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# SYNTACTIC PROPERTIES OF P-CONSEQUENCE*

**Abstract.** p-consequence is intended as a formalization of non-deductive reasoning. So far semantical or general properties have been presented more thoroughly ([2]–[5]). In the present paper we would like to focus on its syntactic properties.

**Keywords**: p-consequence, non-deductive reasoning.

## 1. Introduction

The notion of *p-consequence*, defined in [2] and developed in [4] or [5], introduces formal analysis into plausible reasoning, in the sense of Ajdukiewicz [1]. The best explanation of Ajdukiewicz's view is his own (see [1], Chapter IV, "Subjectively uncertain inference"):

> Subjectively uncertain inference is such in which, on the strength of the acceptance, with some degree of certainty, of the premisses we accept the conclusion with less certainty than that with which we accept the premisses.

So far (see the articles mentioned), the above sentence has been interpreted in terms of semantics. In this paper we would like to focus on syntactical aspects of p-consequence. Some syntactic notions have been introduced earlier, here we will develop them and show their applications.

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## 2. Preliminaries

Assume that $\mathcal{L} = (L, f_1, \ldots, f_n)$ is any sentence language.

Definition 2.1 ([2]). By a *p-consequence operation for* $\mathcal{L}$ we mean any operation $Z\colon \mathcal{P}(L) \to \mathcal{P}(L)$ that fulfils for all $X \in \mathcal{P}(L)$ and $\alpha \in L$:

(i) $X \subseteq Z(X)$,
(ii) if $X \subseteq Y$, then $Z(X) \subseteq Z(Y)$.

Moreover, a p-consequence $Z$ will be called *finitary* iff

(iii) $Z(X) = \bigcup\{Z(Y) : Y \in \mathrm{Fin}(L) \ \& \ Y \subseteq X\}$, where $\mathrm{Fin}(L)$ is the family of all finite sets of formulas.

Naturally, such properties as structurality can be given: $Z$ is *structural* iff $eZ(X) \subseteq Z(eX)$ for every substitution $e$ of $\mathcal{L}$.

So, we can say that p-consequence differs from the ordinary consequence by dropping condition of idempotency, i.e., $Z(Z(X)) \subseteq Z(X)$.

Definition 2.2 ([2]). By a *p-inference for* $\mathcal{L}$, we shall understand any finite sequence $(a_1, \ldots, a_k)$, $k \geqslant 1$, of ordered pairs from the Cartesian product $L \times \{*, 1\}$.

By a *p-rule of inference for* $\mathcal{L}$ we mean an arbitrary nonempty set of p-inferences for $\mathcal{L}$. A p-rule of inference $r$ is called *axiomatic* iff $r$ validates the condition: if $(a_1, \ldots, a_k) \in r$, then $k = 1$. Call a p-rule $r$ *structural* iff for any $\langle \alpha_1, \ldots, \alpha_k \rangle \in r$ and any substitution $e$, $\langle e\alpha_1, \ldots, e\alpha_k \rangle \in r$.

For $L \times \{*, 1\}$, $\mathrm{pr}_1$ and $\mathrm{pr}_2$ will denote the first and the second projection of Cartesian product, i.e., for any $a \in L \times \{*, 1\}$, $a = \langle \mathrm{pr}_1(a), \mathrm{pr}_2(a) \rangle$.

Definition 2.3 ([2]). For any $\alpha \in L$, $x \in \{*, 1\}$, any $X \in \mathcal{P}(L)$ and any set $\mathcal{R}$ of p-rules of inference for $\mathcal{L}$:

(i) A *p-proof of* $\langle \alpha, x \rangle$ *from* $X$ *based on* $\mathcal{R}$ is a p-inference $(a_1, \ldots, a_k)$ for $\mathcal{L}$ such that
  - $a_k = \langle \alpha, x \rangle$,
  - for all $i = 1, 2, \ldots, k$, either $\mathrm{pr}_1(a_i) \in X$ and $\mathrm{pr}_2(a_i) = 1$ or there exists a p-rule $r \in \mathcal{R}$ and a p-inference $(b_1, \ldots, b_j) \in r$ such that $a_i = b_j$ and $\{b_1, \ldots, b_{j-1}\} \subseteq \{a_1, \ldots, a_{i-1}\}$.

(ii) We shall write $X \Longrightarrow_{\mathcal{R}} \langle \alpha, x \rangle$ iff there exists a p-proof of $\langle \alpha, x \rangle$ from $X$ based on $\mathcal{R}$.

(iii) A formula $\alpha$ is *p-derivable from $X$ based $\mathcal{R}$* (in symbols: $X \Vdash_{\mathcal{R}} \alpha$) iff $X \Longrightarrow_{\mathcal{R}} \langle\alpha, 1\rangle$ or $X \Longrightarrow_{\mathcal{R}} \langle\alpha, *\rangle$.

Following the intuitions laying at the base of p-consequence operation, we can say that if $\langle\gamma, 1\rangle$ is an element of a p-proof, then $\gamma$ is well justified (it is the case when, for example — $\gamma$ is an element of the initial set of premisses $X$), otherwise when $\gamma$ occurs with the index $*$, then $\gamma$ is plausible only (but it is at least not rejected). The brief analysis of the notion *of p-proof* allows us to conclude that to enlarge of a p-proof by some formula (say $\delta$), two conditions must be fulfied:

1. all of the assumptions must be provided; this condition is similar to the one known from the ordinary proof theory
2. the assumptions must be proved with "degrees" required by associated indices

We shall write $(a_1, \ldots, a_k, \langle\beta_i, x_i\rangle_{i=1}^m, c_1, \ldots, c_n)$ instead of $(a_1, \ldots, a_k, \langle\beta_1, x_1\rangle, \ldots, \langle\beta_m, x_m\rangle, c_1, \ldots, c_n)$.

For a p-inference $(a_1, \ldots, a_n)$ and $i \in \{1, \ldots, n\}$ define two sets of formulas

$$A_*(i) := \{\mathrm{pr}_1(a_l) : 1 \leqslant l \leqslant i \ \&\ \mathrm{pr}_2(a_l) = *\},$$
$$A_1(i) := \{\mathrm{pr}_1(a_l) : 1 \leqslant l \leqslant i \ \&\ \mathrm{pr}_2(a_l) = 1\}.$$

Then we put: $r \in \mathcal{R}(Z)$ iff for $Y \subseteq L$ and $(a_1, \ldots, a_n) \in r$, $A_*(n-1) \subseteq Z(Y)$, $Z(Y \cup A_1(n-1)) = Z(Y)$ imply that: $(\mathrm{pr}_2(a_n) = * \Rightarrow \mathrm{pr}_1(a_n) \in Z(Y))$ and $(\mathrm{pr}_2(a_n) = 1 \Rightarrow Z(Y, \mathrm{pr}_1(a_n)) = Z(Y))$.

Lemma 2.4 ([2]). *$A_*(k) \subseteq Z(X)$ and $Z(X \cup A_1(k)) = Z(X)$, whenever $(a_1, \ldots, a_k)$ is a p-proof from the set $X$ on the base of $\mathcal{R}(Z)$.*

The above lemma shows that the formulas occurring with 1 in p-proof do not extend the set of conclusions, when are added to the assumptions — formulas proven with 1 behave like the formulas taken from $X$.

Theorem 2.5 ([2]). *For any finitary p-consequence $Z$ on the language $\mathcal{L}$, any $X \subseteq L$ and $\alpha \in L$: $\alpha \in Z(X)$ iff $X \Vdash_{\mathcal{R}(Z)} \alpha$.*

Moreover, a structural p-rule $r$ of the form

$$r = \{(\langle\alpha_1, x_1\rangle, \ldots, \langle\alpha_k, x_k\rangle, \langle\beta, y\rangle) : \alpha_i, \beta \in L \ \&\ x_i, y \in \{*, 1\}\}$$

will be denoted in slightly clearer form

$$(r) \qquad \frac{\langle\alpha_1, x_1\rangle, \ldots, \langle\alpha_k, x_k\rangle}{\langle\beta, y\rangle}$$

For example, the following p-rules will be used in the sequel:

$$\frac{\langle \alpha \to \beta, 1\rangle, \langle \alpha, 1\rangle}{\langle \beta, 1\rangle} \tag{mp$_1$}$$

$$\frac{\langle \alpha \to \beta, 1\rangle, \langle \alpha, *\rangle}{\langle \beta, *\rangle} \tag{mp$_2$}$$

$$\frac{\langle \alpha \to \beta, *\rangle, \langle \alpha, 1\rangle}{\langle \beta, *\rangle} \tag{mp$_3$}$$

$$\frac{\langle \alpha_1, 1\rangle, \ldots, \langle \alpha_k, 1\rangle}{\langle \alpha_1 \wedge \cdots \wedge \alpha_k, *\rangle} \tag{r$^*_\wedge$}$$

For any set of p-rules $\mathcal{R}$ and p-rule $r$ we shall write $\mathcal{R}+r$ rather then $\mathcal{R} \cup \{r\}$.

## 3. Classical logic described as p-consequence

It is widely known that classical logic can be described as an operation (or relation) of consequence in at least two ways. $\alpha \in \mathcal{C}l(X)$ iff for every classical valuation $v$, for which all formulas from $X$ are true, formula $\alpha$ is true, as well. On the other hand, $\mathcal{C}l$ can be defined in syntactic manner. The key is one very important property of classical logic, namely, it cannot be extended to the other structural and consistent consequence.

Let $\mathcal{L} = (L, \neg, \wedge, \vee, \to)$ be a propositional language of the type $(1, 2, 2, 2)$. It is easy to see that a set of structural p-rules defines a structural p-consequence. We put $\mathbf{pAx} := \{(\langle \alpha, 1\rangle) : \ \alpha \in \mathcal{C}l(\emptyset)\}$.

Fact 3.1. *There is no p-axiomatic and structural extension of the set* $\mathbf{pAx}$+(mp$_2$)+(r$^*_\wedge$), *that is there is no the set* $\mathcal{R}$ *of p-rules, such that*

$$\Vdash_{\mathbf{pAx}+(\mathrm{mp}_2)+(\mathrm{r}^*_\wedge)} \subsetneq \Vdash_{\mathbf{pAx}+(\mathrm{mp}_2)+(\mathrm{r}^*_\wedge)+\mathcal{R}} \neq \mathcal{P}(L) \times L\,.$$

Proof. It is easy to see that $\Vdash_{\mathbf{pAx}+(\mathrm{mp}_2)+(\mathrm{r}^*_\wedge)\}} = \vdash_{\mathcal{C}l}$. Assume that $\nvdash_{\mathcal{C}l} \gamma$ and we put $\mathsf{H} = \mathbf{pAx}$+(mp$_2$)+(r$^*_\wedge$)+$\{\langle \beta, *\rangle : \beta \in \mathrm{Sb}(\gamma)\}$, where $\mathrm{Sb}(\gamma)$ is the set of all substitutions of $\gamma$. Let $v$ be a classical valuation such that $v(\gamma) = 0$. For any propositional variable $p$ we put $e(p) := p \to p$, when $v(p) = 1$, and $e(p) := \neg(p \to p)$, when $v(p) = 0$. Obviously, $\mathcal{C}l(h^e(\gamma)) = L$. Thus, for every formula $\alpha \in L$: $\vdash_{\mathcal{C}l} h^e(\gamma) \to \alpha$, and $\Longrightarrow_{\mathbf{pAx}+(\mathrm{mp}_2)+(\mathrm{r}^*_\wedge)} \langle h^e(\gamma) \to \alpha, 1\rangle$. Consequently, $\Longrightarrow_{\mathsf{H}} \langle \alpha, *\rangle$.

The case, when the set $\mathbf{pAx}$+(mp$_2$)+($\mathrm{r}^*_\wedge$) is extended by $\{\langle\beta, 1\rangle : \beta \in \mathrm{Sb}(\gamma)\}$ is rather obvious. ⊣

The above p-logic behaves as ordinary classical logic. However, the next example shows that, for classical logic, it is not a general case:

FACT 3.2. *There are consistent extensions of the set* $\mathbf{pAx}$+(mp$_3$)+($\mathrm{r}^*_\wedge$).

PROOF. We put $\Theta := \{(\langle\alpha \wedge \alpha, *\rangle) : \alpha \in L\}$. It is easy to see that $\mathcal{C}l(\emptyset) \subsetneq Z_{\mathbf{pAx}+(\mathrm{mp}_3)+(\mathrm{r}^*_\wedge)+\Theta}(\emptyset) = \mathcal{C}l(\emptyset) \cup \{\alpha \wedge \alpha : \alpha \in L\} \neq L$. Moreover, $\Theta$ is structural. ⊣

The difference between results contained in facts 3.1 and 3.2 follows from the fact that p-consequence refines division on the sets of formulas. The same set of formulas can be defined in many ways. There is a relevant difference between the fact that $X \Longrightarrow_{\mathcal{R}} \langle\alpha, 1\rangle$ and $X \Longrightarrow_{\mathcal{R}} \langle\alpha, *\rangle$. However, it is not visible in the relation $X \Vdash_{\mathcal{R}} \alpha$. Moreover, $\mathcal{R}\cup\{\{\langle\alpha, 1\rangle\}\}$ and $\mathcal{R}\cup\{\{\langle\alpha, *\rangle\}\}$ can define the same derivability relation, but they can differ when we want to add some extensions to them: they differ potentially only.

For every $n \in \mathbb{N}$ we put $\Theta_n := \{\langle \bigwedge_{i=1}^{2^m} \alpha, *\rangle : \alpha \in L, m \leqslant n\}$. It is easy to see that, if $k < n$, then $\Theta_k \subsetneq \Theta_n$. Thus $Z_{\mathbf{pAx}+(\mathrm{mp}_3)+(\mathrm{r}^*_\wedge)+\Theta_k}(\emptyset) \subsetneq Z_{\mathbf{pAx}+(\mathrm{mp}_3)+(\mathrm{r}^*_\wedge)+\Theta_n}(\emptyset)$ easily follows. Consequently, we obtain strengthening of the Fact 3.2:

FACT 3.3. *There is a countably infinite chain of consistent extensions of the set* $\mathbf{pAx}$+(mp$_3$)+($\mathrm{r}^*_\wedge$).

## 4. Classical and intuitionistic p-logic

Consider the following rules: (mp$_1$) and

$$\frac{}{\langle\alpha \to (\beta \to \alpha), 1\rangle} \tag{r$_1$}$$

$$\frac{}{\langle((\alpha \to \beta) \to ((\alpha \to (\beta \to \gamma)) \to (\alpha \to \gamma)), 1\rangle} \tag{r$_2$}$$

These rules, or their proof-theoretic counterparts, form a basis of pure implicational Heyting calculus. Moreover, they are intuitionistically valid. It is always possible to add some additional axioms, with 1 or $*$

to obtain logics between intuitionistic, and classical one. However, we need to render the formulation of intuitionistic p-logic more precise.

Definition 4.1. By **pINT** we shall mean the union of the following axiomatic p-rules: (mp$_1$), (mp$_2$), (mp$_3$), (r$_1$), (r$_2$) and

$$\frac{}{\langle \alpha \wedge \beta \rightarrow \alpha, 1\rangle} \tag{r$_3$}$$

$$\frac{}{\langle \alpha \wedge \beta \rightarrow \beta, 1\rangle} \tag{r$_4$}$$

$$\frac{}{\langle \alpha \rightarrow \alpha \vee \beta, 1\rangle} \tag{r$_5$}$$

$$\frac{}{\langle \beta \rightarrow \alpha \vee \beta, 1\rangle} \tag{r$_6$}$$

$$\frac{}{\langle (\alpha \rightarrow \gamma) \rightarrow ((\beta \rightarrow \gamma) \rightarrow (\alpha \vee \beta \rightarrow \gamma)), 1\rangle} \tag{r$_7$}$$

$$\frac{}{\langle (\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow \neg\beta) \rightarrow \neg\alpha), 1\rangle} \tag{r$_8$}$$

$$\frac{}{\langle \alpha \rightarrow (\neg\alpha \rightarrow \beta), 1\rangle} \tag{r$_9$}$$

At first, we will prove general version of deduction theorem.

Theorem 4.2. *For every language $\mathcal{L} = (L, f_1, \ldots, f_n)$ such that $\rightarrow \in \{f_1, \ldots, f_n\}$ and for any set of p-rules $\mathcal{R}$ containing* (r$_1$), (r$_2$), (mp$_1$), (mp$_2$), (mp$_3$), *and maybe some axiomatic p-rules:*

$$X, \alpha \Longrightarrow_{\mathcal{R}} \langle \beta, x\rangle \ \textit{ iff } \ X \Longrightarrow_{\mathcal{R}} \langle \alpha \rightarrow \beta, x\rangle,$$

*for every $\alpha, \beta \in L$, $x \in \{1, *\}$ and $X \in \mathcal{P}(L)$.*

Proof. "$\Rightarrow$" Assume that $X, \alpha \Longrightarrow_{\mathcal{R}} \langle \beta, x\rangle$, that is, there exists p-proof $(a_1, \ldots, a_k)$ from $X \cup \{\alpha\}$ based on $\mathcal{R}$ such that $a_k = \langle \beta, x\rangle$. We are going to prove, that for every $i \in \{1, \ldots, k\}$

$$X \Longrightarrow_{\mathcal{R}} \langle \alpha \rightarrow \mathrm{pr}_1(a_i), \mathrm{pr}_2(a_i)\rangle$$

For $i = 1$, if $\mathrm{pr}_1(a_1) \in X$ and $\mathrm{pr}_2(a_1) = 1$, then

$$(\langle \mathrm{pr}_1(a_1) \rightarrow (\alpha \rightarrow \mathrm{pr}_1(a_1)), 1\rangle, \langle \alpha \rightarrow \mathrm{pr}_1(a_1), 1\rangle)$$

forms the required p-proof.

In the case when $\mathrm{pr}_1(a_1) = \alpha$ and $\mathrm{pr}_2(a_1) = 1$ then:

$$\begin{aligned}
&(\langle \alpha \to (\alpha \to \alpha), 1\rangle,\\
&\langle(\alpha \to (\alpha \to \alpha)) \to [(\alpha \to ((\alpha \to \alpha) \to \alpha)) \to (\alpha \to \alpha)], 1\rangle,\\
&\langle(\alpha \to ((\alpha \to \alpha) \to \alpha)) \to (\alpha \to \alpha), 1\rangle,\\
&\langle \alpha \to ((\alpha \to \alpha) \to \alpha), 1\rangle, \langle \alpha \to \alpha, 1\rangle)
\end{aligned}$$

is a p-proof from the empty set, so also from $X$. The similar solution can be provided when $(a_1) \in \bigcup \mathcal{R}$, thus $\Longrightarrow_{\mathcal{R}} \langle \mathrm{pr}_1(a_1) \to (\alpha \to \mathrm{pr}_1(a_1)), 1\rangle$, that is $\Longrightarrow_{\mathcal{R}} \langle \alpha \to \mathrm{pr}_1(a_1), \mathrm{pr}_2(a_1)\rangle$.

Consider the rule ($\mathrm{mp}_2$). The others are treated analogously. Assume that for some $j, m \in \{1, \ldots, i-1\}$: $a_j = \langle \alpha_j, *\rangle$, $a_m = \langle \alpha_j \to \alpha_i, 1\rangle$. By induction assumption one has $X \Longrightarrow_{\mathcal{R}} \langle \alpha \to \alpha_j, *\rangle$ and $X \Longrightarrow_{\mathcal{R}} \langle \alpha \to (\alpha_j \to \alpha_i), 1\rangle$. Existing p-proofs can be joined and extended as follows:

$$\begin{aligned}
&\ldots, \langle(\alpha \to (\alpha_j \to \alpha_i)) \to [(\alpha \to \alpha_j) \to (\alpha \to \alpha_i)], 1\rangle,\\
&\langle(\alpha \to \alpha_j) \to (\alpha \to \alpha_i), 1\rangle, \langle \alpha \to \alpha_i, *\rangle).
\end{aligned}$$

"$\Leftarrow$" It is obvious. $\dashv$

Theorem 4.2 can be applied to **pINT**. However, this case is not really interesting. The reason is simple: $\Vdash_{\mathbf{pINT}}$ equals to $\vdash_{\mathbf{INT}}$, so we would obtain a bit more complex description of intuitionistic logic. We would like to add some axiomatic rule. Consider an axiom of the form $/\langle \alpha, 1\rangle$. In this case we obtain the relation of consequence being in the fact an extension of $\vdash_{\mathbf{INT}}$ by the axiom $/\alpha$. Let us note that if we have axioms labelled by 1 only, then neither of rules, ($\mathrm{mp}_2$) nor ($\mathrm{mp}_3$), is usable.

So, consider the case when the extension is of the form $/\langle \alpha, *\rangle$.

*Example* 4.3. Consider the following axiomatic rule:

$$\frac{}{\langle \alpha \vee \neg\alpha, *\rangle} \qquad (\mathrm{r}_{10})$$

Then $\vdash_{\mathbf{INT}} \subsetneq \Vdash_{\mathbf{pINT}+(\mathrm{r}_{10})}$ and, moreover, $\Vdash_{\mathbf{pINT}+(\mathrm{r}_{10})} \subsetneq \vdash_{\mathcal{C}l}$. It is clear that $\Vdash_{\mathbf{pINT}+(\mathrm{r}_{10})} (p \to q) \vee (q \to p)$, since $\Longrightarrow_{\mathbf{pINT}+(\mathrm{r}_{10})} \langle(p \vee \neg p) \to ((p \to q) \vee (q \to p)), 1\rangle$.

To show the applications of intermediate p-logics, we will introduce some semantics (despite the title of the paper).

Definition 4.4. By a *Heyting algebra* we shall mean an algebra $\mathcal{H} = (H, \wedge, \vee, \to, \mathsf{0}, \mathsf{1})$ such that: $(H, \wedge, \vee, \mathsf{0}, \mathsf{1})$ is a bounded lattice with the least element $\mathsf{0}$ and the largest element $\mathsf{1}$, and for all $a, b, c \in H$:

$$a \wedge c \leq b \ \text{ iff } \ c \leq a \to b,$$

where for all $a, b \in H$: $a \leq b$ iff $a \wedge b = a$.

Equivalently, a Heyting algebra is a bounded lattice $\mathcal{H} = (H, \wedge, \vee, \mathsf{0}, \mathsf{1})$ such that for all $a, b \in H$ there is the greatest element in the set $\{c \in H : a \wedge c \leq b\}$. This greatest element is called the *relative pseudo-complement of $a$ with respect to $b$*, and is denoted by $a \to b$.

Moreover, for any $x \in H$ we put $\neg a := a \to \mathsf{0}$, i.e., $\neg a$ is the greatest element in the set $\{b \in H : a \wedge b = \mathsf{0}\}$.

The class of Heyting algebras forms semantical basis for intuitionistic logic (see e.g. [6]). We are going to modify this semantics for p-version of intuitionistic logic.

Let $A$ be an axiom schema that is not valid in intuitionistic logic.[1] We put $\langle A, * \rangle$ for the set $\{\langle \alpha, * \rangle : \alpha \text{ is an instance of } A\}$. According to p-version of deduction theorem:

$$X \Longrightarrow_{\mathbf{pINT}+\langle A,*\rangle} \langle \alpha \to \beta, x \rangle \quad \text{iff} \quad X, \alpha \Longrightarrow_{\mathbf{pINT}+\langle A,*\rangle} \langle \beta, x \rangle\,.$$

For $A$ there is a polynomial $f_A$ such that $A$ is of the form $f_A(x_1, \ldots, x_m)$. Thus, for any Heyting algebra $\mathcal{H} = (H, \wedge, \vee, \to, \mathsf{0}, \mathsf{1})$ we can put $\mathcal{H}_A := (H, \wedge, \vee, \to, D_1, D_*)$, where

$$D_1 := \{\mathsf{1}\}$$
$$D_*^A := \{a \in H : \exists_{b_1,\ldots,b_m \in H}\ f_A(b_1, \ldots, b_m) \leq a\}.$$[2]

Finally, we put $\mathbf{HA}_A := \{\mathcal{H}_A : \mathcal{H} \text{ is a Heyting algebra}\}$.

We define the p-consequence relation for any $X \in \mathcal{P}(L)$ and $\alpha \in L$:

$X \Vdash_A \alpha$ iff for any $(H, \wedge, \vee, \to, D_1, D_*^A) \in \mathbf{HA}_A$ and any homomorphism $h$ from $\mathcal{L}$ into $(H, \neg, \wedge, \vee, \to)$: $h(X) \subseteq D_1$ implies $h(\alpha) \in D_*^A$.

Theorem 4.5. *For every axiom schema $A$, $\Vdash_A = \Vdash_{\mathbf{pINT}+\langle A,*\rangle}$.*

[1] For example, $A$ can be $\dfrac{}{\alpha \vee \neg\alpha}$ or $\dfrac{}{(\alpha \to \beta) \vee (\beta \to \alpha)}$.

[2] For example, if $A$ is of the form $x \vee \neg x$, then $D_*^A = \{a \in H : \exists_{b \in H}\ b \vee \neg b \leq a\}$.

Proof. "$\subseteq$" We put $\Vdash := \Vdash_{\mathbf{pINT}+\langle A,*\rangle}$, $\Longrightarrow := \Longrightarrow_{\mathbf{pINT}+\langle A,*\rangle}$ and assume that $X \nVdash \alpha$. Then $X \not\Longrightarrow \langle\alpha, *\rangle$.

Define a relation $\approx_X$ on the language $L$:

$$\beta \approx_X \gamma \text{ iff } X \Longrightarrow \langle\beta \to \gamma, 1\rangle \text{ and } X \Longrightarrow \langle\gamma \to \beta, 1\rangle$$

It is easy to see that $\approx_X$ is a congruence on $\mathcal{L}$: $X \Longrightarrow \langle\beta, 1\rangle$ iff $\beta$ follows from $X$ by means of intuitionistic logic. According to that fact, we can state even more: quotient algebra $\mathcal{L}/\approx_X$ is a Heyting algebra with the greatest element $1_{\mathcal{L}/\approx_X}$ fulfilling:

$$\pi/\approx_X (X) \subseteq 1_{\mathcal{L}/\approx_X},$$

where $\pi/\approx_X\colon L \longrightarrow L/\approx_X$ is a canonical projection. Moreover, $[\beta] \leq [\gamma]$ iff $X \Longrightarrow \langle\beta \to \gamma, 1\rangle$.

We are going to show that

$$\{[\beta] : \exists_{[\beta_1],\dots,[\beta_k]\in L/\approx_X} f_A([\beta_1],\dots,[\beta_k]) \leq [\beta]\} = \{[\beta] \in L/_{\approx_X} : X \Longrightarrow \langle\beta, *\rangle\}$$

To have a notation readable we have omitted indexes in symbols of abstract classes w.r.t. $\approx_X$ in the above equation.

Assume that $f_A([\beta_1],\dots,[\beta_k]) \leq [\beta]$, that is $[f_A(\beta_1,\dots,\beta_k)] \leq [\beta]$, that is $X \Longrightarrow \langle f_A(\beta_1,\dots,\beta_k) \to \beta, 1\rangle$. Since $X \Longrightarrow \langle f_A(\beta_1,\dots,\beta_k), *\rangle$ and $(MP_2)$ we obtain $X \Longrightarrow \langle\beta, *\rangle$.

Assume that $X \Longrightarrow \langle\beta, *\rangle$. Then by the definition, there exists p-proof $(\langle\gamma_1, x_1\rangle, \langle\gamma_2, x_2\rangle, \dots, \langle\gamma_k, x_k\rangle, \langle\beta, *\rangle)$ from the set $X$. We can assume that this proof is minimal, in the sense that there is no a subsequence $(\langle\gamma_{t_1}, x_{t_1}\rangle, \langle\gamma_{t_2}, x_{t_2}\rangle, \dots, \langle\gamma_{t_j}, x_{t_j}\rangle, \langle\beta, *\rangle)$ being p-proof from the same set, shorter than the initial sequence and $t_1 < t_2 < \dots < t_j$. Due to the form of the rules ($\text{mp}_2$) and ($\text{mp}_3$) it can be also assumed that all elements from $A_1(k)$ occur before any element of $A_*(k)$ (see the notation introduced immediately before Lemma 2.4). Thus, ($\text{mp}_2$) or ($\text{mp}_3$) has been used at most once, and at the final step of derivation.

Indeed, if, for example, ($\text{mp}_2$) was used twice as in the example:

$$\dots, \langle\delta, *\rangle, \dots, \langle\delta \to \zeta, 1\rangle, \dots, \langle\zeta, *\rangle, \dots, \langle\zeta \to \eta, 1\rangle, \dots, \langle\eta, *\rangle, \dots$$

then the above sequence could be replaced by:

$$\langle(\delta \to \zeta) \to [(\zeta \to \eta) \to (\delta \to \eta)], 1\rangle \dots, \langle\delta, *\rangle, \dots, \langle\delta \to \zeta, 1\rangle,$$
$$\langle(\zeta \to \eta) \to (\delta \to \eta), 1\rangle, \dots, \langle\zeta \to \eta, 1\rangle, \langle\delta \to \eta, 1\rangle, \dots, \langle\eta, *\rangle, \dots$$

In the case when (mp$_3$) precedes (mp$_2$):

$$\ldots, \langle \delta, 1\rangle, \ldots, \langle \delta \to \zeta, *\rangle, \ldots, \langle \zeta, *\rangle, \ldots, \langle \zeta \to \eta, 1\rangle, \ldots, \langle \eta, *\rangle, \ldots$$

the above sequence could be replaced by:

$$\begin{aligned}
&\langle (\zeta \to \eta) \to [(\delta \to \zeta) \to (\delta \to \eta)], 1\rangle, \ldots,\\
&\langle (\zeta \to \eta), 1\rangle, \langle (\delta \to \zeta) \to (\delta \to \eta), 1\rangle, \ldots,\\
&\langle [(\delta \to \zeta) \to (\delta \to \eta)] \to [\delta \to (\zeta \to \eta)], 1\rangle\\
&\qquad\qquad \text{(thesis of intuitionistic logic)},\\
&\langle \delta \to (\zeta \to \eta), 1\rangle, \ldots, \langle [\delta \to (\zeta \to \eta)] \to [\delta \to ((\delta \to \zeta) \to \eta)], 1\rangle\\
&\qquad\qquad \text{(thesis of intuitionistic logic)},\\
&\langle \delta \to ((\delta \to \zeta) \to \eta), 1\rangle, \langle \delta, 1\rangle, \langle (\delta \to \zeta) \to \eta, 1\rangle, \langle \delta \to \zeta, *\rangle, \langle \eta, *\rangle, \ldots
\end{aligned}$$

And similarly in the remaining two cases.

So, we can assume that the last application of (mp) rules has one of the following form:

(a) $\langle \gamma_{k-1}, x_{k-1}\rangle = \langle \delta \to \beta, 1\rangle$, $\langle \gamma_k, x_k\rangle = \langle \delta, *\rangle$
(b) $\langle \gamma_{k-1}, x_{k-1}\rangle = \langle \delta, 1\rangle$, $\langle \gamma_k, x_k\rangle = \langle \delta \to \beta, *\rangle$.

*Ad* (a) $\delta$ is an instance of $A$ and $[\delta] \leq [\beta]$.

*Ad* (b) In this case $\delta \to \beta$ is an instance of $A$, it is easy to see that $X \Longrightarrow \langle (\delta \to \beta) \to \beta, 1\rangle$, since $X, \delta \to \beta \Longrightarrow \langle \beta, 1\rangle$ and Theorem 4.2. Therefore, $[\delta \to \beta] \leq [\beta]$.

"$\supseteq$" This direction is standard. ⊣

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