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5.	$(\sim\triangle\phi)? \rightarrow \neg(\nabla\phi?)$	`P(2)` 3, 4
6.	$(\sim\triangle\phi)? \vee \triangle\phi!$	`K(8)`
7.	$\neg(\nabla\phi?) \vee \triangle\phi!$	`P(8)` 5, 6
8.	$\triangle\phi! \vee \neg(\nabla\phi?)$	`P(18)` 7

$\varphi = \Diamond\phi$. $\triangle\alpha = \triangle\varphi \vee \neg\nabla\varphi = \triangle\Diamond\phi \vee \neg\nabla\Diamond\phi = \triangle\phi? \vee \neg(\nabla\phi!)$.

1.	$\triangle\phi \vee \neg\nabla\phi$	Hyp. Ind.
2.	$\sim\triangle\phi \rightarrow \neg\nabla\phi$	$\sim$(8) 1
3.	$(\sim\triangle\phi)! \rightarrow (\neg\nabla\phi)!$	`K(4)` 2
4.	$(\neg\nabla\phi)! \rightarrow \neg(\nabla\phi!)$	`K2`
5.	$(\sim\triangle\phi)! \rightarrow \neg(\nabla\phi!)$	`P(2)` 3, 4
6.	$(\sim\sim\triangle\phi)? \vee (\sim\triangle\phi)!$	`K(8)`
7.	$\sim\sim\triangle\phi \rightarrow \triangle\phi$	$\sim$(9)
8.	$(\sim\sim\triangle\phi)? \rightarrow \triangle\phi?$	`K(1)` 7
9.	$\triangle\phi? \vee (\sim\triangle\phi)!$	`P(8)` 6, 8
10.	$(\sim\triangle\phi)! \vee \triangle\phi?$	`P(18)` 9
11.	$\neg(\nabla\phi!) \vee \triangle\phi?$	`P(8)` 5, 10
12.	$\triangle\phi? \vee \neg(\nabla\phi!)$	`P(18)` 11

$\varphi = \neg\phi$. $\triangle\alpha = \triangle\varphi \vee \neg\nabla\varphi = \triangle\neg\phi \vee \neg\nabla\neg\phi = \neg\nabla\phi \vee \neg\neg\triangle\phi$.

1.	$\triangle\phi \vee \neg\nabla\phi$	Hyp. Ind.
2.	$\triangle\phi \rightarrow \neg\neg\triangle\phi$	`N4`
3.	$\neg\neg\triangle\phi \vee \neg\nabla\phi$	`P(8)` 1, 2
4.	$\neg\nabla\phi \vee \neg\neg\triangle\phi$	`P(18)` 3

$\varphi = \phi \rightarrow \lambda$. $\triangle\alpha = \triangle\varphi \vee \neg\nabla\varphi = \triangle(\phi \rightarrow \lambda) \vee \neg\nabla(\phi \rightarrow \lambda) = (\triangle\phi \rightarrow \triangle\lambda) \vee \neg(\triangle\phi \rightarrow \nabla\lambda)$. Since ϕ's and λ's sizes are smaller than n, by the hypothesis of induction we have that there exists a $K_?$-derivation of $\triangle\phi \vee \neg\nabla\phi$, of $\triangle\lambda \vee \neg\nabla\lambda$ and of $\neg\nabla\phi \rightarrow (\triangle\phi \rightarrow \triangle\mu)$. Let $\mu = \bot$. $\triangle\mu = \triangle(p \wedge \neg p) = \triangle p \wedge \neg\nabla p = p \wedge \neg p$.

1.	$\triangle\phi \vee \neg\nabla\phi$	Hyp. Ind.
2.	$\triangle\lambda \vee \neg\nabla\lambda$	Hyp. Ind.
3.	$\neg\nabla\lambda \vee \triangle\lambda$	`P(18)` 2
4.	$(\triangle\phi \wedge \neg\nabla\lambda) \vee (\neg\nabla\phi \vee \triangle\lambda)$	`P(19)` 1, 3
5.	$(\triangle\phi \wedge \neg\nabla\lambda) \rightarrow \neg(\triangle\phi \rightarrow \nabla\lambda)$	`N1`
6.	$\neg(\triangle\phi \rightarrow \nabla\lambda) \vee (\neg\nabla\phi \vee \triangle\lambda)$	`P(8)` 4, 5
7.	$\neg\nabla\phi \rightarrow (\triangle\phi \rightarrow \bot)$	Hyp. Ind.
8.	$\triangle\phi \rightarrow (\neg\nabla\phi \rightarrow \bot)$	`P(1)` 7
9.	$(\triangle\phi \rightarrow \sim\neg\nabla\phi) \rightarrow ((\sim\neg\nabla\phi \rightarrow \triangle\lambda) \rightarrow (\triangle\phi \rightarrow \triangle\lambda))$	`P(2)`

10.	$(\sim\neg\nabla\phi \to \triangle\lambda) \to (\triangle\phi \to \triangle\lambda)$	MP 8, 9
11.	$\neg\nabla\phi \vee \triangle\lambda \to (\sim\neg\nabla\phi \to \triangle\lambda)$	$\sim$(8)
12.	$\neg\nabla\phi \vee \triangle\lambda \to (\triangle\phi \to \triangle\lambda)$	`P(2)` 10, 11
13.	$(\neg\nabla\phi \vee \triangle\lambda) \vee \neg(\triangle\phi \to \nabla\lambda)$	`P(18)` 6
14.	$(\triangle\phi \to \triangle\lambda) \vee \neg(\triangle\phi \to \nabla\lambda)$	`P(8)` 12,13

$\varphi = \phi \wedge \lambda$. $\triangle\alpha = \triangle\varphi \vee \neg\nabla\varphi = \triangle(\phi \wedge \lambda) \vee \neg\nabla(\phi \wedge \lambda) = (\triangle\varphi \wedge \triangle\lambda) \vee \neg(\nabla\phi \wedge \nabla\lambda)$.

1.	$\triangle\phi \vee \neg\nabla\phi$	Hyp. Ind.
2.	$\triangle\lambda \vee \neg\nabla\lambda$	Hyp. Ind.
3.	$(\triangle\phi \wedge \triangle\lambda) \vee (\neg\nabla\phi \vee \neg\nabla\lambda)$	`P(19)` 1, 2
4.	$(\neg\nabla\phi \vee \neg\nabla\lambda) \vee (\triangle\phi \wedge \triangle\lambda)$	`P(18)` 3
5.	$\neg\nabla\phi \vee \neg\nabla\lambda \to \neg(\nabla\phi \wedge \nabla\lambda)$	`N2`
6.	$\neg(\nabla\phi \wedge \nabla\lambda) \vee (\triangle\phi \wedge \triangle\lambda)$	`P(8)` 4, 5
7.	$(\triangle\phi \wedge \triangle\lambda) \vee \neg(\nabla\phi \wedge \nabla\lambda)$	`P(18)` 6

$\varphi = \phi \vee \lambda$. $\triangle\alpha = \triangle\varphi \vee \neg\nabla\varphi = \triangle(\phi \vee \lambda) \vee \neg\nabla(\phi \vee \lambda) = (\triangle\varphi \vee \triangle\lambda) \vee \neg(\nabla\phi \vee \nabla\lambda)$.

1.	$\triangle\phi \vee \neg\nabla\phi$	Hyp. Ind.
2.	$\neg\nabla\phi \vee \triangle\phi$	`P(18)` 1
3.	$\triangle\lambda \vee \neg\nabla\lambda$	Hyp. Ind.
4.	$\neg\nabla\lambda \vee \triangle\lambda$	`P(18)` 3
5.	$(\neg\nabla\phi \wedge \neg\nabla\lambda) \vee (\triangle\phi \vee \triangle\lambda)$	`P(19)` 2, 4
6.	$(\neg\nabla\phi \wedge \neg\nabla\lambda) \to \neg(\nabla\phi \vee \nabla\lambda)$	`N3`
7.	$\neg(\nabla\phi \vee \nabla\lambda) \vee (\triangle\phi \vee \triangle\lambda)$	`P(8)` 5, 6
8.	$(\triangle\phi \vee \triangle\lambda) \vee \neg(\nabla\phi \vee \nabla\lambda)$	`P(18)` 7

`P9`: $\alpha = (\varphi \to \beta) \to ((\varphi \to \neg\beta) \to \neg\varphi)$. $\triangle((\varphi \to \beta) \to ((\varphi \to \neg\beta) \to \neg\varphi)) = (\triangle\varphi \to \triangle\beta) \to ((\triangle\varphi \to \neg\nabla\beta) \to \neg\nabla\varphi)$. In order to do prove this case, we shall make use of the result proved above in which for every $\beta \in L_{\Diamond}$ there is a $K_?$-derivation of $\neg\nabla\beta \to (\triangle\beta \to \triangle\varphi)$, where $\varphi \in L_{\Diamond}$ is an arbitrary formula. We shall refer to this result as `P10`*. Again we shall do the proof by induction on the size of φ.

Base of induction: φ has size 1. $\varphi = p$. $\triangle\alpha = (\triangle\varphi \to \triangle\beta) \to ((\triangle\varphi \to \neg\nabla\beta) \to \neg\nabla\varphi) = \triangle\alpha = (\triangle p \to \triangle\beta) \to ((\triangle p \to \neg\nabla\beta) \to \neg\nabla p) = (p \to \triangle\beta) \to ((p \to \neg\nabla\beta) \to \neg p)$.

1.	$(p \to q) \to ((p \to \neg q) \to \neg p)$	A1
2.	$(p \to \bot) \to \neg p$	`P(16)` 1
3.	$\neg\nabla\beta \to (\triangle\beta \to \bot)$	`P10`*

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4.	$\triangle\beta \to (\neg\nabla\beta \to \bot)$	`P(1)` 3
5.	$(p \to \triangle\beta) \to ((p \to \neg\nabla\beta) \to \neg p)$	`P(13)` 2, 4

Hypothesis of Induction: Take an arbitrary formula φ of size n. Suppose that, for formulas ϕ of size $m < n$, there is a $K_?$-derivation of $(\triangle\phi \to \triangle\lambda) \to ((\triangle\phi \to \neg\nabla\lambda) \to \neg\nabla\phi)$. Before considering all forms φ may have, we will prove the following auxiliary result, which shall be used as an inference rule:

Aux: $(\triangle\phi \to \triangle\beta) \to ((\triangle\phi \to \neg\nabla\beta) \to \neg\nabla\phi) \vdash_{K_?} (\triangle\phi \to \bot) \to \neg\nabla\phi$

1.	$(\triangle\phi \to \triangle\beta) \to ((\triangle\phi \to \neg\nabla\beta) \to \neg\nabla\phi)$	Hyp.
2.	$(\triangle\phi \to \triangle\beta) \wedge (\triangle\phi \to \neg\nabla\beta) \to \neg\nabla\phi$	`P(3)` 1
3.	$(\triangle\phi \to \triangle\beta \wedge \neg\nabla\beta) \to (\triangle\phi \to \triangle\beta) \wedge (\triangle\phi \to \neg\nabla\beta)$	`P(12)`
4.	$(\triangle\phi \to \triangle\beta \wedge \neg\nabla\beta) \to \neg\nabla\phi$	`P(2)` 2, 3
5.	$(\triangle\phi \to \bot) \to (\triangle\phi \to \triangle\beta \wedge \neg\nabla\beta)$	$\sim$(2)
6.	$(\triangle\phi \to \bot) \to \neg\nabla\phi$	`P(2)` 4, 5

$\varphi = \Box\phi$. $\triangle\alpha = (\triangle\varphi \to \triangle\beta) \to ((\triangle\varphi \to \neg\nabla\beta) \to \neg\nabla\varphi) = (\triangle\Box\phi \to \triangle\beta) \to ((\triangle\Box\phi \to \neg\nabla\beta) \to \neg\nabla\Box\phi) = (\triangle\phi! \to \triangle\beta) \to ((\triangle\phi! \to \neg\nabla\beta) \to \neg(\nabla\phi?))$. Since ϕ's size is smaller than n, by the hypothesis of induction we have that there is a $K_?$-derivation of $(\triangle\phi \to \triangle\beta) \to ((\triangle\phi \to \neg\nabla\beta) \to \neg\nabla\phi)$.

1.	$(\triangle\phi \to \triangle\beta) \to ((\triangle\phi \to \neg\nabla\beta) \to \neg\nabla\phi)$	Hyp. Ind.
2.	$(\triangle\phi \to \bot) \to \neg\nabla\phi$	Aux 1
3.	$(\triangle\phi \to \bot)? \to (\neg\nabla\phi)?$	`K(1)` 2
4.	$(\triangle\phi! \to \bot) \to (\triangle\phi \to \bot)?$	`K(2)`
5.	$(\triangle\phi! \to \bot) \to (\neg\nabla\phi)?$	`P(2)` 3, 4
6.	$(\neg\nabla\phi)? \to \neg(\nabla\phi?)$	`K3`
7.	$(\triangle\phi! \to \bot) \to \neg(\nabla\phi?)$	`P(2)` 5, 6
8.	$\neg\nabla\beta \to (\triangle\beta \to \bot)$	`P10`*
9.	$\triangle\beta \to (\neg\nabla\beta \to \bot)$	`P(1)` 8
10.	$(\triangle\phi! \to \triangle\beta) \to ((\triangle\phi! \to \neg\nabla\beta) \to \neg(\nabla\phi?))$	`P(13)` 7, 9

$\varphi = \Diamond\phi$. $\triangle\alpha = (\triangle\varphi \to \triangle\beta) \to ((\triangle\varphi \to \neg\nabla\beta) \to \neg\nabla\varphi) = (\triangle\Diamond\phi \to \triangle\beta) \to ((\triangle\Diamond\phi \to \neg\nabla\beta) \to \neg\nabla\Diamond\phi) = (\triangle\phi? \to \triangle\beta) \to ((\triangle\phi? \to \neg\nabla\beta) \to \neg(\nabla\phi!))$.

1.	$(\triangle\phi \to \triangle\beta) \to ((\triangle\phi \to \neg\nabla\beta) \to \neg\nabla\phi)$	Hyp. Ind.
2.	$(\triangle\phi \to \bot) \to \neg\nabla\phi$	Aux 1
3.	$(\triangle\phi \to \bot)! \to (\neg\nabla\phi)!$	`K(4)` 2

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4.	$(\triangle\phi? \to \bot) \to (\triangle\phi \to \bot)!$	K(5)
5.	$(\triangle\phi? \to \bot) \to (\neg\nabla\phi)!$	P(2) 3, 4
6.	$(\neg\nabla\phi)! \to \neg(\nabla\phi!)$	K2
7.	$(\triangle\phi? \to \bot) \to \neg(\nabla\phi!)$	P(2) 5, 6
8.	$\neg\nabla\beta \to (\triangle\beta \to \bot)$	P10*
9.	$\triangle\beta \to (\neg\nabla\beta \to \bot)$	P(1) 8
10.	$(\triangle\phi? \to \triangle\beta) \to ((\triangle\phi? \to \neg\nabla\beta) \to \neg(\nabla\phi!))$	P(13) 7, 9

$\varphi = \neg\phi$. $\triangle\alpha = (\triangle\varphi \to \triangle\beta) \to ((\triangle\varphi \to \neg\nabla\beta) \to \neg\nabla\varphi) = (\triangle\neg\phi \to \triangle\beta) \to ((\triangle\neg\phi \to \neg\nabla\beta) \to \neg\nabla\neg\phi) = (\neg\nabla\phi \to \triangle\beta) \to ((\neg\nabla\phi \to \neg\nabla\beta) \to \neg\neg\triangle\phi)$.

1.	$(\triangle\phi \to \triangle\beta) \to ((\triangle\phi \to \neg\nabla\beta) \to \neg\nabla\phi)$	Hyp. Ind.
2.	$(\triangle\phi \to \bot) \to \neg\nabla\phi$	Aux 1
3.	$((\triangle\phi \to \bot) \to \neg\nabla\phi) \to ((\neg\nabla\phi \to \bot) \to \triangle\phi)$	$\sim$(4)
4.	$(\neg\nabla\phi \to \bot) \to \triangle\phi$	MP 2, 3
5.	$\triangle\phi \to \neg\neg\triangle\phi$	N4
6.	$(\neg\nabla\phi \to \bot) \to \neg\neg\triangle\phi$	P(2) 4, 5
7.	$\neg\nabla\beta \to (\triangle\beta \to \bot)$	P10*
8.	$\triangle\beta \to (\neg\nabla\beta \to \bot)$	P(1) 7
9.	$(\neg\nabla\phi \to \triangle\beta) \to ((\neg\nabla\phi \to \neg\nabla\beta) \to \neg\neg\triangle\phi)$	P(13) 6, 8

$\varphi = \phi \to \lambda$. $\triangle\alpha = (\triangle\varphi \to \triangle\beta) \to ((\triangle\varphi \to \neg\nabla\beta) \to \neg\nabla\varphi) = (\triangle(\phi \to \lambda) \to \triangle\beta) \to ((\triangle(\phi \to \lambda) \to \neg\nabla\beta) \to \neg\nabla(\phi \to \lambda)) = ((\triangle\phi \to \triangle\lambda) \to \triangle\beta) \to (((\triangle\phi \to \triangle\lambda) \to \neg\nabla\beta) \to \neg(\triangle\phi \to \nabla\lambda))$.

1.	$(\sim\triangle\lambda \to \neg\nabla\lambda) \to (\triangle\phi \wedge \sim\triangle\lambda \to \triangle\phi \wedge \neg\nabla\lambda)$	P(15)
2.	$(\triangle\lambda \to \triangle\beta) \to ((\triangle\lambda \to \neg\nabla\beta) \to \neg\nabla\lambda)$	Hyp. Ind.
3.	$\sim\triangle\lambda \to \neg\nabla\lambda$	Aux 2
4.	$\triangle\phi \wedge \sim\triangle\lambda \to \triangle\phi \wedge \neg\nabla\lambda$	MP 1, 3
5.	$\triangle\phi \wedge \neg\nabla\lambda \to \neg(\triangle\phi \to \nabla\lambda)$	N1
6.	$\triangle\phi \wedge \sim\triangle\lambda \to \neg(\triangle\phi \to \nabla\lambda)$	P(2) 4, 5
7.	$\sim(\triangle\phi \to \triangle\lambda) \to \triangle\phi \wedge \sim\triangle\lambda$	$\sim$(5)
8.	$((\triangle\phi \to \triangle\lambda) \to \bot) \to \neg(\triangle\phi \to \nabla\lambda)$	P(2) 6, 7
9.	$\neg\nabla\beta \to (\triangle\beta \to \bot)$	P10*
10.	$\triangle\beta \to (\neg\nabla\beta \to \bot)$	P(1) 9
11.	$((\triangle\phi \to \triangle\lambda) \to \triangle\beta) \to (((\triangle\phi \to \triangle\lambda) \to \neg\nabla\beta) \to \neg(\triangle\phi \to \nabla\lambda))$	P(13) 8, 10

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$\varphi = \phi \wedge \lambda$. $\triangle\alpha = (\triangle\varphi \to \triangle\beta) \to ((\triangle\varphi \to \neg\nabla\beta) \to \neg\nabla\varphi) = (\triangle(\phi \wedge \lambda) \to \triangle\beta) \to ((\triangle(\phi \wedge \lambda) \to \neg\nabla\beta) \to \neg\nabla(\phi \wedge \lambda)) = ((\triangle\phi \wedge \triangle\lambda) \to \triangle\beta) \to (((\triangle\phi \wedge \triangle\lambda) \to \neg\nabla\beta) \to \neg(\nabla\phi \wedge \nabla\lambda))$.

1.	$(\triangle\phi \to \triangle\beta) \to ((\triangle\phi \to \neg\nabla\beta) \to \neg\nabla\phi)$	Hyp. Ind.
2.	$\sim\triangle\phi \to \neg\nabla\phi$	Aux 1
3.	$(\triangle\lambda \to \triangle\beta) \to ((\triangle\lambda \to \neg\nabla\beta) \to \neg\nabla\lambda)$	Hyp. Ind.
4.	$\sim\triangle\lambda \to \neg\nabla\lambda$	Aux 3
5.	$\sim\triangle\phi \vee \sim\triangle\lambda \to \neg\nabla\phi \vee \neg\nabla\lambda$	`P(17)` 2, 4
6.	$\sim(\triangle\phi \wedge \triangle\lambda) \to \sim\triangle\phi \vee \sim\triangle\lambda$	$\sim$(6)
7.	$\sim(\triangle\phi \wedge \triangle\lambda) \to \neg\nabla\phi \vee \neg\nabla\lambda$	`P(2)` 5, 6
8.	$\neg\nabla\phi \vee \neg\nabla\lambda \to \neg(\nabla\phi \wedge \nabla\lambda)$	`N2`
9.	$((\triangle\phi \wedge \triangle\lambda) \to \bot) \to \neg(\nabla\phi \wedge \nabla\lambda)$	`P(2)` 7, 8
10.	$\neg\nabla\beta \to (\triangle\beta \to \bot)$	`P10`*
11.	$\triangle\beta \to (\neg\nabla\beta \to \bot)$	`P(1)` 10
12.	$((\triangle\phi \wedge \triangle\lambda) \to \triangle\beta) \to (((\triangle\phi \wedge \triangle\lambda) \to \neg\nabla\beta) \to \neg(\nabla\phi \wedge \nabla\lambda))$	`P(13)` 9, 11

$\varphi = \phi \vee \lambda$. $\triangle\alpha = (\triangle\varphi \to \triangle\beta) \to ((\triangle\varphi \to \neg\nabla\beta) \to \neg\nabla\varphi) = (\triangle(\phi\vee\lambda) \to \triangle\beta) \to ((\triangle(\phi\vee\lambda) \to \neg\nabla\beta) \to \neg\nabla(\phi\vee\lambda)) = ((\triangle\phi\vee\triangle\lambda) \to \triangle\beta) \to (((\triangle\phi \vee \triangle\lambda) \to \neg\nabla\beta) \to \neg(\nabla\phi \vee \nabla\lambda))$. Here the proof is almost identical to one of the case above; just replace in the derivation "`P(17)` 2, 4" for "`P(21)` 2, 4" in line 5, "$\sim$(6)" for "$\sim$(7)" in line 6 and "`N2`" for "`N3`" in line 8, along with the corresponding changes.

`K`: $\alpha = \Box(\varphi \to \phi) \to (\Box\varphi \to \Box\phi)$. $\triangle\alpha = \triangle(\Box(\varphi \to \phi) \to (\Box\varphi \to \Box\phi)) = (\triangle\varphi \to \triangle\phi)! \to (\triangle\varphi! \to \triangle\phi!)$. We have therefore to prove that there exists a $K_?$-derivation of $(\triangle\varphi \to \triangle\phi)! \to (\triangle\varphi! \to \triangle\phi!)$. Since $(\triangle\varphi \to \triangle\phi)! \to (\triangle\varphi! \to \triangle\phi!)$ is an instance of $K_?$, it itself is the derivation we are looking for.

`NP`: $\alpha = \Diamond\varphi \leftrightarrow \neg\Box\neg\varphi$. $\triangle\alpha = \triangle(\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi) = \triangle((\Diamond\varphi \to \neg\Box\neg\varphi) \wedge (\neg\Box\neg\varphi \to \Diamond\varphi)) = \triangle(\Diamond\varphi \to \neg\Box\neg\varphi) \wedge \triangle(\neg\Box\neg\varphi \to \Diamond\varphi) = (\triangle(\Diamond\varphi) \to \triangle(\neg\Box\neg\varphi)) \wedge (\triangle(\neg\Box\neg\varphi) \to \triangle(\Diamond\varphi)) = (\triangle\varphi? \to \neg\nabla(\Box\neg\varphi)) \wedge \neg(\nabla(\Box\neg\varphi) \to \triangle\varphi?) = (\triangle\varphi? \to \neg((\nabla\neg\varphi)?)) \wedge (\neg((\nabla\neg\varphi)?) \to \triangle\varphi?) = (\triangle\varphi? \to \neg((\neg\triangle\varphi)?)) \wedge (\neg((\neg\triangle\varphi)?) \to \triangle\varphi?)$.

1.	$\triangle\varphi \to \neg\neg\triangle\varphi$	`N4`
2.	$\triangle\varphi? \to (\neg\neg\triangle\varphi)?$	`K(1)` 1
3.	$(\neg\neg\triangle\varphi)? \to \neg((\neg\triangle\varphi)?)$	`K3`
4.	$\triangle\varphi? \to \neg((\neg\triangle\varphi)?)$	`P(2)` 2, 3
5.	$\neg\neg\triangle\varphi \to \triangle\varphi$	`N4`

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6.	$(\neg\neg\triangle\varphi)? \to \triangle\varphi?$	K(1) 5
7.	$\neg((\neg\triangle\varphi)?) \to (\neg\neg\triangle\varphi)?$	K3
8.	$\neg((\neg\triangle\varphi)?) \to \triangle\varphi?$	P(2) 6, 7
9.	$(\triangle\varphi? \to \neg((\neg\triangle\varphi)?)) \wedge (\neg((\neg\triangle\varphi)?) \to \triangle\varphi?)$	P(20) 4, 8

We therefore have proved that in the case where α is an instance of one of the axioms of K, if $A \oplus B \vdash_{\mathrm{K}} \alpha$ then $\triangle(A) \oplus \triangle(B) \vdash_{\mathrm{K?}} \triangle(\alpha)$. This completes the basis of induction of our proof. Now we will proceed to consider the case where the size of the K-derivation of α from A and B is greater than 1.

Hypothesis of induction: Let $n > 1$ be the size of the K-derivation of α from A and B. Suppose that for K-derivations of sizes smaller than n the result holds. That is to say, if $A \oplus B \vdash_{\mathrm{K}} \varphi$ and the size of the derivation of φ from A and B is smaller than n, then $\triangle(A) \oplus \triangle(B) \vdash_{\mathrm{K?}} \triangle(\varphi)$. Let $\alpha_1, \ldots, \alpha_n$ be the K-derivation of α from A and B. Considering we are dealing with the propositional case, by Definition I.3.23, $\alpha_n = \alpha$ should satisfy one of the following conditions:

(i) α_n is an axiom of K;
(ii) α_n is one of the premises ($\alpha \in A \cup B$);
(iii) There are $i, j < n$, such that $\alpha_j = \alpha_i \to \alpha_n$;
(iv) There is an $\alpha_i \in \alpha_1, \ldots, \alpha_{n-1}$, such that $\alpha_n = \Box\alpha_i$, and no element of B appears in the derivation of α_i.

We have just considered cases (i) and (ii) when we dealt with derivations of size 1. Let us now consider the two other cases.
Case (iii): $\alpha_j = \alpha_i \to \alpha_n$. $\triangle(\alpha_j) = \triangle(\alpha_i) \to \triangle(\alpha_n)$. Since the sizes of the K-derivations of α_i and α_j from A and B are smaller than n, by the hypothesis of induction we have that there is a K?-derivation of $\triangle(\alpha_i)$ from $\triangle(A)$ and $\triangle(B)$ and a K?-derivation of $\triangle(\alpha_j) = \triangle(\alpha_i) \to \triangle(\alpha_n)$ from $\triangle(A)$ and $\triangle(B)$. Therefore, taking these two K?-derivations together and considering item d) of Definition I.3.23 (MP rule), we conclude that there is a K? of $\triangle(\alpha_n)$ from $\triangle(A)$ and $\triangle(B)$.
Case (iv): $\alpha_n = \Box\alpha_i$. $\triangle(\alpha_n) = \triangle(\alpha_i)!$. Since no element of B appears in the K-derivation of α_i from A and B, we are sure that there is a K-derivation of α_i from A and $\varnothing$. Since the size of such a derivation is smaller than n, by the hypothesis of induction we have that there is a K?-derivation of $\triangle(\alpha_i)$ from $\triangle(A)$ and $\varnothing$. Given this, and taking this K?-derivation along with item f) of Definition I.3.23 (rule N), we conclude that that there is a K?-derivation of $\triangle(\alpha_n) = \triangle(\alpha_i)!$ from $\triangle(A)$ and $\triangle(B)$. ⊣

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LEMMA 2.8. *The following schemas of relations between sets of formulas and formula are valid:*

$\mathtt{K}_{\Diamond}(1)$: $\vdash_{\mathrm{K}} \neg\alpha \leftrightarrow {\sim}\alpha$

$\mathtt{K}_{\Diamond}(2)$: *if* $\vdash_{\mathrm{K}} \alpha \leftrightarrow \beta$ *and* $\vdash_{\mathrm{K}} \varphi$*, where* α *occurs in* φ*, then* $\vdash_{\mathrm{K}} \varphi[\alpha/\beta]$

$\mathtt{K}_{\Diamond}(3)$: $(\alpha \to \beta) \to ((\alpha \to \varphi) \to \lambda) \vdash_{\mathrm{K}} (\alpha \to \beta \wedge \varphi) \to \lambda$

$\mathtt{K}_{\Diamond}(4)$: $\vdash_{\mathrm{K}} \Box\neg\alpha \leftrightarrow \neg\Diamond\alpha$

$\mathtt{K}_{\Diamond}(5)$: $\vdash_{\mathrm{K}} \Diamond\neg\alpha \leftrightarrow \neg\Box\alpha$.

LEMMA 2.9. *Let* $A, B \subseteq \mathrm{L}_?$ *and* $\alpha \in \mathrm{L}_?$*. If* $A \oplus B \vdash_{\mathrm{K}_?} \alpha$ *then* $\Pi(A) \oplus \Pi(B) \vdash_{\mathrm{K}} \Pi(\alpha)$.

PROOF. To prove this lemma, we shall follow the same path of the proof of lemma 2.7. If $A \oplus B \vdash_{\mathrm{K}_?} \alpha$, then there is a $\mathrm{K}_?$-derivation of α from A and B. We then have to prove that if this is the case, there is K-derivation of $\Pi(\alpha)$ from $\Pi(A)$ and $\Pi(B)$.

Base of induction: derivation of size 1: $\alpha_1 = \alpha$.

Case 1: $\alpha \in A \cup B$.

Trivially, $\Pi(\alpha) \in \Pi(A \cup B)$. Therefore, $\Pi(A) \oplus \Pi(B) \vdash_{\mathrm{K}} \Pi(\alpha)$.

Case 2: α is an axiom of $\mathrm{K}_?$.

To deal with this case, we have to analyze the possibility of α's being an instance of each one of $\mathrm{K}_?$'s axiom-schemas. For each one of these possibilities, we shall show that there is a K-derivation of $\Pi(\alpha)$, which implies that there exists a K-derivation of $\Pi(\alpha)$ from $\Pi(A)$ and $\Pi(B)$ and therefore that $\Pi(A) \oplus \Pi(B) \vdash_{\mathrm{K}} \Pi(\alpha)$. The cases where α is instance of one of $\mathrm{K}_?$'s axioms of positive logic, paranormal classical axioms and additional classical axioms are trivial, for these axiom-schemas are valid in K: `P1`–`P8` and `A1`–`A3` belong to the axiomatic of K and `N1`–`N5` are easily derived from them along with MP (in fact, all of them are theorems of classical logic). Below we consider the cases where α is an instance of `K1`–`K5` or $\mathrm{K}_?$ and show that, in these cases, there is a K-derivation of $\Pi(\alpha)$. In order to simplify our exposition, we show that $\Pi({\sim}\varphi) = {\sim}\Pi(\varphi)$: $\Pi({\sim}\varphi) = \Pi(\varphi \to (p \wedge \neg p)) = \Pi\varphi \to \Pi(p \wedge \neg p) = \Pi\varphi \to (\Pi p \wedge \Pi\neg p) = \Pi\varphi \to (\Pi p \wedge \neg \coprod p) = \Pi\varphi \to (p \wedge \neg p) = {\sim}\Pi\varphi$.

`K1`: $\alpha = \varphi? \leftrightarrow {\sim}(({\sim}\varphi)!)$.

$\Pi\alpha = \Pi(\varphi? \leftrightarrow {\sim}(({\sim}\varphi)!)) = \Pi((\varphi? \to {\sim}(({\sim}\varphi)!)) \wedge ({\sim}(({\sim}\varphi)!) \to \varphi?)) = \Pi((\varphi? \to {\sim}(({\sim}\varphi)!)) \wedge \Pi({\sim}(({\sim}\varphi)!) \to \varphi?)) = (\Pi(\varphi?) \to \Pi({\sim}(({\sim}\varphi)!))) \wedge (\Pi({\sim}(({\sim}\varphi)!)) \to \Pi(\varphi?)) = (\Pi(\varphi?) \to {\sim}\Pi(({\sim}\varphi)!)) \wedge ({\sim}\Pi(({\sim}\varphi)!) \to \Pi(\varphi?)) = (\Diamond\Pi\varphi \to {\sim}\Box\Pi({\sim}\varphi)) \wedge ({\sim}\Box\Pi({\sim}\varphi) \to \Diamond\Pi\varphi) = (\Diamond\Pi\varphi \to {\sim}\Box{\sim}\Pi\varphi) \wedge ({\sim}\Box{\sim}\Pi\varphi \to \Diamond\Pi\varphi) = \Diamond\Pi\varphi \leftrightarrow {\sim}\Box{\sim}\Pi\varphi$.

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1.	$\neg\Pi\varphi \leftrightarrow \sim\Pi\varphi$	$\texttt{K}_{\Diamond}(1)$
2.	$\Diamond\Pi\varphi \leftrightarrow \neg\Box\neg\Pi\varphi$	NP
3.	$\Diamond\Pi\varphi \leftrightarrow \neg\Box\sim\Pi\varphi$	$\texttt{K}_{\Diamond}(2)$ 1, 2
4.	$\neg\Box\sim\Pi\varphi \leftrightarrow \sim\Box\sim\Pi\varphi$	$\texttt{K}_{\Diamond}(1)$
5.	$\Diamond\Pi\varphi \leftrightarrow \sim\Box\sim\Pi\varphi$	$\texttt{K}_{\Diamond}(2)$ 3, 4

K2: $\alpha = (\neg\varphi)! \leftrightarrow \neg(\varphi!)$.
$\Pi\alpha = \Pi(((\neg\varphi)! \to \neg(\varphi!)) \wedge (\neg(\varphi!) \to (\neg\varphi)!)) = \Pi((\neg\varphi)! \to \neg(\varphi!)) \wedge \Pi(\neg(\varphi!) \to (\neg\varphi)!) = (\Pi((\neg\varphi)!) \to \Pi(\neg(\varphi!))) \wedge (\Pi(\neg(\varphi!)) \to \Pi((\neg\varphi)!)) = (\Box\Pi(\neg\varphi) \to \neg\coprod(\varphi!)) \wedge (\neg\coprod(\varphi!) \to \Box\Pi(\neg\varphi)) = (\Box\neg\coprod\varphi \to \neg\Diamond\coprod\varphi) \wedge (\neg\Diamond\coprod\varphi \to \Box\neg\coprod\varphi) = \Box\neg\coprod\varphi \leftrightarrow \neg\Diamond\coprod\varphi$. Since $\Box\neg\coprod\varphi \leftrightarrow \neg\Diamond\coprod\varphi$ is an instance of $\texttt{K}_{\Diamond}(4)$, it itself is the derivation we are looking for.

K3: $\alpha = (\neg\varphi)? \leftrightarrow \neg(\varphi?)$.
$\Pi\alpha = \Pi(((\neg\varphi)? \to \neg(\varphi?)) \wedge (\neg(\varphi?) \to (\neg\varphi)?)) = \Pi((\neg\varphi)? \to \neg(\varphi?)) \wedge \Pi(\neg(\varphi?) \to (\neg\varphi)?) = (\Pi((\neg\varphi)?) \to \Pi(\neg(\varphi?))) \wedge (\Pi(\neg(\varphi?)) \to \Pi((\neg\varphi)?)) = \Diamond\Pi(\neg\varphi) \to \neg\coprod(\varphi?)) \wedge (\neg\coprod(\varphi?) \to \Diamond\Pi(\neg\varphi)) = (\Diamond\neg\coprod\varphi \to \neg\Box\coprod\varphi) \wedge (\neg\Box\coprod\varphi \to \Diamond\neg\coprod\varphi) = \Diamond\neg\coprod\varphi \leftrightarrow \neg\Box\coprod\varphi$. Since $\Diamond\neg\coprod\varphi \leftrightarrow \neg\Box\coprod\varphi$ is an instance of $\texttt{K}_{\Diamond}(5)$, it itself is the K-derivation we are looking for.

$\texttt{K}_?$: $\alpha = (\varphi \to \phi)! \to (\varphi! \to \phi!)$.
$\Pi\alpha = \Pi((\varphi \to \phi)! \to (\varphi! \to \phi!)) = \Pi(\varphi \to \phi)! \to \Pi(\varphi! \to \phi!) = \Box\Pi(\varphi \to \phi) \to (\Pi(\varphi!) \to \Pi(\phi!)) = \Box(\Pi\varphi \to \Pi\phi) \to (\Box\Pi\varphi \to \Box\Pi\phi)$. Since $\Box(\Pi\varphi \to \Pi\phi) \to (\Box\Pi\varphi \to \Box\Pi\phi)$ is an instance of K, it itself is the K-derivation we are searching for.

We have proved then that, in the case where α is an instance of one of the axioms of $\mathrm{K}_?$, if $A \oplus B \vdash_{\mathrm{K}_?} \alpha$ then $\Pi(A) \oplus \Pi(B) \vdash_{\mathrm{K}} \Pi(\alpha)$. This completes the basis of induction of the proof. Let us examine now the case where the size of the K-derivation of α from A and B is greater than 1.

Hypothesis of induction: Let $n > 1$ be the size of the $\mathrm{K}_?$-derivation of α from A and B. Suppose that, for $\mathrm{K}_?$-derivations of sizes smaller than n the result holds. That is to say, if $A \oplus B \vdash_{\mathrm{K}_?} \varphi$ and the size of the derivation of φ from A and B is smaller than n, then $\Pi(A) \oplus \Pi(B) \vdash_{\mathrm{K}} \Pi(\varphi)$. Let $\{\alpha_1, \ldots, \alpha_n\}$ be the $\mathrm{K}_?$-derivation of α from A and B. by Definition I.3.23, $\alpha_n = \alpha$ may have been obtained in one of the following ways:

(i) α_n is an axiom of $\mathrm{K}_?$;
(ii) α_n is one of the premises ($\alpha \in A \cup B$);
(iii) There are $i, j < n$, such that $\alpha_j = \alpha_i \to \alpha_n$;

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(iv) There is $i < n$ such that $\alpha_n = \alpha_i!$ and no element of B appears in the derivation of α_i.

We have just considered the first two cases when we dealt with derivations of size 1. Let us now consider the two other cases.
Case (iii): $\alpha_j = \alpha_i \to \alpha_n$. $\Pi(\alpha_j) = \Pi(\alpha_i) \to \Pi(\alpha_n)$. Since the size of the K$_?$-derivations of α_i and α_j from A and B is smaller than i, by the hypothesis of induction we have that there is a K-derivation of $\Pi(\alpha_i)$ from $\Pi(A)$ and $\Pi(B)$ and a K-derivation of $\Pi(\alpha_j) = \Pi(\alpha_i) \to \Pi(\alpha_n)$ from $\Pi(A)$ and $\Pi(B)$. Therefore, taking these two K-derivations together and considering item d) of Definition I.3.23 (MP rule), we conclude that there is a K-derivation of $\Pi(\alpha_n)$ from $\Pi(A)$ and $\Pi(B)$.
- Case (iv): $\alpha_n = \alpha_i!$, $\Pi(\alpha_n) = \Box\Pi\alpha_i$. Since no element of B appears in the K$_?$-derivation of α_i from A and B, we are sure that there is a K$_?$-derivation of α_i from A and $\varnothing$. Since the size of such derivation is smaller than n, by the hypothesis of induction we have that there is a K-derivation of $\Pi(\alpha_i)$ from $\Pi(A)$ and $\varnothing$. Given this and taking this K-derivation along with f) of Definition I.3.23 (rule N), we conclude that there is a K-derivation of $\Pi(\alpha_n) = \Box\Pi\alpha_i$ from $\Pi(A)$ and $\Pi(B)$. ⊣

LEMMA 2.10. *Let $\alpha \in \mathrm{L}_\Diamond$ be a formula, $M = \langle W, R, v\rangle$ a model and $w \in W$ a world of M. $M, w \Vdash_{\Psi\Diamond} \alpha$ iff $M, w \Vdash_{\Omega?} \triangle(\alpha)$ or, equivalently, $\Psi_{\Diamond M,w}(\alpha) = 1$ iff $\Omega_{?M,w}(\triangle(\alpha)) = 1$.*

PROOF. We will prove this lemma by induction on the size of α.

Base of induction: $\alpha = p$. In this case the result trivially holds, for $\triangle(p) = p$.

Hypothesis of induction: Let α be an arbitrary formula. Suppose the result holds for all formulas φ of size $m < n$, where n is α's size. We have to prove that, if this is the case, the result also holds for α. This will be done by considering all possible forms α may have. The only situation which poses some difficulty is the case where $\alpha = \neg\varphi$. For all others, the proof is trivial. $\alpha = \neg\varphi$. $\triangle\alpha = \triangle(\neg\varphi) = \neg\nabla\varphi$. We have then to prove that $\Psi_{\Diamond M,w}(\neg\varphi) = 1$ iff $\Omega_{?M,w}(\neg\nabla\varphi) = 1$. That will be done by induction on the size of φ.

Basis of Induction: $\varphi = p$. This case is trivial, for $\nabla p = \triangle p = p$.

Hypothesis of induction (which, in order to be distinguished from the first hypothesis of induction, will be referred to as the second hypothesis of induction): Suppose that the result holds for formulas of size smaller than φ's size. We will show that, if this supposition holds, independently of the form of φ, the general result that $\Psi_{\Diamond M,w}(\neg\varphi) = 1$ iff $\Omega_{?M,w}(\neg\nabla\varphi) = 1$ also holds. As usual, we will consider all forms φ may have.

$\varphi = \Box\phi$. $\neg\nabla\varphi = \neg\nabla\Box\phi = \neg(\nabla\phi?)$.
$\Omega_{?M,w}(\neg(\nabla\phi?)) = 1$ iff $\mho_{?M,w}(\nabla\phi?) = 0$ iff, for at least one $w' \in W$ such that wRw', $\mho_{?M,w'}(\nabla\phi) = 0$. $\Psi_{\Diamond M,w}(\neg\Box\phi) = 1$ iff $\Psi_{\Diamond M,w}(\Box\phi) = 0$ iff, for at least one $w' \in W$ such that wRw', $\Psi_{\Diamond M,w'}(\phi) = 0$. If $\Psi_{\Diamond M,w}(\neg\Box\phi) = 1$, then, for at least one $w' \in W$ such that wRw', $\Psi_{\Diamond M,w'}(\phi) = 0$ or, equivalently, $\Psi_{\Diamond M,w'}(\neg\phi) = 1$. Since ϕ's size is smaller than φ's, by our second hypothesis of induction we have that $\Omega_{?M,w'}(\neg\nabla\phi) = 1$. Since $\Omega_{?M,w'}(\neg\nabla\phi) = 1$ iff $\mho_{?M,w'}(\nabla\phi) = 0$, we have that, for at least one $w' \in W$ such that wRw', $\mho_{?M,w'}(\nabla\phi) = 0$, which implies that $\Omega_{?M,w}(\neg(\nabla\phi?)) = 1$. If $\Omega_{?M,w}(\neg(\nabla\phi?)) = 1$, then, for at least one $w' \in W$ such that wRw', $\mho_{?M,w'}(\nabla\phi) = 0$, or, equivalently, $\Omega_{?M,w'}(\neg\nabla\phi) = 1$. Since ϕ's size is smaller than φ's, by our second hypothesis of induction we have that $\Psi_{\Diamond M,w'}(\neg\phi) = 1$. Since $\Psi_{\Diamond M,w'}(\neg\phi) = 1$ iff $\Psi_{\Diamond M,w'}(\phi) = 0$, we have that, for at least one $w' \in W$ such that wRw', $\Psi_{\Diamond M,w'}(\phi) = 0$, which implies that $\Psi_{\Diamond M,w}(\neg\Box\phi) = 1$.

$\varphi = \Diamond\phi$. $\neg\nabla\varphi = \neg\nabla\Diamond\phi = \neg(\nabla\phi!)$. The proof of this case is almost identical to the previous one. We have just to change the occurrences of ! by ?, and of $\Box$ by $\Diamond$, and where it appears the expression "for at least one" we write "for all".

$\varphi = \neg\phi$. $\neg\nabla\neg\phi = \neg\neg\triangle\phi$.
$\Omega_{?M,w}(\neg\neg\triangle\phi)=1$ iff $\mho_{?M,w}(\neg\triangle\phi)=0$ iff $\Omega_{?M,w}(\triangle\phi)=1$. $\Psi_{\Diamond M,w}(\neg\neg\phi) = 1$ iff $\Psi_{\Diamond M,w}(\neg\phi) = 0$ iff $\Psi_{\Diamond M,w}(\phi) = 1$. If $\Psi_{\Diamond M,w}(\neg\neg\phi) = 1$, then $\Psi_{\Diamond M,w}(\phi) = 1$. Since ϕ's size is smaller than α's, by the (first) hypothesis of induction, we have that $\Omega_{?M,w}(\triangle\phi) = 1$. Therefore, $\Omega_{?M,w}(\neg\neg\triangle\phi) = 1$. If $\Omega_{?M,w}(\neg\neg\triangle\phi) = 1$, then we have $\Omega_{?M,w}(\triangle\phi) = 1$. Since ϕ's size is smaller than α's, by the (first) hypothesis of induction, we have that $\Psi_{\Diamond M,w}(\phi) = 1$. Therefore, $\Psi_{\Diamond M,w}(\neg\neg\phi) = 1$.

$\varphi=\phi \to \lambda$. $\neg\nabla(\phi \to \lambda)=\neg(\triangle\phi \to \nabla\lambda)$. $\Omega_{?M,w}(\neg(\triangle\phi \to \nabla\lambda))= 1$ iff $\mho_{?M,w}(\triangle\phi \to \nabla\lambda) = 0$ iff $\Omega_{?M,w}(\triangle\phi) = 1$ and $\mho_{?M,w}(\nabla\lambda) = 0$. $\Psi_{\Diamond M,w}(\neg(\phi \to \lambda)) = 1$ iff $\Psi_{\Diamond M,w}(\phi \to \lambda) = 0$ iff $\Psi_{\Diamond M,w}(\phi) = 1$ and $\Psi_{\Diamond M,w}(\lambda) = 0$. If $\Psi_{\Diamond M,w}(\neg(\phi \to \lambda)) = 1$, then $\Psi_{\Diamond M,w}(\phi) = 1$ and $\Psi_{\Diamond M,w}(\lambda)=0$, which is equivalent to $\Psi_{\Diamond M,w}(\phi)=1$ and $\Psi_{\Diamond M,w}(\neg\lambda)=1$. Since ϕ's size is smaller than α's, by the (first) hypothesis of induction we have that $\Omega_{?M,w}(\triangle\phi)$=1, and since λ's size is smaller than φ's, by the second hypothesis of induction we have that $\Omega_{?M,w}(\neg\nabla\lambda)$=1, which is equivalent to $\mho_{?M,w}(\nabla\lambda)$=0. We therefore have $\Omega_{?M,w}(\neg(\triangle\phi \to \nabla\lambda))$=

1. If $\Omega_{?M,w}(\neg(\triangle\phi \to \nabla\lambda))=1$, then $\Omega_{?M,w}(\triangle\phi)=1$ and $\mho_{?M,w}(\nabla\lambda)=0$, which is equivalent to $\Omega_{?M,w}(\triangle\phi)=1$ and $\Omega_{?M,w}(\neg\nabla\lambda)=1$. Since ϕ's size is smaller than α's, by the first hypothesis of induction we have that $\varPsi_{\Diamond M,w}(\phi)=1$, and since λ's size is smaller than φ, by the second hypothesis of induction we have that $\varPsi_{\Diamond M,w}(\neg\lambda)=1$, which is equivalent to $\varPsi_{\Diamond M,w}(\phi)=1$ and $\varPsi_{\Diamond M,w}(\lambda)=0$. Therefore, $\varPsi_{\Diamond M,w}(\neg(\phi \to \lambda))=1$.

$\varphi = \phi \wedge \lambda$. $\neg\nabla(\phi \wedge \lambda) = \neg(\nabla\phi \wedge \nabla\lambda)$. $\Omega_{?M,w}(\neg(\nabla\phi \wedge \nabla\lambda))=1$ iff $\mho_{?M,w}(\nabla\phi \wedge \nabla\lambda)=0$ iff $\mho_{?M,w}(\nabla\phi)=0$ or $\mho_{?M,w}(\nabla\lambda)=0$. $\varPsi_{\Diamond M,w}(\neg(\phi \wedge \lambda)) = 1$ iff $\varPsi_{\Diamond M,w}(\phi \wedge \lambda) = 0$ iff $\varPsi_{\Diamond M,w}(\phi) = 0$ or $\varPsi_{\Diamond M,w}(\lambda) = 0$. If $\varPsi_{\Diamond M,w}(\neg(\phi \wedge \lambda)) = 1$, then $\varPsi_{\Diamond M,w}(\phi) = 0$ or $\varPsi_{\Diamond M,w}(\lambda) = 0$, which is equivalent to $\varPsi_{\Diamond M,w}(\neg\phi) = 1$ or $\varPsi_{\Diamond M,w}(\neg\lambda) = 1$. Since ϕ's and λ's sizes are smaller than φ's, by the second hypothesis of induction we have that $\Omega_{?M,w}(\neg\nabla\phi)=1$ or $\Omega_{?M,w}(\neg\nabla\lambda)=1$, which is equivalent to $\mho_{?M,w}(\nabla\phi)=0$ or $\mho_{?M,w}(\nabla\lambda)=0$. Therefore, $\Omega_{?M,w}(\neg(\nabla\phi \wedge \nabla\lambda))=1$. If $\Omega_{?M,w}(\neg(\nabla\phi \wedge \nabla\lambda))=1$, then $\mho_{?M,w}(\nabla\phi)=0$ or $\mho_{?M,w}(\nabla\lambda)=0$, which is equivalent to $\Omega_{?M,w}(\neg\nabla\phi)=1$ or $\Omega_{?M,w}(\neg\nabla\lambda)=1$. Since ϕ's and λ's sizes are smaller than φ, by the second hypothesis of induction we have that $\varPsi_{\Diamond M,w}(\neg\phi)=1$ or $\varPsi_{\Diamond M,w}(\neg\lambda)=1$, which is equivalent to $\varPsi_{\Diamond M,w}(\phi)=0$ or $\varPsi_{\Diamond M,w}(\lambda)=0$. Therefore, $\varPsi_{\Diamond M,w}(\neg(\phi \wedge \lambda))=1$.

$\varphi = \phi \vee \lambda$. The proof of this case is almost identical to the previous one. We have just to replace all occurrences of $\wedge$ by $\vee$ and the relevant occurrences of “and” by “or”. $\dashv$

LEMMA 2.11. *Let $\alpha \in \mathrm{L}_?$, M be a model and w a world of M. $M, w \Vdash_{\Omega_?} \alpha$ iff $M, w \Vdash_{\varPsi_\Diamond} \Pi(\alpha)$ or, equivalently, $\Omega_{?M,w}(\alpha)=1$ iff $\varPsi_{\Diamond M,w}(\Pi(\alpha))=1$.*

PROOF. The proof of this lemma is almost identical to lemma 2.10's. All we have to do is to properly erase the occurrences of $\triangle$ and consider function Π along with $\varPsi$. $\dashv$

THEOREM 2.5. *For any $A, B \subseteq \mathrm{L}_\Diamond$ and $\alpha \in \mathrm{L}_\Diamond$:*

$$A \oplus B \vdash_{\mathrm{K}} \alpha \quad \text{iff} \quad \triangle(A) \oplus \triangle(B) \vdash_{\mathrm{K}_?} \triangle(\alpha)\,,$$
$$A \oplus B \vDash_{\mathrm{K}} \alpha \quad \text{iff} \quad \triangle(A) \oplus \triangle(B) \vDash_{\mathrm{K}_?} \triangle(\alpha)\,.$$

PROOF. By lemma 2.7, if $A \oplus B \vdash_{\mathrm{K}} \alpha$ then $\triangle(A) \oplus \triangle(B) \vdash_{\mathrm{K}_?} \triangle(\alpha)$. By contraposition: if $A \oplus B \nvdash_{\mathrm{K}} \alpha$, by lemma 2.4, we have that $\Pi(\triangle(A)) \oplus \Pi(\triangle(B)) \nvdash_{\mathrm{K}} \Pi(\triangle(\alpha))$. By lemma 2.9, we have then that $\triangle(A) \oplus \triangle(B) \nvdash_{\mathrm{K}_?} \triangle(\alpha)$.

For the the second equivalence suppose that $\triangle(A) \oplus \triangle(B) \nvDash_{\mathrm{K}_?} \triangle(\alpha)$; then there is a model M and a world w of M such that $M \Vdash_{\Omega_?} \triangle(\phi)$, for

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all $\triangle(\phi) \in \triangle(A)$, $M, w \Vdash_{\Omega_?} \triangle(\lambda)$, for all $\triangle(\lambda) \in \triangle(B)$, and $M, w \nVdash_{\Omega_?} \triangle(\alpha)$. But if $M \Vdash_{\Omega_?} \triangle(\phi)$ for all $\triangle(\phi) \in \triangle(A)$, $M, w \Vdash_{\Omega_?} \triangle(\lambda)$ for all $\triangle(\lambda) \in \triangle(B)$ and $M, w \nVdash_{\Omega_?} \triangle(\alpha)$, by lemma 2.10 we have that $M \Vdash_{\Psi_\Diamond} \phi$ for all $\phi \in A$, $M, w \Vdash_{\Psi_\Diamond} \lambda$ for all $\lambda \in B$ and $M, w \nVdash_{\Psi_\Diamond} \alpha$. Consequently, $A \oplus B \nvDash_{\mathrm{K}} \alpha$. Therefore, if $A \oplus B \vDash_{\mathrm{K}} \alpha$, $\triangle(A) \oplus \triangle(B) \vDash_{\mathrm{K}_?} \triangle(\alpha)$.

Suppose now that $A \oplus B \nvDash_{\mathrm{K}} \alpha$; then there is a model M and a world w of M such that $M \Vdash_{\Psi_\Diamond} \phi$ for all $\phi \in A$, $M, w \Vdash_{\Psi_\Diamond} \lambda$ for all $\lambda \in B$ and $M, w \nVdash_{\Psi_\Diamond} \alpha$. But if $M \Vdash_{\Psi_\Diamond} \phi$ for all $\phi \in A$, $M, w \Vdash_{\Psi_\Diamond} \lambda$ for all $\lambda \in B$ and $M, w \nVdash_{\Psi_\Diamond} \alpha$, then by lemma 2.10: $M \Vdash_{\Omega_?} \triangle(\phi)$ for all $\triangle(\phi) \in \triangle(A)$, $M, w \Vdash_{\Omega_?} \triangle(\lambda)$ for all $\triangle(\lambda) \in \triangle(B)$ and $M, w \nVdash_{\Omega_?} \triangle(\alpha)$. Consequently, $\triangle(A) \oplus \triangle(B) \nvDash_{\mathrm{K}_?} \triangle(\alpha)$. Therefore, if $\triangle(A) \oplus \triangle(B) \vDash_{\mathrm{K}_?} \triangle(\alpha)$ then $A \oplus B \vDash_{\mathrm{K}} \alpha$. ⊣

THEOREM 2.6. *For any* $A, B \subseteq \mathrm{L}_?$ *and* $\alpha \in \mathrm{L}_?$:

$$A \oplus B \vdash_{\mathrm{K}_?} \alpha \quad \text{iff} \quad \prod(A) \oplus \prod(B) \vdash_{\mathrm{K}} \prod(\alpha)\,,$$
$$A \oplus B \vDash_{\mathrm{K}_?} \alpha \quad \text{iff} \quad \prod(A) \oplus \prod(B) \vDash_{\mathrm{K}} \prod(\alpha)\,.$$

PROOF. The proof of the first equivalence is almost identical to the proof of the first equivalence in theorem 2.5: switch $\vdash_{\mathrm{K}_?}$ and $\vdash_{\mathrm{K}}$, lemma 2.9 and lemma 2.7 and replace $\triangle$ for Π.

Respectively, the proof of the second equivalence follows the idea of the proof of the second equivalence in theorem 2.5: properly erase the occurrences of $\triangle$ and consider function Π along with Ψ as well as use lemma 2.11 instead of lemma 2.10. ⊣

Theorems 2.5 and 2.6 show that both from a proof-theoretical and from a semantic point of view, K and $\mathrm{K}_?$ are intertranslatable. In other words, while by using function $\triangle$ we can translate any inferential relation in K into an inferential relation in $\mathrm{K}_?$, using $\prod$ we can translate any inferential relation in $\mathrm{K}_?$ into an inferential relation in K. As a consequence of this, we can say normal and paranormal logics can be fully embedded inside each other: with paranormal modal logic at hand we can obtain normal modal logic, and vice versa. The implications of this are obvious. For instance, since formulas resulting from the application of $\triangle$ can be seen as abbreviations inside $\mathrm{L}_\Diamond$, it might be said that there is a formal paraconsistent and paracomplete inferential relation (in addition to a conceptual one) based on a true paranormal modality-dependent negation inside normal modal logic. This we think strengths the thesis we have mentioned in Section 1 of [4] about normal modal logic being paranormal.

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This result make give room for a sort of objection that questions the whole worthiness of our endeavor: if all the expressive and inferential power of paranormal modal logic is already contained in normal modal logic, what is the point of developing and studying it? We may reply by turning the question around. Since, as theorems 2.5 and 2.6 show, all the expressive and inferential power of normal modal logic is contained in paranormal modal logic, why not question instead the supremacy (if we may use this term) of normal modal logic instead? After all, given the equivalence stated above between the two systems, the only real reason for this supremacy is the historical one that one logic was discovered, created, or whatever, before the other. Of course things are not so simple. In fact, the whole situation parallels the possibility of translating S5 into the monadic fragment of classical first-order logic with only one variable, and vice versa.

As far as we are concerned, we prefer to stick to the formal apparatus available to us and think of two different but very strongly connected formal systems; so strongly connected that they might be taken as different aspects of the same thing. A comparison that comes to mind here is with those transformer toys which at one time look like a car and at other time look like a completely different object, such as an airplane. Despite the toy being, at one specific time, from the point of view of the child who plays with it at that time, only a car, all the materials needed for the airplane are already there, inside the car.

3. Other facts about $K_?$

In this section we shall lay down some important facts about $K_?$. More specifically, we shall present and prove $K_?$'s soundness and completeness, some important logical theorems of $K_?$ as well as some interesting formulas which are not $K_?$-theorems and $K_?$'s theorems of deduction.

Theorem 3.1. $K_?$ *is sound and complete, i.e., for any* $A, B \subseteq L_?$, $\alpha \in L_?$:

$$A \oplus B \vdash_{K_?} \alpha \quad \textit{iff} \quad A \oplus B \vDash_{K_?} \alpha\,.$$

Proof. Let us first prove the left-right direction (soundness): for any $A, B \subseteq L_?$ and $\alpha \in L_?$, if $A \oplus B \vdash_{K_?} \alpha$ then $A \oplus B \vDash_{K_?} \alpha$. Suppose that $A \oplus B \nvDash_{K_?} \alpha$; by theorem 2.6 we have that $\prod(A) \oplus \prod(B) \nvDash_{K} \prod(\alpha)$. By the soundness theorem of normal modal logic K[7], we have

[7] For the proof of soundness and completeness of normal modal logic K with local and global premises see [2]. See also [1] and [3].

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$\prod(A) \oplus \prod(B) \nvdash_{\mathrm{K}} \prod(\alpha)$. From that, along with theorem 2.6, we have that $A \oplus B \nvDash_{\mathrm{K}_?} \alpha$. Therefore, if $A \oplus B \vdash_{\mathrm{K}_?} \alpha$ then $A \oplus B \vDash_{\mathrm{K}_?} \alpha$.

The right-left direction (completeness) is proved through the same reference to normal modal logic K. Suppose that $A \oplus B \nvdash_{\mathrm{K}_?} \alpha$; by theorem 2.6 we have that $\prod(A) \oplus \prod(B) \nvdash_{\mathrm{K}} \prod(\alpha)$. By the completeness theorem of normal modal logic K, we have then that $\prod(A) \oplus \prod(B) \nvDash_{\mathrm{K}} \prod(\alpha)$. From that, along with theorem 2.6, we have that $A \oplus B \nvDash_{\mathrm{K}_?} \alpha$. Therefore, if $A \oplus B \vDash_{\mathrm{K}_?} \alpha$ then $A \oplus B \vdash_{\mathrm{K}_?} \alpha$. ⊣

THEOREM 3.2. *For every schema below, there is a formula of* $\mathrm{L}_?$ *falling under it that is not* $\mathrm{K}_?$*-valid (and consequently not* $\mathrm{K}_?$*-theorem):*

$(\alpha \to \beta) \to (\alpha \to \neg\beta) \to \neg\alpha$	$\neg\alpha \to (\alpha \to \beta)$
$\neg\alpha \vee \alpha$	$\neg(\alpha \wedge \neg\alpha)$
$(\alpha \to \beta) \to (\neg\beta \to \neg\alpha)$	$(\neg\beta \to \neg\alpha) \to (\alpha \to \beta)$
$\neg\alpha \vee \beta \to (\alpha \to \beta)$	$(\alpha \to \beta) \to \neg\alpha \vee \beta$
$(\alpha \to \beta) \to \neg(\alpha \wedge \neg\beta)$	$\neg(\alpha \wedge \neg\beta) \to (\alpha \to \beta)$
$\neg\alpha \to (\alpha \to \neg\beta)$	$(\alpha \to \neg\alpha) \to \neg\alpha$

PROOF. In order to prove this theorem, it suffices to show an instance of each one of these formula-schemas that is not $\mathrm{K}_?$-valid. (From theorem 3.1 it follows that it is not $\mathrm{K}_?$-theorem.) Having picked such a formula, we need then only to show a model $M = \langle W, R, v\rangle$ and a world $w \in W$ such that the formula is not $\Omega_?$-satisfied by M at w. For all formulas, there will be two such models: Model 1: $w \in W$ such that $v_w(q) = 1$ and there are $w', w'' \in W$ such that wRw' and wRw'' such that $v_{w'}(p) = 1$ and $v_{w''}(p) = 0$; and Model 2: $w \in W$ such that $v_w(q) = 0$ and there are w', $w'' \in W$ such that wRw' and wRw'' such that $v_{w'}(p) = 1$ and $v_{w''}(p) = 0$. We then show, for each one of the schemas, the mentioned instance along with the corresponding falsifying model: $(q \to p?) \to ((q \to \neg(p?)) \to \neg q)$ [Model 1]; $\neg(p!) \vee p!$ [either Model 1 or 2]; $(q \to p?) \to (\neg(p?) \to \neg q)$ [Model 1]; $\neg(p?) \vee q \to (p? \to q)$ [Model 2]; $(q \to p?) \to \neg(q \wedge \neg(p?))$ [Model 1]; $\neg(p?) \to (p? \to \neg q)$ [Model 1]; $\neg(p?) \to (p? \to q)$ [Model 2]; $\neg(p? \wedge \neg(p?))$ [either Model 1 or 2]; $(\neg(p!) \to \neg q) \to (q \to p!)$ [Model 1]; $(p! \to q) \to \neg(p!) \vee q$ [Model 2]; $\neg(q \wedge \neg(p!)) \to (q \to p!)$ [Model 1]; $(p! \to \neg(p!)) \to \neg(p!)$ [either Model 1 or 2]. ⊣

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Theorem 3.2 indicates from a proof-theoretical point of view in which respects paranormal logic differs from classical logic (and consequently from normal modal logic.) Taking the {P1–P8, A1–A3} corresponding axiomatization of classical logic, all the above formula-schemas need one of the axioms A1–A3 to be derived. But since A1–A3 can be used only if certain restrictions are satisfied, none of the above schemas can be unrestrictedly derived.

THEOREM 3.3. *All formulas of* $L_?$ *falling under one of the schemas from Theorem 3.2, wherein* α *and* β *are* ?!*-free formulas, are* $K_?$*-valid (and consequently* $K_?$*-theorems).*

THEOREM 3.4. *All formulas of* $L_?$ *falling under one of the following schemas are* $K_?$*-valid (and consequently* $K_?$*-theorems):*

$$(\alpha \to \beta) \to (\alpha \to \sim\beta) \to \sim\alpha \qquad \sim\alpha \to (\alpha \to \beta)$$
$$\sim\alpha \vee \alpha \qquad \sim(\alpha \wedge \sim\alpha)$$
$$(\alpha \to \beta) \to (\sim\beta \to \sim\alpha) \qquad (\sim\beta \to \sim\alpha) \to (\alpha \to \beta)$$
$$\sim\alpha \vee \beta \to (\alpha \to \beta) \qquad (\alpha \to \beta) \to \sim\alpha \vee \beta$$
$$(\alpha \to \beta) \to \sim(\alpha \wedge \sim\beta) \qquad \sim(\alpha \wedge \sim\beta) \to (\alpha \to \beta)$$
$$\sim\alpha \to (\alpha \to \sim\beta) \qquad (\alpha \to \sim\alpha) \to \sim\alpha$$

While Theorem 3.3 shows that paranormal modal logic behaves like classical logic when only !?-free formulas are taken into account, Theorem 3.4 shows that in fact paranormal modal logic behaves like classical logic when we consider just classical negation $\sim$.

THEOREM 3.5. *All formulas of* $L_?$ *falling under one of the following schemas are* $K_?$*-valid (and consequently* $K_?$*-theorems):*

$$(\alpha \to \beta)! \to (\alpha! \to \beta!) \qquad (\alpha \to \beta)! \to (\alpha? \to \beta?)$$
$$(\alpha \wedge \beta)! \leftrightarrow \alpha! \wedge \beta! \qquad (\alpha \wedge \beta)? \to (\alpha? \wedge \beta?)$$
$$\alpha! \vee \beta! \to (\alpha \vee \beta)! \qquad (\alpha? \vee \beta?) \leftrightarrow (\alpha \vee \beta)?$$
$$(\alpha \to \beta)? \leftrightarrow (\alpha! \to \beta?) \qquad (\alpha \vee \beta)! \to (\alpha! \vee \beta?)$$
$$\sim(\alpha!) \leftrightarrow (\sim\alpha)? \qquad \sim(\alpha?) \leftrightarrow (\sim\alpha)!$$
$$\sim(\alpha!) \vee (\sim\alpha)? \qquad \sim(\alpha?) \vee (\sim\alpha)!$$

THEOREM 3.6. *The following schemas of relations between sets of formulas and formula are sound:*

$$\{\alpha \to \beta\} \vdash_{K_?} (\alpha \to \beta)! \qquad \{\alpha \to \beta\} \vDash_{K_?} (\alpha \to \beta)!$$

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$$\{\alpha \to \beta\} \vdash_{K_?} \alpha! \to \beta! \qquad \{\alpha \to \beta\} \vDash_{K_?} \alpha! \to \beta!$$
$$\{\alpha \to \beta\} \vdash_{K_?} \alpha? \to \beta? \qquad \{\alpha \to \beta\} \vDash_{K_?} \alpha? \to \beta?$$

Theorems 3.5 and 3.6 show the similarity between ! and ? and normal modal operators $\Box$ and $\Diamond$. First, when only positive formula-schemas are considered, every theorem of normal modal logic is also a theorem in paranormal modal logic. (Here we are considering just $K_?$, but clearly, as it will become evident later, this applies to all extensions of $K_?$). Second, when we consider external negation $\sim$, all theorems of normal modal logic, without exception, are also theorems of paranormal modal logic. The difference between ! and ? and $\Box$ and $\Diamond$ will appear only when we consider the paranormal negation $\neg$.

Theorem 3.7. *For every schema below, there is a formula of* $L_?$ *falling under it that is not* $K_?$*-valid (and consequently not a* $K_?$*-theorem):*

$$\alpha? \to \neg((\neg\alpha)!) \qquad \alpha! \to \neg((\neg\alpha)?)$$
$$\neg((\neg\alpha)!) \to \alpha? \qquad \neg((\neg\alpha)?) \to \alpha!$$
$$\neg(\alpha!) \to (\neg\alpha)? \qquad \neg(\alpha?) \to (\neg\alpha)!$$
$$(\neg\alpha)? \to \neg(\alpha!) \qquad (\neg\alpha)! \to \neg(\alpha?)$$
$$\alpha! \vee \neg(\alpha!) \qquad \neg(\alpha! \wedge \neg(\alpha!))$$
$$\neg(\alpha? \wedge \neg(\alpha?)) \qquad \alpha? \vee \neg(\alpha?)$$

Theorem 3.7 shows the distinguishing features of ! and ? when taken in connection with $\neg$. It is interesting to note that it is not only ? that disrespects the principle of non-contradiction in its intra-logical form: ! does not satisfy it either; and it is not only ! that disrespects the middle excluded principle: ? does not satisfy it either. This is because the following sorts of formulas are not $K_?$-theorems:

$$\neg(\alpha?! \wedge \neg(\alpha?!)) \qquad \alpha!? \vee \neg(\alpha!?)\,.$$

Definition 3.1. Let $\mathfrak{L}$ be a language and ϑ a ?-modal logic basis. For any $\alpha \in \mathfrak{L}_\vartheta$ and $n \geqslant 0$ we define the following abbreviation:

$$\alpha!^0 := \alpha$$
$$\alpha!^{n+1} := (\alpha!^n)!$$

Theorem 3.8. *Let* $A, B \subseteq L_?$ *and* $\alpha, \varphi \in L_?$*. Then*

$$A \oplus B \cup \{\varphi\} \vdash_{K_?} \alpha \quad \textit{iff} \quad A \oplus B \vdash_{K_?} \varphi \to \alpha\,.$$

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PROOF. Suppose that $A \oplus B \cup \varphi \vdash_{K_?} \alpha$[8] but $A \oplus B \nvdash_{K_?} \varphi \to \alpha$. If $A \oplus B \nvdash_{K_?} \varphi \to \alpha$, by theorem 2.6, $\Pi(A) \oplus \Pi(B) \nvdash_{K} \Pi(\varphi \to \alpha) = \Pi(\varphi) \to \Pi(\alpha)$. Then, by K's local deduction theorem[9], we have that $\Pi(A) \oplus \Pi(B) \cup \Pi(\varphi) \nvdash_{K} \Pi(\alpha)$, which is the same as $\Pi(A) \oplus \Pi(B \cup \varphi) \nvdash_{K} \Pi(\alpha)$. But then, by theorem 2.6 again, we have that $A \oplus B \cup \varphi \nvdash_{K_?} \alpha$, which is a contradiction. Therefore, if $A \oplus B \cup \varphi \vdash_{K_?} \alpha$ then $A \oplus B \vdash_{K_?} \varphi \to \alpha$. Suppose now that $A \oplus B \vdash_{K_?} \varphi \to \alpha$ but $A \oplus B \cup \varphi \nvdash_{K_?} \alpha$. If $A \oplus B \cup \varphi \nvdash_{K_?} \alpha$, by theorem 2.6, $\Pi(A) \oplus \Pi(B \cup \varphi) \nvdash_{K} \Pi(\alpha)$, which is the same as $\Pi(A) \oplus \Pi(B) \cup \Pi(\varphi) \nvdash_{K} \Pi(\alpha)$. Then, by K's local deduction theorem, $\Pi(A) \oplus \Pi(B) \nvdash_{K} \Pi(\varphi) \to \Pi(\alpha) = \Pi(\varphi \to \alpha)$. But by theorem 2.6, we have that $A \oplus B \nvdash_{K_?} \varphi \to \alpha$, which is a contradiction. Therefore, if $A \oplus B \vdash_{K_?} \varphi \to \alpha$ then $A \oplus B \cup \varphi \vdash_{K_?} \alpha$. ⊣

THEOREM 3.9. *Let* $A, B \subseteq L_?$ *and* $\alpha, \varphi \in L_?$. *Then*

$$A \cup \{\varphi\} \oplus B \vdash_{K_?} \alpha \quad \textit{iff} \quad \textit{for some } n \geqslant 0,\ A \oplus B \cup \{\varphi!^0, \varphi!^1, \ldots, \varphi!^n\} \vdash_{K_?} \alpha\,.$$

THEOREM 3.10. *Let* $A, B \subseteq L_?$ *and* $\alpha, \beta \in L_?$. *Then*

$$A \oplus B \cup \{\beta\} \vDash_{K_?} \alpha \quad \textit{iff} \quad A \oplus B \vDash_{K_?} \beta \to \alpha\,.$$

THEOREM 3.11. *Let* $A, B \subseteq L_?$ *and* $\alpha, \varphi \in L_?$. *Then*

$$A \cup \{\varphi\} \oplus B \vDash_{K_?} \alpha \quad \textit{iff} \quad \textit{for some } n \geqslant 0,\ A \oplus B \cup \{\varphi!^0, \varphi!^1, \ldots, \varphi!^n\} \vDash_{K_?} \alpha\,.$$

Theorems 3.8–3.11 lay down the syntactic and semantic forms of both local (theorems 3.8 and 3.10) and global (theorems 3.9 and 3.11) deduction theorems of paranormal modal logic. They are equivalent to deduction theorems of normal modal logic as stated, for example, in [2].

4. Other Paranormal Modal Logics

In this section we show how $K_?$ can be extended in such a way as to obtain other paranormal modal logics. In the next subsection we consider how we can do this by adding extra axioms on the axiomatic side or by restricting the class of frames on the semantic side. The procedure is exactly identical to the way we extend K and obtain other normal modal

[8] We shall here ignore the symbols "{" and "}" when writing down unary sets of formulas.

[9] See [2].

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logics. In Subsection 4.2 we proceed to consider first order paranormal modal logic. Since the definition of other first-order systems is identical to the propositional case, we just consider first-order paranormal modal logic $\mathrm{K}_?$. Finally, in Subsection 4.3, we consider multi-modal logics which contain both normal and paranormal modal operators; we call them multi-normal modal logics.

4.1. Extensions of $\mathbf{K}_?$

In this subsection we present some of the most important propositional paranormal modal logics. As it shall become clear when we start our exposition, for each normal modal system N there is a corresponding paranormal system $N_?$. And the way $N_?$ is obtained from $\mathrm{K}_?$ is identical to the way N is obtained from K. For instance, in the same way that we obtain T from K by taking into account only reflexive frames on the semantic side, and by adding the following axiom-schema

$\mathtt{T}$: $\Box\alpha \to \alpha$

to K's axiomatic on the axiomatic side, we obtain $\mathrm{T}_?$ from $\mathrm{K}_?$ by restricting ourselves to reflexive frames and by adding the axiom-schema $\mathtt{T}_?$ to $\mathrm{K}_?$'s axiomatic (see Section 1 of [4]).

In this subsection let $\mathfrak{L}$ be any language and let ϑ be any ?-modal logic basis. Notice that in Definition I.4.9 we put $\mathrm{K}_? := \langle\vartheta_?, \Omega_?, \mathcal{F}_\mathrm{K}, \varSigma^\star_{\mathrm{K}_?}\rangle$.

The System $\mathbf{D}_?$

Definition 4.1. The $\mathtt{D}_?$-*axioms* $\varSigma_{\mathtt{D}_?}$ *in* $\mathfrak{L}_\vartheta$ is the set composed by all formulas of $\mathfrak{L}_\vartheta$ falling under the following schema:

$\mathtt{D}_?$: $\alpha! \to \alpha?$

Definition 4.2. The propositional paranormal modal logic $\mathrm{D}_?$ is the propositional modal system $\langle\vartheta_?, \Omega_?, \mathcal{F}_\mathrm{D}, \varSigma^\star_{\mathrm{K}_?} \cup \varSigma_{\mathtt{D}_?}\rangle$, where $\mathcal{F}_\mathrm{D}$ is the class of all serial frames and $\varSigma_{\mathtt{D}_?}$ are the $\mathtt{D}_?$-axioms in $\mathrm{L}_?$.

Theorem 4.1. *For any* $A, B \subseteq \mathrm{L}_?$ *and* $\alpha \in \mathrm{L}_?$:

$$A \oplus B \vdash_{\mathrm{D}_?} \alpha \quad \text{iff} \quad A \cup \varSigma_{\mathrm{D}_?} \oplus B \vdash_{\mathrm{K}_?} \alpha\,,$$
$$A \oplus B \vDash_{\mathrm{D}_?} \alpha \quad \text{iff} \quad A \cup \varSigma_{\mathrm{D}_?} \oplus B \vDash_{\mathrm{K}_?} \alpha\,.$$

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PROOF. Suppose that $A \oplus B \vdash_{D_?} \alpha$. We can easily extend the proof of theorem 2.6 in such a way as to prove that $A \oplus B \vdash_{D_?} \alpha$ iff $\Pi(A) \oplus \Pi(B) \vdash_D \Pi(\alpha)$, where D is the normal modal extension of K obtained by the addition of axiom D: $\Box\alpha \to \Diamond\alpha$. With this result, we have that $\Pi(A) \oplus \Pi(B) \vdash_D \Pi(\alpha)$. Given then the known result that $A \oplus B \vdash_D \alpha$ iff $A \cup \Sigma_D \oplus B \vdash_K \alpha$ (where Σ_D is the set of all instances of axiom D in $L_\Diamond$)[10] we have then that $\Pi(A) \cup \Sigma_D \oplus \Pi(B) \vdash_K \Pi(\alpha)$. Since $\Sigma_D = \Pi(\Sigma_{D_?})$, $\Pi(A) \cup \Sigma_D \oplus \Pi(B) \vdash_K \Pi(\alpha)$ is the same as $\Pi(A) \cup \Pi(\Sigma_{D_?}) \oplus \Pi(B) \vdash_K \Pi(\alpha)$ or $\Pi(A \cup \Sigma_{D_?}) \oplus \Pi(B) \vdash_K \Pi(\alpha)$. By the D-version of theorem 2.6 therefore, we have that $A \cup \Sigma_{D_?} \oplus B \vdash_{K_?} \alpha$. The right-left side of the proof follows the same reasoning.

The second equivalence is obtained similarly — this time we use the other equivalence of Theorem 2.6. ⊣

THEOREM 4.2. $D_?$ *is sound and complete.*

THEOREM 4.3. *All formulas of* $L_?$ *falling under one of the following schemas are* $D_?$*-valid (and consequently* $D_?$*- theorems).*

$$(\alpha \to \alpha)? \qquad\qquad ((\alpha \to \beta)! \to (\alpha! \to \beta!))?$$

The System $T_?$

DEFINITION 4.3. The $T_?$*-axioms* $\Sigma_{T_?}$ *in* $\mathfrak{L}_\vartheta$ is the set composed by all formulas of $\mathfrak{L}_\vartheta$ falling under the following schema:

$T_?$: $\alpha! \to \alpha$

DEFINITION 4.4. The propositional paranormal modal logic $T_?$ is the propositional modal system $\langle \vartheta_?, \Omega_?, \mathcal{F}_T, \Sigma^\star_{K_?} \cup \Sigma_{T_?} \rangle$, where $\mathcal{F}_T$ is the class of all reflexive frames and $\Sigma_{T_?}$ are the $T_?$-axioms in $L_?$.

THEOREM 4.4. *For any* $A, B \subseteq L_?$ *and* $\alpha \in L_?$:

$$A \oplus B \vdash_{T_?} \alpha \quad \textit{iff} \quad A \cup \Sigma_{T_?} \oplus B \vdash_{K_?} \alpha\,,$$
$$A \oplus B \vDash_{T_?} \alpha \quad \textit{iff} \quad A \cup \Sigma_{T_?} \oplus B \vDash_{K_?} \alpha\,.$$

THEOREM 4.5. $T_?$ *is sound and complete.*

THEOREM 4.6. *All formulas of* $L_?$ *falling under one of the following schemas are* $T_?$*-valid (and consequently* $T_?$*-theorems):*

$$\alpha \to \alpha? \qquad\qquad (\alpha \to \alpha!)?$$

[10] See [2].

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The System $B_?$

DEFINITION 4.5. The $\mathtt{B}_?$-*axioms* $\Sigma_{\mathtt{B}_?}$ *in* $\mathfrak{L}_\vartheta$ is the set composed by all formulas of $\mathfrak{L}_\vartheta$ falling under the following schema:

$\mathtt{B}_?$: $\alpha \to \alpha?!$

DEFINITION 4.6. The propositional paranormal modal logic $\mathrm{B}_?$ is the propositional modal system $\langle \vartheta_?, \Omega_?, \mathcal{F}_{\mathrm{B}}, \Sigma^{\star}_{\mathrm{K}_?} \cup \Sigma_{\mathtt{T}_?} \cup \Sigma_{\mathtt{B}_?} \rangle$, where $\mathcal{F}_{\mathrm{B}}$ is the class of all reflexive and symmetric frames, $\Sigma_{\mathtt{T}_?}$ are the $\mathtt{T}_?$-axioms and $\Sigma_{\mathtt{B}_?}$ are the $\mathtt{B}_?$-axioms in $\mathrm{L}_?$.

THEOREM 4.7. *For any* $A, B \subseteq \mathrm{L}_?$ *and* $\alpha \in \mathrm{L}_?$:

$$A \oplus B \vdash_{\mathrm{B}_?} \alpha \quad \textit{iff} \quad A \cup \Sigma_{\mathtt{T}_?} \cup \Sigma_{\mathtt{B}_?} \oplus B \vdash_{\mathrm{K}_?} \alpha\,,$$
$$A \oplus B \vDash_{\mathrm{B}_?} \alpha \quad \textit{iff} \quad A \cup \Sigma_{\mathtt{T}_?} \cup \Sigma_{\mathtt{B}_?} \oplus B \vDash_{\mathrm{K}_?} \alpha\,.$$

THEOREM 4.8. $\mathrm{B}_?$ *is sound and complete.*

The System $S4_?$

DEFINITION 4.7. The $\mathtt{4}_?$-*axioms* $\Sigma_{\mathtt{4}_?}$ *in* $\mathfrak{L}_\vartheta$ is the set composed by all formulas of $\mathfrak{L}_\vartheta$ falling under the following schema:

$\mathtt{4}_?$: $\alpha! \to \alpha!!$

DEFINITION 4.8. The propositional paranormal modal logic $\mathrm{S4}_?$ is the propositional modal system $\langle \vartheta_?, \Omega_?, \mathcal{F}_{\mathrm{S4}}, \Sigma^{\star}_{\mathrm{K}_?} \cup \Sigma_{\mathtt{T}_?} \cup \Sigma_{\mathtt{4}_?} \rangle$, where $\mathcal{F}_{\mathrm{S4}}$ is the class of all reflexive and transitive frames, $\Sigma_{\mathtt{T}_?}$ are the $\mathtt{T}_?$-axioms and $\Sigma_{\mathtt{4}_?}$ are the $\mathtt{4}_?$-axioms in $\mathrm{L}_?$.

THEOREM 4.9. *For any* $A, B \subseteq \mathrm{L}_?$ *and* $\alpha \in \mathrm{L}_?$:

$$A \oplus B \vdash_{\mathrm{S4}_?} \alpha \quad \textit{iff} \quad A \cup \Sigma_{\mathtt{T}_?} \cup \Sigma_{\mathtt{4}_?} \oplus B \vdash_{\mathrm{K}_?} \alpha\,,$$
$$A \oplus B \vDash_{\mathrm{S4}_?} \alpha \quad \textit{iff} \quad A \cup \Sigma_{\mathtt{T}_?} \cup \Sigma_{\mathtt{4}_?} \oplus B \vDash_{\mathrm{K}_?} \alpha\,.$$

THEOREM 4.10. $\mathrm{S4}_?$ *is sound and complete.*

THEOREM 4.11. *All formulas of* $\mathrm{L}_?$ *falling under one of the following schemas are* $\mathrm{S4}_?$-*valid* (*and consequently* $\mathrm{S4}_?$-*theorems*):

$$\alpha?? \to \alpha? \qquad\qquad \alpha?!? \to \alpha?$$
$$\alpha! \leftrightarrow \alpha!! \qquad\qquad \alpha? \leftrightarrow \alpha??$$
$$\alpha?! \leftrightarrow \alpha?!?! \qquad\qquad \alpha!? \leftrightarrow \alpha!?!?$$

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The System S5$_?$

Definition 4.9. The propositional paranormal modal logic S5$_?$ is the propositional modal system $\langle \vartheta_?, \Omega_?, \mathcal{F}_{S5}, \Sigma^\star_{K_?} \cup \Sigma_{T_?} \cup \Sigma_{B_?} \cup \Sigma_{4_?} \rangle$, where $\mathcal{F}_{S5}$ is the class of all reflexive, transitive and symmetric frames, $\Sigma_{T_?}$ are the $\mathtt{T}_?$-axioms, $\Sigma_{B_?}$ are the $\mathtt{B}_?$-axioms and $\Sigma_{4_?}$ are the $\mathtt{4}_?$-axioms in $L_?$.

Theorem 4.12. *For any* $A, B \subseteq L_?$ *and* $\alpha \in L_?$:

$$A \oplus B \vdash_{S5_?} \alpha \quad \text{iff} \quad A \cup \Sigma_{T_?} \cup \Sigma_{B_?} \cup \Sigma_{4_?} \oplus B \vdash_{K_?} \alpha\,,$$
$$A \oplus B \models_{S5_?} \alpha \quad \text{iff} \quad A \cup \Sigma_{T_?} \cup \Sigma_{B_?} \cup \Sigma_{4_?} \oplus B \models_{K_?} \alpha\,.$$

Theorem 4.13. S5$_?$ *is sound and complete.*

Theorem 4.14. *All formulas of* $L_?$ *falling under one of the following schemas are* S5$_?$*-valid (and consequently* S5$_?$*-theorems):*

$$\alpha? \leftrightarrow \alpha?! \qquad \alpha! \leftrightarrow \alpha!?$$
$$\alpha? \leftrightarrow \alpha?? \qquad \alpha! \leftrightarrow \alpha!!$$
$$(\alpha \vee \beta!)! \leftrightarrow (\alpha! \vee \beta!) \qquad (\alpha \vee \beta?)! \leftrightarrow (\alpha! \vee \beta?)$$
$$(\alpha \wedge \beta?)? \leftrightarrow (\alpha? \wedge \beta?) \qquad (\alpha \wedge \beta!)? \leftrightarrow (\alpha? \wedge \beta!)$$

4.2. First-order Paranormal Modal Logic

In this Subsection let $F = \langle W, R_1, \ldots, R_n \rangle$ be any n-frame and $\mathfrak{L}$ be any first-order language.

Definition 4.10. A *first-order modal interpretation* ν *in* F, which is a modal interpretation of $\mathfrak{L}$ in F, is a quadruple $\langle D, V_C, V_F, V_R \rangle$, where D is a function which maps each $w \in W$ to some non-empty set called the domain of w, V_C is a function which assigns to each $w \in W$ and $c \in U_C$ an element of $D(w)$, V_F is a function which assigns to each n-ary function symbol $f \in U_F$ and $w \in W$ a function from $D(w)^n$ to $D(w)$ and V_R is a function which assigns to each n-ary relation symbol $r \in U_R$ and world $w \in W$ a subset of $D(w)^n$.

Definition 4.11. Let $M = \langle W, R_1, \ldots, R_n, \nu \rangle$ be an n-model. We say M is a *first-order model of arity* n (or simply a first-order n-model) iff ν is a first-order modal interpretation.

In the following definitions let $\nu = \langle D, V_C, V_F, V_R \rangle$ be any first-order modal interpretation in F, i.e., $M = \langle W, R_1, \ldots, R_n, \nu \rangle$ is a first-order n-model.

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DEFINITION 4.12. We say ν is *monotonic* iff for every $w, w' \in W$, if $w\, R_i\, w'$ then $D(w) \subseteq D(w')$, for any $i = 1, \ldots, n$. We call the first-order n-model M based on F a *monotonic first-order n-model.*

DEFINITION 4.13. We say ν is a *rigid* first order modal interpretation iff for each $c \in U_C$ and $w, w' \in W$, $V_C(w, c) = V_C(w', c)$ and for every $f \in U_F$ and $w, w' \in W$, $V_F(f, w) = V_F(f, w')$. We call the first-order n-model M based on F a *rigid first-order n-model.*

From now on in this section we shall consider only monotonic and rigid first-order n-models, in such a way that when we speak of a first-order n-model we mean a monotonic and rigid first-order n-model.

DEFINITION 4.14. An *assignment* in M is a function s that assigns to each $x \in U_V$ an element $s(x) \in D(w)$, for some $w \in W$. We write $s[x|a]$ for the assignment that is like s on all variables except x and which maps x to a[11] .

DEFINITION 4.15. For any world $w \in W$ and any assignment s in M, the *denotation function* $\sum_{M.w,s}$ is defined as follows:

- if $c \in U_C$ then $\sum_{M.w,s}(c) = V_C(w, c)$,
- if $x \in U_V$ then $\sum_{M.w,s}(x) = s(x)$,
- if $f \in U_F$ and $t_1, \ldots, t_n$ are terms in U, then $\sum_{M.w,s}(f(t_1, \ldots, t_n)) = V_F(w, f)(\sum_{M.w,s}(t_1), \ldots, \sum_{M.w,s}(t_n))$.

DEFINITION 4.16. Let ϑ be a ?-modal logic basis of arity n. A *first-order Ω_k-modal valuation in ϑ* and a *first-order $\mho_k$-modal valuation in ϑ*, which will also be referred to as the first-order max-min k-modal valuations in ϑ, are max-min k-modal valuations $\Omega_{M,w,s}$ and $\mho_{M,w,s}$ in $\mathscr{L}$[12] and ϑ, which, given a first-order n-model $M = \langle W, R_1, \ldots, R_n, \nu\rangle$ with $\nu = \langle D, V_C, V_F, V_R\rangle$, an assignment s in M, a world $w \in W$, a formula $\alpha \in \mathscr{L}_\vartheta$, an m-ary relation symbol $r \in U_R$ and an m-tuple of terms in U $t_1, \ldots, t_m$, satisfy the following conditions:

- $\Omega_{M,w,s}(r(t_1, \ldots, t_m)) = \mho_{M,w,s}(r(t_1, \ldots, t_m)) = 1$ iff $\langle \sum_{M.w,s}(t_1), \ldots, \sum_{M.w,s}(t_m)\rangle \in V_R(w, r)$,
- $\Omega_{M,w,s}(\forall x\alpha) = 1$ iff for any $d \in D(w)$, $\Omega_{M,w,s[x,d]}(\alpha) = 1$,
- $\mho_{M,w,s}(\forall x\alpha) = 1$ iff for any $d \in D(w)$, $\mho_{M,w,s[x,d]}(\alpha) = 1$.

[11] See e.g. [2], p. 422.

[12] $\mathscr{L}$ is the first order language L_U built upon an arbitrary vocabulary U. See Section 3 of [4].

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A *first-order* Ω_k*-valuation* in ϑ and a *first-order* $\mho_k$*-valuation* in ϑ have as parameters a propositional n-model M, a world w of M, an assignment s in M and a formula α of $\mathscr{L}_\vartheta$.

DEFINITION 4.17. Let ϑ be an n-modal logic basis. The *quantifier axioms* $\Sigma_{\mathtt{Q}}$ *in* $\mathscr{L}_\vartheta$ is the set composed by all formulas of $\mathscr{L}_\vartheta$ falling under the following formula-schema:

Q: $\forall x \alpha(x) \to \alpha(t)$ where the substitution of t for x is admissible

In Definition I.4.1 we put $\vartheta_? := \langle \{!, ?\}, \{!\} \rangle$ (the paranormal modal logic basis) as well as $\mathscr{L}_? := \mathscr{L}_{\vartheta_?}$. Let Ω^1 be the first-order Ω_1-modal valuation in $\vartheta_?$ (see Definition 4.16).

DEFINITION 4.18. The first-order paranormal modal logic $\mathrm{K}^1_?$ is the first-order modal system $\langle \vartheta_?, \Omega^1, \mathcal{F}_{\mathrm{K}^1_?}, \Sigma_{\mathrm{K}^1_?} \rangle$, where $\mathcal{F}_{\mathrm{K}^1_?}$ is the class of all frames and

$$\Sigma_{\mathrm{K}^1_?} := \Sigma_{\mathrm{P}} \cup \Sigma_{\mathrm{A}} \cup \Sigma_{\mathrm{N}} \cup \Sigma_{\mathrm{M}} \cup \Sigma_{\mathtt{K}_?} \cup \Sigma_{\mathtt{Q}}\,,$$

where Σ_{P} are the axioms of positive logic in $\mathscr{L}_?$, Σ_{A} the paranormal classical axioms in $\mathscr{L}_?$, Σ_{N} the additional classical axioms in $\mathscr{L}_?$, Σ_{M} the paranormal modal axioms in $\mathscr{L}_?$, $\Sigma_{\mathtt{K}_?}$ the $\mathtt{K}_?$-axioms in $\mathscr{L}_?$ and $\Sigma_{\mathtt{Q}}$ the quantifier axioms in $\mathscr{L}_?$.

THEOREM 4.15. $\mathrm{K}^1_?$ *is sound and complete, i.e., for any* $A, B \subseteq \mathscr{L}_?$ *and* $\alpha \in \mathscr{L}_?$:

$$A \oplus B \vdash_{\mathrm{K}^?_1} \alpha \quad \textit{iff} \quad A \oplus B \vDash_{\mathrm{K}^1_?} \alpha\,.$$

With $\mathrm{K}^1_?$-theorems and $\mathrm{K}^1_?$-valid formulas, things work exactly as in propositional paranormal modal logic $\mathrm{K}_?$: all theorems from 3.2 to 3.7 restated in terms of $\mathrm{K}^1_?$ and $\mathscr{L}$ are valid. A not-so-straightforward observation is that, both from a proof-theoretical and from a semantic point of view, the differences between $\mathrm{K}_?$ and $\mathrm{K}^1_?$ are equivalent to the differences between the propositional and the first-order cases of normal modal logic K. More generally, given a specific propositional paranormal modal logic $P_?$, the way it is extended into first-order paranormal modal logic $P^1_?$ is exactly the same as the way propositional normal modal logic P is extended to first-order normal modal logic P^1. Therefore, given the amount of literature about the connections between (normal) propositional and first-order modal logic, we will proceed without elaborating further on the first-order features of $\mathrm{K}^1_?$.

An important issue of first-order modal logic is designation of terms across different worlds. As far as we are concerned, we are considering

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only *monotonic* and rigid first-order models. This means first that every individual constant symbol c and function symbol f name the same *things* no matter what plausible world we are considering, and second that everything that exists in a given world also exists in any world accessible from it. From a proof-theoretical point of view, this is the sort of model we obtain when we extend propositional modal logic into first-order modal logic through the simplest way: by adding axiom Q and generalization rule. In this formulation, even though the converse Barcan formula holds

$$\Box\forall x\alpha \rightarrow \forall x\Box\alpha$$

the so-called Barcan formula

$$\forall x\Box\alpha \rightarrow \Box\forall x\alpha$$

does not. However, when we consider logics with symmetric frames such as S5 and B, both Barcan formulas are valid. The justification for choosing this specific way of extending $K_?$ in $K_?^1$ rests on the same point of the preceding paragraph. Since it is not our purpose to get into first-order details such as technical and philosophical discussions about the way quantifiers should be treated, for the sake of completeness we have just focused on the simplest way of extending propositional paranormal modal logic into first-order paranormal modal logic.

4.3. Multi-normal Modal Logic

What we call multi-normal modal logic is any modal logic which contains both normal and paranormal modalities. It therefore includes modal systems of arity greater than or equal to 2. For the sake of simplicity, we will consider here only the simplest case where there exist two pairs of dual modal operators, one normal and the other paranormal.

In this subsection let $\mathfrak{L}$ be any language.

DEFINITION 4.19. We define the notions of $?\Diamond$-modal logic basis and multi-normal modal logic basis as follows:

(i) A $?\Diamond$-*modal logic basis* is any pair $\langle\Theta, \Theta_{\mathrm{d}}\rangle$ in which $\{!, ?, \Box, \Diamond\} \subseteq \Theta$ and $\{!, \Box\} \subseteq \Theta_{\mathrm{d}}$. $\Box$ and $\Diamond$ are used as prefix operators, and ! and ? are used as postfix operators.

(ii) We call the $?\Diamond$-modal logic basis $\vartheta_{?\Diamond} := \langle\{\Box, \Diamond, !, ?\}, \{\Box, !\}\rangle$ the *multi-normal modal logic basis.*

(iii) We refer to the modal logic language based on $\mathfrak{L}$ and $\vartheta_{?\Diamond}$ by $\mathfrak{L}_{?\Diamond}$.

In multi-normal modal logic, ! and ? are the paranormal modal operators and $\Box$ and $\Diamond$ are the normal operators. Therefore, the same negation operator $\neg$ will behave sometimes as a modality-dependent paranormal negation and sometimes as a (modal) classical one.

DEFINITION 4.20. A *multi-normal modal Ω-valuation in $\mathfrak{L}$* and a *multi-normal modal $\mho$-valuation in $\mathfrak{L}$*, which will also be referred to as the multi-normal max-min modal valuations in $\mathfrak{L}$, are, respectively, a Ω_2-modal valuation $\Omega_{?\Diamond M,w,\ldots}$ in $\mathfrak{L}$ and $\vartheta_{?\Diamond}$, and a $\mho_2$-modal valuation $\mho_{?\Diamond M,w,\ldots}$ in $\mathfrak{L}$ and $\vartheta_{?\Diamond}$ which, given a 2-model $M = \langle W, R_{\Diamond}, R_{?}, \nu\rangle$, a world $w \in W$ and any formula $\alpha \in \mathfrak{L}_{?\Diamond}$, and possibly other parameters, satisfy the following conditions:

- $\Omega_{?\Diamond M,w,\ldots}(\Diamond\,\alpha) = 1$ iff for some $w' \in W$ such that $w\,R_{\Diamond}\,w'$, $\Omega_{?\Diamond M,w',\ldots}(\alpha) = 1$,
- $\mho_{?\Diamond M,w,\ldots}(\Diamond\,\alpha) = 1$ iff for some $w' \in W$ such that $w\,R_{\Diamond}\,w'$, $\mho_{?\Diamond M,w',\ldots}(\alpha) = 1$,
- $\Omega_{?\Diamond M,w,\ldots}(\Box\,\alpha) = 1$ iff for any $w' \in W$ such that $w\,R_{\Diamond}\,w'$, $\Omega_{?\Diamond M,w',\ldots}(\alpha) = 1$,
- $\mho_{?\Diamond M,w,\ldots}(\Box\,\alpha) = 1$ iff for any $w' \in W$ such that $w\,R_{\Diamond}\,w'$, $\mho_{?\Diamond M,w',\ldots}(\alpha) = 1$.

A model of multi-normal modal logic is then a 2-model M with two accessibility relations where one is used to evaluate ? and !-marked formulas and the other to evaluate $\Box$ and $\Diamond$-marked ones.

DEFINITION 4.21. The *first-order multi-normal modal Ω-valuation* is the modal valuation $\Omega^1_{?\Diamond}$ which is both a first-order Ω_2-modal valuation in $\vartheta_{?\Diamond}$ and a multi-normal modal Ω-valuation in $\mathscr{L}$.

DEFINITION 4.22. Let ϑ be a $\Diamond$-modal logic basis. The *negation necessity axioms Σ_{NN} in $\mathfrak{L}_\vartheta$* is the set composed by all formulas of $\mathfrak{L}_\vartheta$ falling under the following schema:

NN: $\sim\Box\sim\alpha \leftrightarrow \neg\Box\neg\alpha$

NN is needed in multi-normal modal logic in order to set the normal behavior of $\Box$ (and, consequently, of $\Diamond$). Notice that from NN and NP along with P2 we get:

NP˜: $\Diamond\,\alpha \leftrightarrow \sim\Box\sim\alpha$

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Definition 4.23. The first-order multi-normal modal logic $\mathrm{K_?K}$ is the first-order 2-modal system $\langle \vartheta_{?\Diamond}, \Omega^1_{?\Diamond}, \mathcal{F}, \Sigma_{\mathrm{K_?K}} \rangle$, where $\Omega^1_{?\Diamond}$ is the first-order multi-normal modal Ω-valuation, $\mathcal{F}$ is the class of all 2-frames and

$$\Sigma_{\mathrm{K_?K}} := \Sigma_{\mathrm{P}} \cup \Sigma_{\mathrm{A}} \cup \Sigma_{\mathrm{N}} \cup \Sigma_{\mathrm{M}} \cup \Sigma_{\mathtt{K_?}} \cup \Sigma_{\mathtt{NP}} \cup \Sigma_{\mathtt{K}} \cup \Sigma_{\mathtt{Q}} \cup \Sigma_{\mathtt{NN}} \,,$$

where Σ_{P} are the axioms of positive logic in $\mathscr{L}_{?\Diamond}$, Σ_{A} the paranormal classical axioms in $\mathscr{L}_{?\Diamond}$, Σ_{N} the additional classical axioms in $\mathscr{L}_{?\Diamond}$, Σ_{M} the paranormal modal axioms in $\mathscr{L}_{?\Diamond}$, $\Sigma_{\mathtt{K_?}}$ the $\mathtt{K_?}$-axioms in $\mathscr{L}_{?\Diamond}$, $\Sigma_{\mathtt{NP}}$ the possibility-necessity axioms in $\mathscr{L}_{?\Diamond}$, $\Sigma_{\mathtt{K}}$ the $\mathtt{K}$-axioms in $\mathscr{L}_{?\Diamond}$, $\Sigma_{\mathtt{Q}}$ the quantifier axioms in $\mathscr{L}_{?\Diamond}$ and $\Sigma_{\mathtt{NN}}$ the negation necessity axioms in $\mathscr{L}_{?\Diamond}$.

Theorem 4.16. $\mathrm{K_?K}$ *is sound and complete.*

$\mathrm{K_?K}$ is the most basic first-order multi-normal modal system and, strictly speaking, does not assign a definite meaning to its modal symbols. If we want to take $\Box$ and $\Diamond$ along with their traditional meanings of necessity and possibility, and ! and ? as meaning our skeptical plausibility and credulous plausibility, it seems that at least the axiom-schema below should be added to $\mathrm{K_?K}$.

Definition 4.24. Let ϑ be a $?\Diamond$-modal logic basis. The *possibility-plausibility axioms* $\Sigma_{\mathtt{PP}}$ *in* $\mathfrak{L}_\vartheta$ is the set composed by all formulas of $\mathfrak{L}_\vartheta$ falling under the following schema:

PP: $\quad \Box\,\alpha \to \alpha!$

Intuitively, PP means that if α is necessary, then it is also skeptically plausible. Now notice that in Lemma 2.2 we proved that from $\Sigma_{\mathrm{K_?}}$ we can derive all formulas of $\mathfrak{L}_\vartheta$ falling under the following schema: $(\alpha \to \beta) \to ({\sim}\,\beta \to {\sim}\,\alpha)$. Hence, from PP we get: ${\sim}(({\sim}\,\alpha)!) \to {\sim}\Box{\sim}\,\alpha$. Hence, by NP, K1 and P2, we obtain that from $\Sigma_{\mathrm{K_?K}} \cup \Sigma_{\mathtt{PP}}$ we can derive all formulas of $\mathfrak{L}_\vartheta$ falling under the following schema:

$\mathtt{PP}^{?\Diamond}$: $\quad \alpha? \to \Diamond\,\alpha$

Intuitively, $\mathtt{PP}^{?\Diamond}$ means that if α is credulously plausible, then it is possible. From a semantic point of view, this implies taking only those 2-frames $\langle W, R_\Diamond, R_? \rangle$ in which $R_? \subseteq R_\Diamond$, i.e., for any $w, w' \in W$, if $w\,R_?\,w'$, then $w\,R_\Diamond\,w'$, i.e., every plausible world (of w) is also a plausible world (of w).

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DEFINITION 4.25. The first-order multi-normal modal logic PPK$_?$K is the first-order 2-modal system $\langle \vartheta_{?\Diamond}, \Omega_{?\Diamond^1}, \mathcal{F}_{\text{PP}}, \Sigma_{\text{K}_?\text{K}} \cup \Sigma_{\text{PP}} \rangle$, where $\mathcal{F}_{\text{PP}}$ is the class of all 2-frames $\langle W, R_\Diamond, R_? \rangle$ in which $R_? \subseteq R_\Diamond$, and Σ_{PP} are the possibility-plausibility axioms in $\mathscr{L}_{?\Diamond}$.

THEOREM 4.17. PPK$_?$K *is sound and complete.*

According to this interpretation of our modal symbols, PPK$_?$K can be said to be the most basic logic of plausibility and possibility. From it, many logics such as PPB$_?$S4 and PPS5$_?$S5 can be defined. For other multi-normal modal logics, including one which tries to address in full the problem of formalizing the notions of skeptical and credulous plausibility and epistemic certainty (rather than necessity) see [5], Chapter 6.

5. Conclusion

In this two-parts paper we have presented paranormal modal logic inside a general framework in which a wide range of logics, including classical logic and traditional normal modal logic, can be defined. By doing that we think we have achieved a couple of goals. First, as a version of LEI, we made explicit the combining aspect of paranormal modal logic as well as to what extent it can be said to depart from traditional modal logic. Second, we were able to show an important relation that exists between these two classes of logics: despite contrary appearances, paranormal modal logic is both from a representational as well as from an inferential point of view equivalent to normal modal logic. This we think throws some light upon the relations that hold between modal logic and paraconsistent and paracomplete logic. Furthermore, we gave a philosophical justification to paranormal modal logic as a logic of skeptical and credulous plausibility; in particular we justified what is surely one of its distinguishing features as well as its most controversial feature: axioms `K2` and `K3`.

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