Remark 2. Duplication of main formulas in premises of (sub), $(\Rightarrow\mathsf{D})$, $(\Rightarrow\mathsf{E})$, $(\Rightarrow\Diamond)$ and $(\Rightarrow\Diamond^-)$ (as well as $j : j$ in $(\Rightarrow\mathsf{D})$ and $(\Rightarrow\mathsf{E})$) is necessary for keeping completeness due to lack of contraction and using sets (not multisets) of formulas in sequents.

(ax) $\Gamma \Rightarrow \Delta$, where $\Gamma \cap \Delta \neq \emptyset$

$(\neg\Rightarrow)$ $\dfrac{\Gamma \Rightarrow \Delta, i : \varphi}{i : \neg\varphi, \Gamma \Rightarrow \Delta}$ $\qquad$ $(\Rightarrow\neg)$ $\dfrac{i : \varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, i : \neg\varphi}$

$(\wedge\Rightarrow)$ $\dfrac{i : \varphi, i : \psi, \Gamma \Rightarrow \Delta}{i : \varphi \wedge \psi, \Gamma \Rightarrow \Delta}$ $\qquad$ $(\Rightarrow\wedge)$ $\dfrac{\Gamma \Rightarrow \Delta, i : \varphi \mid \Gamma \Rightarrow \Delta, i : \psi}{\Gamma \Rightarrow \Delta, i : \varphi \wedge \psi}$

$(@\Rightarrow)$ $\dfrac{j : \varphi, \Gamma \Rightarrow \Delta}{i : @_j\varphi, \Gamma \Rightarrow \Delta}$ $\qquad$ $(\Rightarrow @)$ $\dfrac{\Gamma \Rightarrow \Delta, j : \varphi}{\Gamma \Rightarrow \Delta, i : @_j\varphi}$

$(\mathsf{ref})^2$ $\dfrac{i : i, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$ $\qquad$ (sub) $\dfrac{j : \varphi, i : j, i : \varphi, \Gamma \Rightarrow \Delta}{i : j, i : \varphi, \Gamma \Rightarrow \Delta}$

$(\Diamond\Rightarrow)^1$ $\dfrac{i : \Diamond j, j : \varphi, \Gamma \Rightarrow \Delta}{i : \Diamond\varphi, \Gamma \Rightarrow \Delta}$ $\qquad$ $(\Rightarrow\Diamond)$ $\dfrac{i : \Diamond j, \Gamma \Rightarrow \Delta, i : \Diamond\varphi, j : \varphi}{i : \Diamond j, \Gamma \Rightarrow \Delta, i : \Diamond\varphi}$

$(\Diamond^-\Rightarrow)^1$ $\dfrac{j : \Diamond i, j : \varphi, \Gamma \Rightarrow \Delta}{i : \Diamond^-\varphi, \Gamma \Rightarrow \Delta}$ $\qquad$ $(\Rightarrow\Diamond^-)$ $\dfrac{j : \Diamond i, \Gamma \Rightarrow \Delta, i : \Diamond^-\varphi, j : \varphi}{j : \Diamond i, \Gamma \Rightarrow \Delta, i : \Diamond^-\varphi}$

$(\mathsf{E}\Rightarrow)^1$ $\dfrac{j : \varphi, \Gamma \Rightarrow \Delta}{i : \mathsf{E}\varphi, \Gamma \Rightarrow \Delta}$ $\qquad$ $(\Rightarrow\mathsf{E})$ $\dfrac{j : j, \Gamma \Rightarrow \Delta, i : \mathsf{E}\varphi, j : \varphi}{j : j, \Gamma \Rightarrow \Delta, i : \mathsf{E}\varphi}$

$(\mathsf{D}\Rightarrow)^1$ $\dfrac{j : \varphi, \Gamma \Rightarrow \Delta, i : j}{i : \mathsf{D}\varphi, \Gamma \Rightarrow \Delta}$ $\qquad$ $(\Rightarrow\mathsf{D})$ $\dfrac{\Gamma \Rightarrow \Delta, i : \mathsf{D}\varphi, i : j, j : \varphi}{\Gamma \Rightarrow \Delta, i : \mathsf{D}\varphi, i : j}$

[1] where j does not occur in $\Gamma \cup \Delta \cup \{\varphi\}$;
[2] provided that i occurs in $\Gamma \cup \Delta$.

Figure 3. Rules for the sequent calculus $\mathcal{S}_{\mathcal{H}(@,\mathsf{E},\mathsf{D},\Diamond^-)}$

Completeness of this calculus follows from the completeness result established for the tableau calculus, since one can easily rewrite each tableau proof as a proof in the sequent calculus specified above.

One can easily prove in the standard way (cf. e.g. Negri, von Plato [18]) that rules of weakening in antecedent and succedent are admissible in this calculus. Moreover, admissibility of (cut) may be proven constructively, in the similar way as for Myers' and Pattinson's sequent calculus in [17] if we replace $(\Rightarrow\mathsf{D})$ with its two-premise counterpart corresponding to $(*)$:

$$\frac{j : j, \Gamma \Rightarrow \Delta, i : \mathsf{D}\varphi, j : \varphi \mid i : j, j : j, \Gamma \Rightarrow \Delta, i : \mathsf{D}\varphi}{j : j, \Gamma \Rightarrow \Delta, i : \mathsf{D}\varphi}$$

In fact, from the standpoint of syntactical purity of the rules, normally required from sequent calculi, such a rule is better. Well-behaved rules of sequent calculi should have only (one) occurrence of the main formula in conclusion-sequent and this requirement is not satisfied by our calculus since in some rules we have also explicit occurences of some additional formulas. In order to avoid this situation one can delete occurences of $j : j$ in the antecedents of ($\Rightarrow$E), ($\Rightarrow$D) and (ub) in favour of suitable side condition, and replace ($\Rightarrow\Diamond$) and ($\Rightarrow \Diamond^-$) with two-premise variants of the form:

$$\frac{\Gamma \Rightarrow \Delta, i : \Diamond\varphi, i : \Diamond j \mid \Gamma \Rightarrow \Delta, i : \Diamond\varphi, j : \varphi}{\Gamma \Rightarrow \Delta, i : \Diamond\varphi}$$

In this way we obtain a calculus with bigger branching factor but with rules which better fit to ordinary shape of sequent calculus rules and with no need of (cut) (including (ub)). But from the standpoint of simpler proof-trees the reduction of branching factor is essential and we can move in the other direction as well. One can introduce the full analytic (cut) as the only branching rule (which covers (ub) as a special case) and get rid with all other branching rules. In this case we additionaly replace ($\Rightarrow\wedge$) with its one-premise equivalents:

$$\frac{i : \varphi, \Gamma \Rightarrow \Delta, i : \psi}{i : \varphi, \Gamma \Rightarrow \Delta, i : \varphi \wedge \psi} \qquad \text{or} \qquad \frac{i : \psi, \Gamma \Rightarrow \Delta, i : \varphi}{i : \psi, \Gamma \Rightarrow \Delta, i : \varphi \wedge \psi}$$

Such a system is a sequent calculus counterpart of Mondadori and D'Agostini [1] KE system. In such a system (ub), and generally, analytical (cut), is not eliminable but such a system still provides decision procedure, and moreover, the length of proofs is essentially smaller (see [1]).

6. Termination of $\mathcal{T}_{\mathcal{H}(@,\mathsf{E},\mathsf{D},\Diamond^-)}$

Exploiting the (ub) rule and the conditions (c1)–(c3) we show that $\mathcal{T}_{\mathcal{H}(@,\mathsf{E},\mathsf{D},\Diamond^-)}$ is terminating for the logic $\mathcal{H}(@, \mathsf{E}, \mathsf{D}, \Diamond^-)$, provided that it has the *finite model property* for a certain class of frames.

First, we make a remark that will be useful afterwards (cf. [20]).

Remark 3. For each $[i]_{\approx_{\mathcal{B}}}$ the number of applications of the rules introducing a new label, namely ($\Diamond$), (E), (D), ($\Diamond^-$) to members of $[i]_{\approx_{\mathcal{B}}}$ is finite.

Proof. Indeed, if the (ub) is eagerly applied and the conditions (c2) and (c3) are fulfilled, it ensures that no superfluous application of ($\Diamond$),

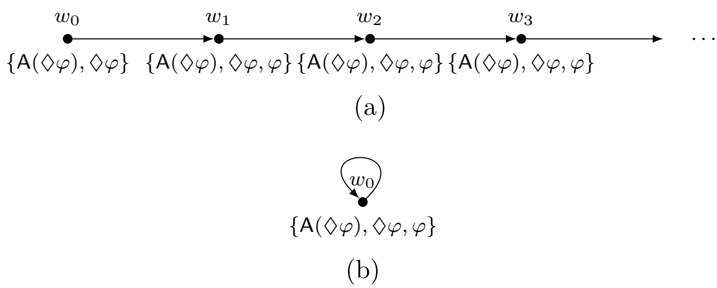

Figure 4. (a) and (b) present, respectively, an infinite and a finite (minimal) model (not a tableau) for the formula $\mathsf{A}(\Diamond\varphi)$. Both of them can be obtained from a tableau if the (ub) rule is involved, since it allows merging worlds in an arbitrary way, provided that the consistency is preserved.

(E), (D), $(\Diamond^-)$ is performed, since they are only applied to one member of $[i]_{\approx_\mathcal{B}}$ and are not applied to accessibility formulas (otherwise it would lead to an infinite derivation that could not be subjected to blocking). Since the input formula φ is assumed to be finite, therefore for each i that occurred in $\mathcal{B}$, the number of $(\Diamond)$, (E), (D), $(\Diamond^-)$ applications to $[i]_{\approx_\mathcal{B}}$ is finite. $\dashv$

COROLLARY 1. *For each $\mathcal{T}_{\mathcal{H}(@,\mathsf{E},\mathsf{D},\Diamond^-)}$ tableau branch $\mathcal{B}$ is finite iff W of $\mathfrak{M}(\mathcal{B})$ is finite.*

Now we are ready to state the lemma that is essential for termination of $\mathcal{T}_{\mathcal{H}(@,\mathsf{E},\mathsf{D},\Diamond^-)}$. However, before we do this, we explain informally how the (ub) rule works. Our tableau calculus by default handles all distinct nominals that were introduced to a branch as labelling distinct worlds. It leads to a situation where a satisfiable formula having a simple model generates an infinite tableau (see Fig. 4). The (ub) rule compares all labels that occurred in a branch and its left conclusion merges each pair unless it leads to the inconsistency. As a consequence, if a formula has a model $\mathfrak{M}$ of a certain cardinality, it will be reflected by a finite, fully expanded open branch of a $\mathcal{T}_{\mathcal{H}(@,\mathsf{E},\mathsf{D},\Diamond^-)}$ tableau. The reason is that the left conclusion of the (ub) rule decreases the cardinality of a model whenever possible, so a model of the cardinality not-greater than the cardinality of $\mathfrak{M}$ will eventually be obtainable from one of the branches of a tableau. The formal argument is presented in the following lemma.

LEMMA 2. *Suppose that a finite model $\mathfrak{N} = \langle W', R', V' \rangle$ satisfies a formula φ. Then there exists an open branch $\mathcal{B}$ in a $\mathcal{T}_{\mathcal{H}(@,\mathsf{E},\mathsf{D},\Diamond^-)}$ tableau and $\mathfrak{M}(\mathcal{B}) = \langle W, R, V \rangle$ such that $\mathsf{Card}(W) \leq \mathsf{Card}(W')$.*

PROOF. We proceed by induction on the number of steps in the derivation. During the derivation we construct a branch $\mathcal{B}$ in such a way that $\mathfrak{M}(\mathcal{B})$ is partially isomorphic to $\mathfrak{N}$ (cf. [20]).

Basic step: φ is satisfiable on $\mathfrak{N}$, so there must exist $w \in W'$ such that $\mathfrak{N}, w \models \varphi$. If also $\mathfrak{N}, w \models i$ such that i does not occur in φ, we put at the initial node of the derivation $i : \varphi$. If no such nominal holds in w, we conservatively extend $\mathfrak{N}$ by adding fresh nominal i to w and put at the initial node of the derivation $i : \varphi$.

Inductive step: Application of each tableau rule should be considered as a separate case. Nevertheless, only five rules seem to be essential for this proof, namely $(\Diamond)$, (E), (D), $(\Diamond^-)$ and (ub), i.e. rules that either introduce a new label to a branch or identify labels already present on a branch. We consider each of them.

Case: $(\Diamond)$. Suppose that a formula $\Diamond\psi$ occurred at the nth node of the derivation. It means that we associated the label i of this node with a world in W' that satisfies $\Diamond\psi$ and i. It follows that there must exist a world v such that wRv and $\mathfrak{N}, v \models \psi$. If v does not satisfy any nominal l that has not yet occurred on the branch either as a label or as a subformula, we conservatively extend $\mathfrak{N}$ by ascribing l to v. Applying $(\Diamond)$ to $\Diamond\psi$ we obtain $i : \Diamond j$ and $j : \psi$. We put l in place of j.

Case: (E). Suppose that a formula $\mathsf{E}\psi$ occurred at the nth node of the derivation. It means that we associated the label i of this node with a world in W' that satisfies $\mathsf{E}\psi$ and i. Therefore, there exists a world v such that $\mathfrak{N}, v \models \psi$. If v does not satisfy any nominal l that has not yet occurred on a branch either as a label or as a subformula, we conservatively extend $\mathfrak{N}$ by ascribing l to v. Applying (E) to $\mathsf{E}\psi$ we obtain $j : \psi$. We put l in place of j.

Case: (D). Suppose that a formula $\mathsf{D}\psi$ occurred at the nth node of the derivation. It means that we affixed the label i of this node to a world in W' that satisfies $\mathsf{D}\psi$ and i. Therefore, there exists a world v such that $\mathfrak{N}, v \models \psi \wedge \neg i$. If v does not satisfy any nominal l that has not yet occurred on a branch either as a label or as a subformula, we conservatively extend $\mathfrak{N}$ by ascribing l to v. Applying (D). to $\mathsf{D}\psi$ we obtain $j : \neg i$ and $j : \psi$. We put l in place of j.

Case: $(\Diamond^-)$. Suppose that a formula $\Diamond^-\psi$ occurred at the nth node of the derivation. It means that we associated the label i of this node with a world in W' that satisfies $\Diamond^-\psi$ and i. It follows that there must exist a world v such that vRw and $\mathfrak{N}, v \models \psi$. If v does not satisfy any nominal l that has not yet occurred on the branch either as a label or as a subformula, we conservatively extend $\mathfrak{N}$ by ascribing l to v. Applying $(\Diamond)$ to $\Diamond\psi$ we obtain $i : \Diamond j$ and $j : \psi$. We put l in place of j.

Case: (ub). Suppose that during the derivation two labels i and j have been introduced to $\mathcal{B}$. By the inductive hypothesis we mapped these labels to worlds w and v of (the conservative extension of) W'. Either the world w satisfies $i \wedge j$ (which would mean that w and v are the same world) or it satisfies $i \wedge \neg j$ (which indicates that w and v are distinct). If the former is the case, we pick the left conclusion of (ub) and add it to $\mathcal{B}$, if the latter is the case, we choose the right conclusion of (ub) and add it to $\mathcal{B}$.

Since $\mathbf{B}$ is open, we can construct a model $\mathfrak{M}(\mathcal{B}) = \langle W, R, V\rangle$ out of it. Now we show that $\mathsf{Card}(W) \leq \mathsf{Card}(W')$ (we consider $\mathfrak{N}$ as already conservatively extended in progress of constructing $\mathcal{B}$). We set a function $f\colon W' \longrightarrow W$ as follows

$$f(w) = \begin{cases} [i]_{\approx_{\mathcal{B}}}, & \text{if there is } i : i \in \mathcal{B} \text{ such that } i \text{ was} \\ & \text{affixed to } w \text{ during the derivation} \\ \text{arbitrary element of } W, & \text{otherwise} \end{cases}$$

f is injective and if we cut it to these elements of W' to which we assigned a nominal during the derivation, it is also an isomorphism. That concludes the proof. $\dashv$

To conclude our considerations it is sufficient to prove that the logic $\mathcal{H}(@, \mathsf{E}, \mathsf{D}, \Diamond^-)$ has the finite model property. The following proposition deals with it.

Proposition 2. *The logic* $\mathcal{H}(@, \mathsf{E}, \mathsf{D}, \Diamond^-)$ *has the* effective finite model property *with the bounding function* $\mu = 2^{\mathsf{Card}(\mathsf{Sub}(\varphi))+1}$, *where* $\mathsf{Sub}(\varphi)$ *is a set of all subformulas of a formula* φ.

Proof. We use the standard, filtration-based argument. Suppose that a formula φ is satisfied on a (possibly infinite) model $\mathfrak{M} = \langle W, R, V\rangle$. It means that there exists $w \in W$ such that $\mathfrak{M}, w \models \varphi$. We show that there exists a finite model $\mathfrak{M}'$ that satisfies φ and whose cardinality does not exceed $2^{\mathsf{Card}(\mathsf{Sub}(\varphi))+1}$.

First, we set the relation $\approx_\varphi$ on W in the following way:

$$w \approx_\varphi v \qquad \text{iff} \qquad \text{for all } \psi \in \mathsf{Sub}(\varphi)\ \mathfrak{M}, w \models \psi \text{ iff } \mathfrak{M}, v \models \psi.$$

It is straightforward that $\approx_\varphi$ is the equivalence relation

Now we are ready to construct our finite model that will satisfy ϕ. Let $\mathfrak{M}' = \langle W', R', V' \rangle$ such that:

$W' = W/_{\approx_\varphi} \uplus W/_{\approx_\varphi}$;
$R' = \{(|v|_{\approx_\varphi}, |u|_{\approx_\varphi}) : R(v, u)\}$;
$V'(p) = \{|v|_{\approx_\varphi} : v \in V(p)\}$ for all proposition letters in φ;
$V'(i) = \{|v|_{\approx_\varphi} : v \in V(i)\}$ for all nominals in φ.

We prove that $\mathfrak{M}'$ satisfies φ by induction on the complexity of subformulas of φ. Since the proof for the modal part of $\mathcal{H}(@, \mathsf{E}, \mathsf{D}, \Diamond^-)$ is well known (cf. [5]) and the case of $\psi = i$ follows immediately from the definition of V', we confine ourselves to proving the cases of $@_i\chi$, $\mathsf{E}\chi$, $\mathsf{D}\chi$ and $\Diamond^-\chi$.

Case: $\psi = @_i\chi$. Suppose that $\mathfrak{M}, v \models @_i\chi$. It means that χ holds at a world u at which also i holds. This world is transformed to a singleton equivalence class $\{u\}$ in W'. By the inductive hypothesis it follows that $\mathfrak{M}', \{u\} \models i$ and $\mathfrak{M}', \{u\} \models \chi$. Hence $\mathfrak{M}', |v| \models @_i\chi$.

Case: $\psi = \mathsf{E}\chi$. Suppose that $\mathfrak{M}, v \models \mathsf{E}\chi$. It means that there exists a world u at which χ holds. By the inductive hypothesis $\mathfrak{M}', |u| \models \chi$. Hence $\mathfrak{M}', |v| \models \mathsf{E}\chi$.

Case: $\psi = \mathsf{D}\chi$. Suppose that $\mathfrak{M}, v \models \mathsf{D}\chi$. It means that there exists a world u different than v, at which χ holds. By the inductive hypothesis $\mathfrak{M}', |u| \models \chi$. Two complementary cases might occur. If $|v| \neq |u|$, then we obtain $\mathfrak{M}', |v| \models \chi$. If, however, $|v| = |u|$, it means that χ is also satisfied by a copy of $|v|$ that we pasted to W' at the stage of the construction of $\mathfrak{M}'$. Since $|v|$ and its copy are distinct, we obtain $\mathfrak{M}', |v| \models \chi$.

Case: $\psi = \Diamond^-\chi$. Suppose that $\mathfrak{M}, v \models \Diamond^-\chi$. It means that χ holds at a world u such that uRv. By the inductive hypothesis it follows that $\mathfrak{M}', |u| \models \chi$ and by the definition of R' we have that $|u|R'|v|$. Hence $\mathfrak{M}', |v| \models \Diamond^-\chi$.

Observe that pasting a distinct copy of $W/_{\approx_\varphi}$ to W' is only necessary if D is involved. Therefore, in other cases the bounding function $\mu = 2^{\mathsf{Card}(\mathsf{Sub}(\varphi))}$. $\dashv$

Consequently, we obtain the following result:

Theorem 4. *$\mathcal{T}_{\mathcal{H}(@,\mathsf{E},\mathsf{D},\Diamond^-)}$ is terminating.*

Proof. Follows from Corollary 1, Lemma 2 and Proposition 2. ⊣

Obviously, the bounding function μ from Proposition 2 can be reduced (cf. [16, 2]), however, the main aim of this paper is not optimising the complexity of $\mathcal{T}_{\mathcal{H}(@,\mathsf{E},\mathsf{D},\Diamond^-)}$. Besides, the filtration-based argument can be easily adapted for different types of frames. Thus, we formulate the following strategy-condition for performing the derivation in $\mathcal{T}_{\mathcal{H}(@,\mathsf{E},\mathsf{D},\Diamond^-)}$:

(tm) Expand a branch of $\mathcal{T}_{\mathcal{H}(@,\mathsf{E},\mathsf{D},\Diamond^-)}$-tableau until the number of equivalence classes of individuals in $\mathcal{B}$ exceeds the bound given by the μ function. Then stop.

It turns our tableau calculus into a deterministic decision procedure.

7. Concluding Remarks

In this paper we presented an internalised tableau-based decision procedure for the logic $\mathcal{H}(@,\mathsf{E},\mathsf{D},\Diamond^-)$. Tableau calculus $\mathcal{T}_{\mathcal{H}(@,\mathsf{E},\mathsf{D},\Diamond^-)}$ was proven to be sound, complete and terminating. In the existing literature of the subject several approaches to systematic treatment of decision procedures for hybrid logics can be found. We recall two of them. In [4] and [6] Blackburn, Bolander and Braüner provide a terminating internalised tableau-based decision procedure for the logic $\mathcal{H}(@,\mathsf{E})$. However, their main concern is different from ours. Their attempts are focused on tailoring a suitable tableau calculus for each logic separately. Therefore, they introduce two different blocking mechanisms, namely *subset blocking* and *equality blocking* for the logics $\mathcal{H}(@)$ and $\mathcal{H}(@,\mathsf{E},\Diamond^-)$ and modify the notion of *urfather* subject to a particular logic. The resulting calculus is conceptually complex but seems to avoid any superfluous performances of the rules. In [12] Götzmann, Kaminski and Smolka describe *Spartacus*, which is a tableau prover for hybrid logics with @ operators and universal modality. Thanks to the application of advanced blocking and optimisation techniques, namely *pattern-based blocking* and *lazy branching* the system is very efficient in terms of complexity. Also, some thorough examination of the problem of nominal equality in tableau reasoning can be found in the works by Cerrito and Cialdea-Mayer ([8, 9, 15]).

The decision procedure introduced in this paper presents the approach which is different from the aforementioned. It introduces (ub) as an explicit tableau rule which is sound and, together with the conditions (c1)-(c3), ensures termination of the whole calculus. (ub) allows a direct comparison of every pair of labels that occurred on a branch and, therefore, subsumes any other blocking mechanisms. (ub) is a generic rule which means that it generates every possible configuration of labels occurring on a branch. In comparison to [4] and [12] many of these configurations are superfluous. However, the huge advantage of this approach is conceptual simplicity which allows to avoid introducing complicated strategies of searching for a pair of labels that are liable to blocking mechanism. Additionally, for each satisfiable formula φ (ub) ensures that a minimal model for φ (in terms of a domain size) will be generated, which cannot be guaranteed by the systems of [4] and [12]. Moreover, $\mathcal{T}_{\mathcal{H}(@,\mathsf{E},\mathsf{D},\Diamond^-)}$ provides a uniform approach to all hybrid logics mentioned in the paper and covers the case of difference modality which is omitted in [4]. At the current stage of research a systematic comparison between the implementation of the presented calculus and implementations of other existing calculi would not bring any scientific profit. A naïve application of the (ub) makes the implementation far from optimal in terms of running speed. Certain possibilities of imposing some additional conditions on application of the (ub), that would restrict the number of branching points in a tableau, are presently investigated by R. Schmidt, D. Tishkovsky and M. Khodadadi from The University of Manchester. However, applying the results obtained in this field lies outside the scope of this paper, simultaneously being considered as a serious direction of future work.

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