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ADMISSIBILITY OF CUT IN CONGRUENT MODAL LOGICS*

Abstract. We present a detailed proof of the admissibility of cut in sequent calculus for some congruent modal logics. The result was announced much earlier during the Trends in Logic Conference, Toruń 2006 and the proof for monotonic modal logics was provided already in Indrzejczak [5]. Also some tableau and natural deduction formalizations presented in Indrzejczak [6] and Indrzejczak [7] were based on this result but the proof itself was not published so far. In this paper we are going to fill this gap. The delay was partly due to the fact that the author from time to time was trying to improve the result and extend it to some additional logics by testing other methods of proving cut elimination. Unfortunately all these attempts failed and cut elimination holds only for these logics which were proved to satisfy this property already in 2005.

Keywords: modal logics, proof methods, sequent calculi, cut elimination

1. Languages and Logics

In this paper we deal exclusively with some modal congruent logics (also called classical — cf. Chellas [2]). The investigations on this class of logics are rather neglected although both epistemic and doxastic interpretation of modal constants in the context of normal or regular logics lead to unintuitive results and weaker logics (congruent and monotonic) are often considered as better candidates for these applications. Despite that, neither model theory nor proof theory of weak modal logics (congruent

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and monotonic) is well established. As for the former, except classics like Segerberg [10] and Chellas [2], one can find some results in Hansen [4] Proof theory is touched on in Lavendhomme and Lucas [8] where cut-free sequent calculus for **E** and **M**, the weakest congruent and monotonic systems is provided. Sequent calculi for monotonic logics obtained axiomatically by all combinations of axioms D, T, 4, B and 5 over **M** is presented in Indrzejczak [5], whereas tableau and natural deduction approaches to these logics and their congruent counterparts are provided by Indrzejczak [6] and Indrzejczak [7] respectively. In what follows we recall syntactic and semantic formalizations of congruent modal logics usually axiomatized by combinig axioms D, T, 4, B and 5 over **E** or **EN**, i.e. the weakest congruent logic closed under the rule of necessitation. Then, we present their modular formalization on the basis of standard sequent calculus, and finally provide the proof of cut admissibility for some of the considered systems.

Let us start with some basic definitions. $\mathbf{L_M}$ denotes a standard modal propositional language, i.e., an abstract algebra of formulae $\langle$FOR, $\neg, \wedge, \vee, \rightarrow, \Box\rangle$, with denumerable set of propositional variables: VAR := $\{p, q, r, \ldots, p_1, q_1, \ldots\} \subseteq$ FOR, where $\neg$, $\wedge$, $\vee$, $\rightarrow$ denote boolean negation, conjunction, disjunction and implication. $\Box$ denotes an unary modal operator of (*alethic*) necessity, but of course many other interpretations of epistemic or deontic character may be applied.

To represent elements of FOR we will use φ, ψ, χ; $\varGamma$, $\varDelta$ will denote sets or multisets of formulae of standard modal language. FOR is defined in ordinary way, i.e. as the least set satisfying the following conditions:

- VAR $\subseteq$ FOR;
- if $\varphi \in$ FOR and $\odot \in \{\neg, \Box\}$, then $\odot\varphi \in$ FOR;
- if $\varphi, \psi \in$ FOR and $\odot \in \{\wedge, \vee, \rightarrow\}$, then $(\varphi \odot \psi) \in$ FOR.

Every modal logic may be defined as a class of formulae in $\mathbf{L_M}$ containing all tautologies of **CPL** (classical propositional logic) and closed under MP (modus ponens) and substitution. Every congruent modal logic is closed under RE, i.e.:

RE if $\varphi \leftrightarrow \psi \in \mathbf{L}$, then $\Box\varphi \leftrightarrow \Box\psi \in \mathbf{L}$.

The weakest congruent modal logic will be denoted as **E**, and all its extensions will be called shortly E-logics. Additionally we will consider congruent modal logics closed under RN, i.e.:

RN if $\varphi \in \mathbf{L}$, then $\Box\varphi \in \mathbf{L}$.

The weakest E-logic closed under RN will be called **EN** and all its extensions will be called shortly EN-logics. We will consider E- and EN-logics which may be obtained axiomatically by means of the following well known axioms:

- D $\Box\varphi \to \neg\Box\neg\varphi$
- T $\Box\varphi \to \varphi$
- 4 $\Box\varphi \to \Box\Box\varphi$
- B $\varphi \to \Box\neg\Box\neg\varphi$
- 5 $\neg\Box\varphi \to \Box\neg\Box\varphi$

How many different E- and EN-logics axiomatized in this way exist? Not so many due to the

Lemma 1 (The Dependence Theorem). *The following holds true*:

$\mathbf{CPL} + \mathrm{T} \vdash \mathrm{D}$
$\mathbf{CPL} + \mathrm{T} + 5 \vdash \mathrm{B}$
$\mathbf{CPL} + \mathrm{D} + 4 + \mathrm{B} \vdash \mathrm{T}$
$\mathbf{E} + \mathrm{T} + 5 \vdash 4$
$\mathbf{E} + \mathrm{B} + 4 + \mathrm{D} \vdash 5$
$\mathbf{E} + \mathrm{B} + 5 + \mathrm{T} \vdash 4$
$\mathbf{E} + \mathrm{B} + \mathrm{T} \vdash \mathrm{N}$

where $\vdash$ means that suitable formula is derivable from the combination of those listed on the left of $\vdash$.

In the first three cases we have used the form **CPL**+X instead of **E**+X because we want to stress that there is no use of RE in the proof. Thus one may count that there is 18 different E-logics and 16 EN-logics axiomatized by means of D, T, 4, B, 5 over **E** and over **EN**.

2. Neighbourhood Semantics

The standard semantic approach to weak modal logic, congruent in particular, is based on the use of neighbourhood frames (models) called also minimal frames (see Chellas [2]).

Definition. A Neighbourhood Frame $\mathfrak{F} = \langle \mathcal{W}, \mathcal{N} \rangle$, where $\mathcal{W} \neq \varnothing$ is a nonempty set of states (possible worlds), and $\mathcal{N}$ is a function $\mathcal{N} \colon \mathcal{W} \longrightarrow \mathcal{P}(\mathcal{P}(\mathcal{W}))$. A model on the frame $\mathfrak{F}$ is any structure $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$, where V is a valuation function on atoms ($V \colon \mathrm{VAR} \longrightarrow \mathcal{P}(\mathcal{W})$).

Satisfaction of a formula in a state of a model is defined as in Kripke models with the exception of $\Box\varphi$ which is evaluated by means of $\mathcal{N}$:

$$
\begin{array}{lll}
\mathfrak{M}, w \vDash \varphi & \text{iff} & w \in V(\varphi), \text{ for any } \varphi \in \mathrm{VAR}, \\
\mathfrak{M}, w \vDash \neg\varphi & \text{iff} & \mathfrak{M}, w \nvDash \varphi, \\
\mathfrak{M}, w \vDash \varphi \wedge \psi & \text{iff} & \mathfrak{M}, w \vDash \varphi \text{ and } \mathfrak{M}, w \vDash \psi, \\
\mathfrak{M}, w \vDash \varphi \vee \psi & \text{iff} & \mathfrak{M}, w \vDash \varphi \text{ or } \mathfrak{M}, w \vDash \psi, \\
\mathfrak{M}, w \vDash \varphi \to \psi & \text{iff} & \mathfrak{M}, w \nvDash \varphi \text{ or } \mathfrak{M}, w \vDash \psi, \\
\mathfrak{M}, w \vDash \Box\varphi & \text{iff} & \|\varphi\| \in \mathcal{N}(w),
\end{array}
$$

where $\|\varphi\|$ is the set of all states in a model where φ is satisfied.

The preceding definition establishes conditions for local (at a state in a model) satisfiability. We may generalize them in many ways, for our aims the following two notions are suitable:

- If $\|\varphi\| = \mathcal{W}$ in some model $\mathfrak{M}$, then φ is valid on $\mathfrak{M}$.
- If φ is valid on all models built on some class of frames $\mathcal{F}$, then φ is $\mathcal{F}$-valid.

The following weak adequacy theorem holds (see, e.g. [2]):

Theorem 1. 1. $\varphi \in \mathbf{E}$ *iff it is valid on all neighbourhood frames.*

2. $\varphi \in \mathbf{EN}$ *iff it is valid on all neighbourhood frames satisfying condition*:

(n) $\mathcal{W} \in \mathcal{N}(w)$.

As for the extensions of **E** and **EN**, note that the following conditions on neighbourhood frames correspond to respective axioms:

(d) if $X \in \mathcal{N}(w)$, then $-X \notin \mathcal{N}(w)$;
(t) if $X \in \mathcal{N}(w)$, then $w \in X$;
(4) if $X \in \mathcal{N}(w)$, then $\{w' : X \in \mathcal{N}(w')\} \in \mathcal{N}(w)$;
(b) if $w \in X$, then $\{w' : -X \notin \mathcal{N}(w')\} \in \mathcal{N}(w)$;
(5) if $X \notin \mathcal{N}(w)$, then $\{w' : X \notin \mathcal{N}(w')\} \in \mathcal{N}(w)$.

Below we will use the shorthand to the effect that E-logic (or EN-logic) containing an axiom X (say D, T, etc.) will be denoted as **EX** (**ENX** respectively) and the class of neighbourhood frames satisfying condition (x) will be denoted as $\mathcal{X}$. For instance, E-logic axiomatized by means of D and 4 is denoted as **ED4**, whereas the class of frames satisfying both (d) and (4), as $\mathcal{D}4$.

The following adequacy theorem holds for all logics under consideration.

Theorem 2. 1. $\varphi \in$ **EXY..Z** *iff it is* $\mathcal{X}\mathcal{Y}..\mathcal{Z}$*-valid.*
2. $\varphi \in$ **ENXY..Z** *iff it is* $\mathcal{X}\mathcal{Y}..\mathcal{Z}$*-valid on all neighbourhood frames satisfying condition* (n).

Note that the results stated in two preceding theorems can be extended to strong adequacy for (local) consequence relation defined as follows:

$$\Gamma \models_L \varphi \text{ iff } \|\Gamma\| \subseteq \|\varphi\| \text{ in all models based on neighbourhood frames from the class determining logic } \mathbf{L}.$$

3. Sequent Calculi

In the following we will examine sequent formalizations of the E-logics under consideration. As a basis we use a standard version of sequent calculus for **CPL** denoted by SC, where sequents are ordered pairs of (finite) multisets of formulae. Except of a pair of rules for every (non-modal) constant we have structural rules of contraction, weakening (for both sides of a sequent) and cut.

General rules

(AX) $\varphi, \Gamma \Rightarrow \Delta, \varphi$

(Cut) $\dfrac{\Gamma \Rightarrow \Delta, \varphi \qquad \varphi, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}$

(W⇒) $\dfrac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}$

(⇒W) $\dfrac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}$

(C⇒) $\dfrac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}$

(⇒C) $\dfrac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi}$

(¬⇒) $\dfrac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma \Rightarrow \Delta}$

(⇒¬) $\dfrac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi}$

(∧⇒) $\dfrac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}$

(⇒∧) $\dfrac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}$

(∨⇒) $\dfrac{\varphi, \Gamma \Rightarrow \Delta \qquad \psi, \Gamma \Rightarrow \Delta}{\varphi \vee \psi, \Gamma \Rightarrow \Delta}$

(⇒∨) $\dfrac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$

(→⇒) $\dfrac{\Gamma \Rightarrow \Delta, \varphi \qquad \psi, \Gamma \Rightarrow \Delta}{\varphi \to \psi, \Gamma \Rightarrow \Delta}$

(⇒→) $\dfrac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \to \psi}$

In order to get SC-**L**, where **L** denotes any of the 18 + 16 logics under consideration, we must add the following rules defined for modal constants.

Modal rules

$$(\text{E})\quad \frac{\varphi \Rightarrow \psi \qquad \psi \Rightarrow \varphi}{\Box\varphi, \Gamma \Rightarrow \Delta, \Box\psi}$$

$$(\text{N})\quad \frac{\Rightarrow \varphi}{\Gamma \Rightarrow \Delta, \Box\varphi} \qquad (\text{T})\quad \frac{\varphi,\ \Gamma \Rightarrow \Delta}{\Box\varphi,\ \Gamma \Rightarrow \Delta}$$

$$(\text{D-2})\quad \frac{\Rightarrow \varphi, \psi \qquad \varphi, \psi \Rightarrow}{\Box\varphi, \Box\psi, \Gamma \Rightarrow \Delta} \qquad (\text{4-2})\quad \frac{\Box\varphi \Rightarrow \psi \qquad \psi \Rightarrow \Box\varphi}{\Box\varphi, \Gamma \Rightarrow \Delta, \Box\psi}$$

$$(\text{5-2})\quad \frac{\Rightarrow \Box\varphi, \psi \qquad \Box\varphi, \psi \Rightarrow}{\Gamma \Rightarrow \Delta, \Box\varphi, \Box\psi} \qquad (\text{B-2})\quad \frac{\Rightarrow \Box\varphi, \psi \qquad \Box\varphi, \psi \Rightarrow}{\Rightarrow \varphi, \Box\psi}$$

Note that all modal rules are modular, i.e. they correspond to respective rules and axioms in a unique way.

The proof of a sequent $\Gamma \Rightarrow \Delta$ is defined in a standard way as a tree of sequents with $\Gamma \Rightarrow \Delta$ as a root, axioms as leaves and all edges as obtained by an application of a rule. In the case $\Gamma \Rightarrow \Delta$ is derivable in the system SC-**L** we write $\Gamma \Rightarrow \Delta \in$ SC-**L** or SC-**L** $\vdash \Gamma \Rightarrow \Delta$.

Completeness is easy to prove; it is enough to provide proofs of respective axioms, since primitive rules MP, RE and RN are represented by (Cut), (E) and (N) respectively. We demonstrate the case of 5 derivability, i.e. we show that (5-2) is sufficient to derive 5.

$$\dfrac{\dfrac{\dfrac{\dfrac{\Box\varphi \Rightarrow \Box\varphi}{\Rightarrow \Box\varphi, \neg\Box\varphi}\,(\Rightarrow\neg) \qquad \dfrac{\Box\varphi \Rightarrow \Box\varphi}{\Box\varphi, \neg\Box\varphi \Rightarrow}\,(\neg\Rightarrow)}{\Rightarrow \Box\varphi, \Box\neg\Box\varphi}\,(\text{5-2})}{\neg\Box\varphi \Rightarrow \Box\neg\Box\varphi}\,(\neg\Rightarrow)}{\Rightarrow \neg\Box\varphi \to \Box\neg\Box\varphi}\,(\Rightarrow\to)$$

A reader should have no problems with proving the rest of the axioms. As a result of these and of theorems 1 and 2 we have:

Theorem 3 (Completeness). *If* $\varphi \in \mathbf{L}$, *then* $\varphi \in$ SC-**L**.

In order to prove soundness we must show that all the modal rules preserve validity from premise-sequents to conclusion-sequent in respective classes of models. Validity of sequents is meant in a standard way, i.e. $\models \Gamma \Rightarrow \Delta$ means that in every world of every respective model, either at least one element of Γ is false or at lest one element od Δ is true; hence $\not\models \Gamma \Rightarrow \Delta$ means that there is a model with a world where all elements of Γ are true and all elements of Δ are false.

Theorem 4 (Soundness). *If* $\varphi \in$ SC-**L**, *then* $\models \varphi$ *in respective class of* **L**-*frames.*

Proof. The cases for nonmodal rules are standard. We demonstrate two cases of modal rules. For (E) we have:

Assume that (i) $\models \varphi \Rightarrow \psi$ and $\models \psi \Rightarrow \varphi$ but (ii) $\not\models \Box\varphi, \Gamma \Rightarrow \Delta, \Box\psi$. From (i) we obtain (iii) $\|\varphi\| = \|\psi\|$ and from (ii) follows that there is a model with a world w such that (iv) $w \vDash \Box\varphi$ and (v) $w \nvDash \Box\psi$. So, by (iv) $\|\varphi\| \in \mathcal{N}(w)$ but then, by (iii), $\|\psi\| \in \mathcal{N}(w)$ which means that $w \vDash \Box\psi$ and contradicts (v).

The case of (5-2).

Assume that (i) $\models\ \Rightarrow \Box\varphi, \psi$ and $\models \Box\varphi, \psi \Rightarrow$ but (ii) $\not\models \Gamma \Rightarrow \Delta, \Box\varphi, \Box\psi$. From (i) we obtain (iii) $\|\neg\Box\varphi\| = \|\psi\|$ and from (ii) there is a model with a world w such that (iv) $w \nvDash \Box\varphi$ and (v) $w \nvDash \Box\psi$. So, by (iv) $\|\varphi\| \notin \mathcal{N}w$ which, by (5) implies $\{w' : \|\varphi\| \notin \mathcal{N}(w')\} \in \mathcal{N}(w)$ which means $\{w' : w' \nvDash \Box\varphi\} \in \mathcal{N}(w)$, hence $\|\neg\Box\varphi\| \in \mathcal{N}(w)$. The latter, by (iii) implies that $\|\psi\| \in \mathcal{N}(w)$ which means that $w \vDash \Box\psi$ and contradicts (v). $\dashv$

As a result we obtain weak adequacy of our sequent calculi.

Theorem 5 (Adequacy).

$$\models \varphi \textit{ in respective class of } \mathbf{L}\textit{-frames iff } \text{SC-}\mathbf{L} \vdash\ \Rightarrow \varphi.$$

4. Cut-elimination

The calculus we have described is redundant. First of all, due to the shape of modal rules and an axiom we may easily show that both rules of weakening are admissible. In case of contraction the result does not hold in general. Note, that in D-logics we may need to derive $\Box\varphi, \Box\varphi, \Gamma \Rightarrow \Delta$ by means of (D-2). In such cases the proof of contraction elimination fails. Clearly, what is the most interesting is the admissibility of cut.

We provide the proof of cut admissibility applying the method due to Dragalin [3] (see the presentation for classical logic in Troelstra and Schwichtenberg [11] or in Negri and von Plato [9]). It goes by double induction on the cut-height and on the grade of cut-formula. Recall that the cut-formula is simply the formula which is eliminated from both premises and the grade of the cut-formula is its length (the number of distinct symbols).

In order to define cut-height we must first introduce the notion of the height of the proof of a sequent. It is defined inductively as follows: the proof of an axiom has height 0; if the proven sequent is obtained by means of one-premise rule and the height of the proof of the premise is n, then the height of the proof is $n+1$. If the proven sequent is obtained by means of two-premise rule and the height of the proofs of the premises is n and m, then the height of the proof is $\max(n, m) + 1$. Now, the cut-height of an instance of cut application is the sum of heights of the proofs of both premises.

By a principal formula of a rule application we mean a new formula which appear in a conclusion-sequent and is displayed as such in the schema of the rule. All the remaining formulae are called parametric. Note the important facts:[1]

1. In all modal rules the elements of multisets Γ, Δ which appear in conclusion-sequent are parametric (hence except parametric formulae which are just rewritten from premise-sequents in classical rules we have a category of new parametric formulae!)
2. In (E) and (D-2) we have two principal formulae.
3. In (4-2) and (5-2) there is one principal formula; the second (boxed) formula which is rewritten from premise-sequents is counted as parametric.

We omit a detailed analysis of the whole proof referring a reader to [9] or [11] and focus on representative cases for modal rules. Recall that in Dragalin's proof we first show that cut is completely eliminable when one of the premises is an instance of axiom, then we consider three exhaustive cases:

1. cut-formula not principal in the left premise
2. cut-formula principal in the left premise only
3. cut-formula principal in both premises

In all these cases one must consider subcases according to what rule was applied and show that either cut-height is reduced or its grade (even

[1] We do not consider (B-2) since no SC with this rule admits cut elimination.

if the height of new applications of cut is bigger) or it is completely eliminable.

Let us start with SC-**ED**.

Case 1 (cut-formula not principal in the left premise).

Case 1.1. left premise obtained by (E)

$$\dfrac{(\mathrm{E})\ \dfrac{\varphi \Rightarrow \psi \qquad \psi \Rightarrow \varphi}{\Box\varphi, \Gamma \Rightarrow \Delta, \Box\psi, \chi} \qquad \chi, \Pi \Rightarrow \Sigma}{\Box\varphi, \Gamma, \Pi \Rightarrow \Delta, \Sigma, \Box\psi}\ (\mathrm{Cut})$$

is replaced by a proof with no cut.

$$(\mathrm{E})\ \dfrac{\varphi \Rightarrow \psi \qquad \psi \Rightarrow \varphi}{\Box\varphi, \Gamma, \Pi \Rightarrow \Delta, \Sigma, \Box\psi}$$

Case 1.2. left premise obtained by (D-2) is dealt with analogously.

Case 2 (cut-formula principal in the left premise only).

In this case generally reduction is performed on the height of the right premise with no analysis of the the way in which the left premise is obtained. One may notice that again, similarly as in the in the preceding case, if (E) or (D-2) is applied in the right premise, then cut is totally eliminable due to the fact that parameters are newly introduced in the conclusion-sequent not rewritten from premises as in the extensional rules.

Let us consider as an example the case with (E) on the right.

$$(\mathrm{Cut})\ \dfrac{\Pi \Rightarrow \Sigma, \chi \qquad \dfrac{\varphi \Rightarrow \psi \qquad \psi \Rightarrow \varphi}{\chi, \Box\varphi, \Gamma \Rightarrow \Delta, \Box\psi}\ (\mathrm{E})}{\Box\varphi, \Gamma, \Pi \Rightarrow \Delta, \Sigma, \Box\psi}$$

It is replaced by a proof with no cut.

$$(\mathrm{E})\ \dfrac{\varphi \Rightarrow \psi \qquad \psi \Rightarrow \varphi}{\Box\varphi, \Gamma, \Pi \Rightarrow \Delta, \Sigma, \Box\psi}$$

If the right premise is obtained by (D-2) we have an analogous transformation.

Case 3 (cut-formula principal in both premises).

Case 3.1. Both cut-formulae principal obtained by the application of (E)

$$\dfrac{\text{(E)}\,\dfrac{\varphi \Rightarrow \psi \qquad \psi \Rightarrow \varphi}{\Box\varphi, \Gamma \Rightarrow \Delta, \Box\psi} \qquad \dfrac{\psi \Rightarrow \chi \qquad \chi \Rightarrow \psi}{\Box\psi, \Pi \Rightarrow \Sigma, \Box\chi}\,\text{(E)}}{\Box\varphi, \Gamma, \Pi \Rightarrow \Delta, \Sigma, \Box\chi}\,\text{(Cut)}$$

is replaced by a proof with two cuts of lesser grade.

$$\dfrac{\text{(Cut)}\,\dfrac{\varphi \Rightarrow \psi \qquad \psi \Rightarrow \chi}{\varphi \Rightarrow \chi} \qquad \dfrac{\chi \Rightarrow \psi \qquad \psi \Rightarrow \varphi}{\chi \Rightarrow \varphi}\,\text{(Cut)}}{\Box\varphi, \Gamma, \Pi \Rightarrow \Delta, \Sigma, \Box\chi}\,\text{(E)}$$

The case with (E) on the left and (D-2) on the right is done similarly. Neither the case with (D-2) on both sides nor with (D-2) on the left and (E) on the right does not apply since in (D-2) principal formulae occur only in the antecedent.

From the above cut admissibility may be extracted for SC-**E** alone.

5. The case of EN, E5 and EN5

Now let us consider SC-**EN5**

Case 1 (cut-formula not principal in the left premise).

Cases where cut-formula is a new parameter introduced by means of (N) or (5-2) on the left are dealt with exactly as in the subcase 1.1 from the preceding section, i.e. new proof is obtained with no cut, just from the premises of respective (N) or (5-2) application (cut-formula is not introduced). But one must remember that in the case of (5-2) we have also a parameter rewritten from premises and this case must be treated separately. There are two subcases to consider:

a. cut-formula on the right is not new (=parametric). In this case a reduction of cut-height is possible for the right premise in all the cases. Let us consider the case with some two-premise rule, e.g. ($\to\Rightarrow$) on the right. The proof

$$\dfrac{\text{(5-2)}\,\dfrac{\Rightarrow \Box\varphi, \psi \qquad \Box\varphi, \psi \Rightarrow}{\Gamma \Rightarrow \Delta, \Box\varphi, \Box\psi} \qquad \dfrac{\Box\varphi, \Pi \Rightarrow \Sigma, \chi \qquad \delta, \Box\varphi, \Pi \Rightarrow \Sigma}{\chi \to \delta, \Box\varphi, \Pi \Rightarrow \Sigma}\,(\to\Rightarrow)}{\chi \to \delta, \Gamma, \Pi \Rightarrow \Delta, \Sigma, \Box\psi}\,\text{(Cut)}$$

is replaced by a proof with two cuts of lesser heigth

$$
\text{(W)}\,\dfrac{\text{(}{\to}{\Rightarrow}\text{)}\,\dfrac{\text{(Cut)}\,\dfrac{\Rightarrow \Box\varphi, \Box\psi \qquad \Box\varphi, \Pi \Rightarrow \Sigma, \chi}{\Pi \Rightarrow \Sigma, \Box\psi, \chi} \qquad \dfrac{\Rightarrow \Box\varphi, \Box\psi \qquad \delta, \Box\varphi, \Pi \Rightarrow \Sigma}{\delta, \Pi \Rightarrow \Sigma, \Box\psi}\,\text{(Cut)}}{\chi \to \delta, \Pi \Rightarrow \Sigma, \Box\psi}}{\chi \to \delta, \Gamma, \Pi \Rightarrow \Delta, \Sigma, \Box\psi}
$$

where (W) is a shorthand for any (including empty) application of (W⇒) or (⇒W).

b. cut-formula on the right is new: either parametric in (E), (N) or (5-2) or principal in (E) In the former case we obtain a proof without cut; conclusion-sequent is derived just from premises of respective (E), (N) or (5-2) application (see also an example in Case 2 below). In the latter case we have the following:

$$
\text{(Cut)}\,\dfrac{\text{(5-2)}\,\dfrac{\Rightarrow \Box\varphi, \psi \qquad \Box\varphi, \psi \Rightarrow}{\Gamma \Rightarrow \Delta, \Box\varphi, \Box\psi} \qquad \dfrac{\varphi \Rightarrow \chi \qquad \chi \Rightarrow \varphi}{\Box\varphi, \Pi \Rightarrow \Sigma, \Box\chi}\,\text{(E)}}{\Gamma, \Pi \Rightarrow \Delta, \Sigma, \Box\psi, \Box\chi}
$$

which is replaced by:

$$
\dfrac{\text{(Cut)}\,\dfrac{\Rightarrow \Box\varphi, \psi \qquad \text{(E)}\,\dfrac{\varphi \Rightarrow \chi \qquad \chi \Rightarrow \varphi}{\Box\varphi \Rightarrow \Box\chi}}{\Rightarrow \Box\chi, \psi} \qquad \dfrac{\text{(E)}\,\dfrac{\chi \Rightarrow \varphi \qquad \varphi \Rightarrow \chi}{\Box\chi \Rightarrow \Box\varphi} \qquad \Box\varphi, \psi \Rightarrow}{\Box\chi, \psi \Rightarrow}\,\text{(Cut)}}{\Gamma, \Pi \Rightarrow \Delta, \Sigma, \Box\psi, \Box\chi}\,\text{(5-2)}
$$

One may easily notice that both cut applications have lesser height.

Case 2 (cut-formula principal in the left premise only).

In two cases where cut-formula on the left is principal and obtained by the application of (N) or (5-2) we have a similar situation on the right: either cut-formula is a rewritten (from premis(es) of instances of extensional rules) parameter or a new parameter (the case of the application of modal rules). In the former situation it is a reduction on the height of cut in the right premise (exactly as in the point a. above). In the latter case we just obtain a proof without cut directly from premises (exactly as in b. above). As an example consider the following:

$$
\text{(Cut)}\,\dfrac{\text{(N)}\,\dfrac{\Rightarrow \varphi}{\Gamma \Rightarrow \Delta, \Box\varphi} \qquad \dfrac{\Rightarrow \Box\psi, \chi \qquad \chi, \Box\psi \Rightarrow}{\Box\varphi, \Pi \Rightarrow \Sigma, \Box\psi, \Box\chi}\,\text{(5-2)}}{\Gamma, \Pi \Rightarrow \Delta, \Sigma, \Box\psi, \Box\chi}
$$

which is replaced by

$$
\dfrac{\Rightarrow \Box\psi, \chi \qquad \chi, \Box\psi \Rightarrow}{\Gamma, \Pi \Rightarrow \Delta, \Sigma, \Box\psi, \Box\chi}\,\text{(5-2)}
$$

Analogously in all remaining cases (all combinations of (N), (E) and (5-2) on the left (cut-formula principal) and the right (cut-formula as a new parameter).

Case 3 (cut-formula principal in both premises).

Taking into account that in (N) and (5-2) principal formulae are only in the succedent we have only 2 subcases, namely (N) versus (E) and (5-2) versus (E) (and clearly with both cut-formulae obtained through (E) which was considered already in the previous section). In both cases we have a reduction of cut-degree. Let us illustrate the point with the second (more difficult) case:

$$\text{(Cut)}\ \dfrac{\text{(5-2)}\ \dfrac{\Rightarrow \Box\varphi, \psi \qquad \Box\varphi, \psi \Rightarrow}{\Gamma \Rightarrow \Delta, \Box\varphi, \Box\psi} \qquad \dfrac{\psi \Rightarrow \chi \qquad \chi \Rightarrow \psi}{\Box\psi, \Pi \Rightarrow \Sigma, \Box\chi}\ \text{(E)}}{\Gamma, \Pi \Rightarrow \Delta, \Sigma, \Box\varphi, \Box\chi}$$

is replaced by:

$$\text{(5-2)}\ \dfrac{\text{(Cut)}\ \dfrac{\Rightarrow \Box\varphi, \psi \qquad \psi \Rightarrow \chi}{\Rightarrow \Box\varphi, \chi} \qquad \dfrac{\chi \Rightarrow \psi \qquad \Box\varphi, \psi \Rightarrow}{\chi, \Box\varphi \Rightarrow}\ \text{(Cut)}}{\Gamma, \Pi \Rightarrow \Delta, \Sigma, \Box\varphi, \Box\chi}$$

From the above proof cut admissibility may be extracted also for SC-**EN** and SC-**E5**.

6. The case of some congruent T-logics

Now let us consider SC-**EN4T**

Case 1 (cut-formula not principal in the left premise).

If the left premise is obtained by (4-2) it is dealt with exactly as in Case 1.1, from Section 4, i.e. we can derive conclusion-sequent directly from premises of (4-2) application with no cut (the cases on (E) and (N) were already considered). If the left premise is obtained by (T) it is dealt with exactly as in the case of extensional rules, i.e. we permute (T) with (Cut) and reduce in this way cut-height.

$$\dfrac{\dfrac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Box\varphi, \Gamma \Rightarrow \Delta, \psi}\ \text{(T)} \qquad \psi, \Pi \Rightarrow \Sigma}{\Box\varphi, \Gamma, \Pi \Rightarrow \Delta, \Sigma}\ \text{(Cut)}$$

is replaced by

$$\dfrac{\dfrac{\varphi, \Gamma \Rightarrow \Delta, \psi \qquad \psi, \Pi \Rightarrow \Sigma}{\varphi, \Gamma, \Pi \Rightarrow \Delta, \Sigma}\,(\mathrm{Cut})}{\Box\varphi, \Gamma, \Pi \Rightarrow \Delta, \Sigma}\,(\mathrm{T})$$

Case 2 (cut-formula principal in the left premise only).

The cases where the cut-formula on the left is principal formula of the application of (E), (N) or (4-2) (the case of (T) does not apply since the principal formula is in the antecedent) and on the right we have an application of (T) with cut-formula parametric is reduced on height exactly as in cases of extensional rules.

The cases where a parametric cut-formula on the right is new (an application of any modal rule except (T)) is dealt with exactly as in point b. above, i.e. no cut is needed to obtain conclusion-sequent directly from the premises of the left part of the proof.

We must consider separately a case where parametric cut-formula on the right is not new but obtained through the application of (4-2). Consider a subcase with the left premise obtained by means of the same rule.

$$(\mathrm{Cut})\,\dfrac{(\text{4-2})\,\dfrac{\Box\varphi \Rightarrow \psi \qquad \psi \Rightarrow \Box\varphi}{\Box\varphi, \Gamma \Rightarrow \Delta, \Box\psi} \qquad \dfrac{\Box\psi \Rightarrow \chi \qquad \chi \Rightarrow \Box\psi}{\Box\psi, \Pi \Rightarrow \Sigma, \Box\chi}\,(\text{4-2})}{\Box\varphi, \Gamma, \Pi \Rightarrow \Delta, \Sigma, \Box\chi}$$

is replaced by

$$(\text{4-2})\,\dfrac{(\mathrm{Cut})\,\dfrac{(\text{4-2})\,\dfrac{\Box\varphi \Rightarrow \psi \qquad \psi \Rightarrow \Box\varphi}{\Box\varphi \Rightarrow \Box\psi} \qquad \Box\psi \Rightarrow \chi}{\Box\varphi \Rightarrow \chi} \qquad \dfrac{\chi \Rightarrow \Box\psi \qquad \dfrac{\psi \Rightarrow \Box\varphi}{\Box\psi \Rightarrow \Box\varphi}\,(\mathrm{T})}{\chi \Rightarrow \Box\varphi}\,(\mathrm{Cut})}{\Box\varphi, \Gamma, \Pi \Rightarrow \Delta, \Sigma, \Box\chi}$$

Note that both applications of cut have lesser height and that (T) was necessary to transform the proof. The latter observation shows why we cannot demonstrate cut admissibility for SC-**E4**.

The subcases where the left premise is obtained by means of (E) or (N) and parametric cut-formula on the right is not new but obtained through the application of (4-2) are dealt with similarly.

Case 3 (cut-formula principal in both premises).

Neither (N) nor (4-2) have principal formula in the antecedent so the subcases with these rules on the right do not apply and (E) was considered previously. The subcases where cut-formula on the right is the principal formula of (T) application and on the left we have one of the (E), (N), (4-2) ((T) does not apply) are dealt with similarly, by reduction of cut-height and weakening (if needed). Let us illustrate the point with one example.

$$\dfrac{\dfrac{\varphi \Rightarrow \psi \qquad \psi \Rightarrow \varphi}{\Box\varphi, \Gamma \Rightarrow \Delta, \Box\psi}\,(E) \qquad \dfrac{\psi, \Pi \Rightarrow \Sigma}{\Box\psi, \Pi \Rightarrow \Sigma}\,(T)}{\Box\varphi, \Gamma, \Pi \Rightarrow \Delta, \Sigma}\,(Cut)$$

is replaced by

$$\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \psi \qquad \psi, \Pi \Rightarrow \Sigma}{\varphi, \Pi \Rightarrow \Sigma}\,(Cut)}{\Box\varphi, \Pi \Rightarrow \Sigma}\,(T)}{\Box\varphi, \Gamma, \Pi \Rightarrow \Delta, \Sigma}\,(W)$$

where (W) is a shorthand for any (including empty) application of (W$\Rightarrow$) or ($\Rightarrow$W).

Note that from the above proof we can extract cut admissibility for SC-**E4T**, SC-**ENT** and SC-**ET**.

Summing up we have obtained the following

Theorem 6. (Cut) *is admissible for* SC-**E**, SC-**E5**, SC-**ET**, SC-**E4T**, *their* N-*counterparts, and for* SC-**ED**.

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