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WHITEHEAD'S POINTFREE GEOMETRY AND DIAMETRIC POSETS

Abstract. This note is motivated by Whitehead's researches in inclusion-based point-free geometry as exposed in *An Inquiry Concerning the Principles of Natural Knowledge* and in *The concept of Nature*. More precisely, we observe that Whitehead's definition of point, based on the notions of abstractive class and covering, is not adequate. Indeed, if we admit such a definition it is also questionable that a point exists. On the contrary our approach, in which the diameter is a further primitive, enables us to avoid such a drawback. Moreover, since such a notion enables us to define a metric in the set of points, our proposal looks to be a good starting point for a foundation of the geometry metrical in nature (as proposed, for example, by L. M. Blumenthal).

Keywords: Whitehead, pointfree geometry, lattices, diameters.

1. Introduction

The first researches in pointfree geometry go back to three books, philosophical in nature, of the mathematician and philosopher A. N. Whitehead. Indeed, the content of these books (independently from Whitehead's motivations) could be interpreted as an attempt to define the usual Euclidean geometry by assuming as a primitive the notion of region and either the one of inclusion [13], [14] or the one (topological in nature) of connection [15]. Points and lines are defined by suitable classes of regions (named "abstractive processes").

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Successively, several approaches to pointfree geometry were proposed (for a brief survey see [6]). In particular, several attempts where metrical in nature. For example in [5] and [8] the notions of distance between regions and diameter are assumed as primitives. Moreover, in [4] one refers to the interval-distance. Now, a very interesting series of researches for representation theorems for lattices with a diameter can be interpreted in the framework of pointfree geometry (see the papers of B. Banaschewski and A. Pultr ([1], [2], [11], [12]) and, also, of F. Previale ([9], [10]). In all these approaches the starting point is a lattice equipped with a suitable "diameter" and one defines a notion of point and a distance between points by obtaining a metric space.

In this note we consider a notion of diameter but we refer to posets which are not necessarily lattices and to a notion of point rather close to the one proposed by Whitehead.

The aim is foundational in nature and the final task is to individuate a suitable set of axioms in order to obtain that such a metrical space is isometric with the canonical metric space of an Euclidean space. The possibility of such an enterprise is suggested by the existing proposals for a metrical foundations of geometry (see for example L. M. Blumenthal [3]).

2. Diametric posets

Given a poset $\boldsymbol{X} = (X, \leq)$, we say that an element x is the *minimum* (or *least element*) of $\boldsymbol{X}$ iff for every $y \in X$, $x \leq y$. Moreover, we define the *overlapping relation* O by setting $x \mathsf{O} y$ iff there is an element z such that z is not the minimum in $\boldsymbol{X}$ and $z \leq x$ and $z \leq y$. Of course, O is symmetric. Let $X_+ := \{x \in P \mid x \text{ is not the minimum of } \boldsymbol{X}\}$. We have $\mathsf{O} \subseteq X_+^2$ and for all $x, y \in X_+$: $x \mathsf{O} x$ and if $x \leq y$ then $x \mathsf{O} y$.

Given two posets $(X, \leq)$ and $(X', \leq')$ and $n \in \mathbb{N}$, we say that a map $h\colon X^n \to X'$ is *order-preserving* (resp. *order-reversing*) provided that for all $x_1, \ldots, x_n, y_1, \ldots, y_n$ from X:

$$x_1 \leq y_1, \ldots, x_n \leq y_n \implies h(x_1, \ldots, x_n) \leq' h(y_1, \ldots, y_n)$$
$$(h(y_1, \ldots, y_n) \leq' h(x_1, \ldots, x_n), \text{ respectively}).$$

Definition 2.1. A *diametric poset* is a structure $(\mathfrak{R}, \leq, \delta)$, where $(\mathfrak{R}, \leq)$ is a poset without a minimum and the *diameter* $\delta\colon \mathfrak{R} \to [0, +\infty]$ is a function such that,

A₁ for each $x, y \in \mathfrak{R}$: $x \leq y \implies \delta(x) \leqslant \delta(y)$ (δ is order-preserving),

A₂ for each $x, y \in \mathfrak{R}$: $x \mathsf{O} y \implies$ there exists r such that $x \leq r$, $y \leq r$ and $\delta(r) \leqslant \delta(x) + \delta(y)$,

A₃ given $x \in \mathfrak{R}$, for every $\varepsilon > 0$ there is $r \leq x$ such that $\delta(r) \leqslant \varepsilon$,

A₄ for each $x, y \in \mathfrak{R}$ there is z such that $\delta(z) < +\infty$, $x \mathsf{O} z$ and $y \mathsf{O} z$.

Since we will give a geometric interpretation of a diametric poset, we call *regions* the elements of $\mathfrak{R}$, *inclusion* the relation $\leq$ (obviously, $\leq$ is not the set-theoretical inclusion, in general). A *bounded region* is a region with a finite diameter.

Examples. Consider a poset without a minimum and with the property that for every x and y there exists z such that $x \mathsf{O} z$ and $y \mathsf{O} z$. Then, such a poset can be viewed as a diametric poset if we define δ as the function identically zero. Another example is obtained by assuming that δ is a measure in an Euclidean space and that $\mathfrak{R}$ is the related class of measurable nonempty subsets.

We obtain more meaningful examples by setting $\mathfrak{R}$ equal to a suitable class $\mathcal{C}$ of nonempty subsets of a pseudo-metric space (S, d) and $\delta \colon \mathcal{C} \to [0, \infty]$ be the usual diameter defined by setting, for every $X \in \mathcal{C}$,

$$\delta(X) := \sup\{d(x, y) \mid x, y \in X\}. \tag{2.1}$$

We call *canonical* the diametric posets obtained in such a way. An example is given by the lattice of all open nonempty subsets of (S, d). Another example is obtained by the family of closed disks of an Euclidean space.

The property expressed by **A₂** extends to a finite number of regions.

PROPOSITION 2.2. *Given a diametric poset* $(\mathfrak{R}, \leq, \delta)$, *if* x_1, x_2, ..., x_n *are regions such that* $x_1 \mathsf{O} x_2$, ..., $x_{n-1} \mathsf{O} x_n$, *then a region* r *exists such that*

$$x_1 \leq r, \dots, x_n \leq r \;\; \& \;\; \delta(r) \leqslant \delta(x_1) + \dots + \delta(x_n).$$

PROOF. In the case $n = 1$ the proved thesis is obvious. Assume that it holds true in the case $n - 1$, i.e., there exists r_0 such that

$$x_1 \leq r_0, \dots, x_{n-1} \leq r_0 \;\; \text{and} \;\; \delta(r_0) \leqslant \delta(x_1) + \dots + \delta(x_{n-1}).$$

Then $x_{n-1} \leq r_0$ and $x_{n-1} \mathsf{O} x_n$ entail $r_0 \mathsf{O} x_n$ and by **A₂** an upper bound r of both r_0 and x_n exists such that $\delta(r) \leqslant \delta(r_0) + \delta(x_n)$. Hence r is an upper bound of x_1, ..., x_n such that $\delta(r) \leqslant \delta(x_1) + \dots + \delta(x_n)$. ⊣

From $\mathbf{A_4}$ and Proposition 2.2 we obtain the following fact.

Proposition 2.3. *For each $x, y \in \mathfrak{R}$ there are $z, r \in \mathfrak{R}$ such that*

$$x \leq r, y \leq r, z \leq r \;\; \& \;\; \delta(z) < +\infty \;\; \& \;\; \delta(r) \leqslant \delta(x) + \delta(y) + \delta(z).$$

Consequently, every diametric poset is upward directed, i.e., for every $x, y \in \mathfrak{R}$ there is $r \in \mathfrak{R}$ such that $x \leq r$ and $y \leq r$.

Proof. By $\mathbf{A_4}$, for every $x, y \in \mathfrak{R}$ there is z such that $x \mathrel{\mathsf{O}} z$, $z \mathrel{\mathsf{O}} y$ and $\delta(z) < +\infty$. So we use Proposition 2.2. ⊣

Lemma 2.4. *For each $x, y \in \mathfrak{R}$ the set $\{\delta(z) \mid z \mathrel{\mathsf{O}} x \;\&\; z \mathrel{\mathsf{O}} y\}$ is nonempty and has the infimum (the greatest lower bound) which is finite.*

Proof. By $\mathbf{A_4}$, $Z := \{z \mid \delta(z) < +\infty \;\&\; x \mathrel{\mathsf{O}} z \;\&\; y \mathrel{\mathsf{O}} z\} \neq \emptyset$. Note that 0 is a lower bound of the set $\{\delta(z) \mid z \in Z\} \subseteq \mathbb{R}$. So this set has the finite infimum and $\inf\{\delta(z) \mid z \in Z\} = \inf\{\delta(z) \mid z \mathrel{\mathsf{O}} x \;\&\; z \mathrel{\mathsf{O}} y\}$. ⊣

Now, we define the notion of "lower distance" between two regions. From Lemma 2.4 the following function σ is well-defined, i.e., it takes only finite values.

Definition 2.5. *Lower distance* is the function $\sigma\colon \mathfrak{R} \times \mathfrak{R} \to [0, +\infty)$ defined as follows

$$\sigma(x, y) := \inf\{\delta(z) \mid z \mathrel{\mathsf{O}} x \text{ and } z \mathrel{\mathsf{O}} y\}.$$

It is immediate that σ is order-reversing. Moreover, we obtain:

Lemma 2.6. *For each $x, y \in \mathfrak{R}$: if $x \mathrel{\mathsf{O}} y$, then $\sigma(x, y) = 0$.*

Proof. Let $x \mathrel{\mathsf{O}} y$ and z be a region included in x and in y. Then from $\mathbf{A_3}$ it follows that for every $\varepsilon > 0$ there is $r \leq z$ such that $\delta(r) \leqslant \varepsilon$. Since $r \mathrel{\mathsf{O}} x$ and $r \mathrel{\mathsf{O}} y$, we obtain $\sigma(x, y) \leqslant \delta(r) \leqslant \varepsilon$. As a consequence $\sigma(x, y) = 0$. ⊣

Notice that the function σ satisfies conditions similar to those of a pseudometric space:

Theorem 2.7. *For every $x, y, z \in \mathfrak{R}$,*

(i) $\sigma(x, y) = \sigma(y, x)$,

(ii) $\sigma(x, x) = 0$,

(iii) $\sigma(x, y) \leqslant \sigma(x, z) + \sigma(z, y) + \delta(z)$.

PROOF. The property (i) is obvious. From Lemma 2.6 we obtain (ii). To prove (iii), by $\mathbf{A_4}$, let u be a region such that $u \mathrel{\mathsf{O}} x$ and $u \mathrel{\mathsf{O}} z$, and v be a region such that $v \mathrel{\mathsf{O}} z$ and $v \mathrel{\mathsf{O}} y$. Since $u \mathrel{\mathsf{O}} z$ and $v \mathrel{\mathsf{O}} z$, by Proposition 2.2, there is an upper bound r of u, v and z such that $\delta(r) \leqslant \delta(u) + \delta(v) + \delta(z)$. Since $r \mathrel{\mathsf{O}} x$ and $r \mathrel{\mathsf{O}} y$, $\sigma(x, y) \leqslant \delta(r) \leqslant \delta(u) + \delta(v) + \delta(z)$. Thus

$$\begin{aligned}\sigma(x,y) \leqslant \inf\{\delta(u) \mid u \mathrel{\mathsf{O}} x, u \mathrel{\mathsf{O}} z\} + \inf\{\delta(v) \mid v \mathrel{\mathsf{O}} z, v \mathrel{\mathsf{O}} y\} + \delta(z) \\ = \sigma(x,z) + \sigma(z,y) + \delta(z). \quad \dashv\end{aligned}$$

We consider also a notion of "upper distance" between two regions. Note that since every diametric poset is upward directed, the following function Σ is well-defined.

DEFINITION 2.8. *Upper distance* is the function $\Sigma\colon \mathfrak{R} \times \mathfrak{R} \to [0, +\infty]$ defined as follows

$$\Sigma(x,y) := \inf\{\delta(z) \mid x \leq z \text{ and } y \leq z\}.$$

It is immediate that Σ is order-preserving. For Σ we prove chosen properties similar to those of a pseudometric space:

THEOREM 2.9. *For every x, y and z from $\mathfrak{R}$,*

(i) $\Sigma(x, y) = \Sigma(y, x)$,

(ii) $\Sigma(x, x) = \delta(x)$,

(iii) $\Sigma(x, y) \leqslant \Sigma(x, z) + \Sigma(z, y)$.

PROOF. The properties (i) and (ii) are obvious. Since every diametric poset is upward directed, to prove (iii) we consider two regions u and w such that $x \leq u$, $z \leq u$ and $z \leq w$, $y \leq w$. Then $u \mathrel{\mathsf{O}} w$ and, by $\mathbf{A_2}$, there exists r such that $u \leq r$, $w \leq r$ and $\delta(r) \leqslant \delta(u) + \delta(w)$. Since $x \leq r$ and $y \leq r$, then $\Sigma(x, y) \leqslant \delta(r) \leqslant \delta(u) + \delta(w)$. Thus,

$$\begin{aligned}\Sigma(x,y) \leqslant \inf\{\delta(u) \mid x \leq u \;\&\; z \leq u\} + \inf\{\delta(w) \mid z \leq w \;\&\; y \leq w\} \\ = \Sigma(x,z) + \Sigma(z,y). \quad \dashv\end{aligned}$$

Note that if there is a region x such that $\delta(x) \neq 0$, then σ is not a pseudo-distance, since (iii) is not a triangular inequality. Besides Σ is not a pseudo-distance, since $\Sigma(x, x) = \delta(x) \neq 0$.

Theorem 2.10. *For every pair x and y of regions,*

$$0 \leqslant \Sigma(x,y) - \sigma(x,y) \leqslant \delta(x) + \delta(y).$$

Moreover,

$$\Sigma(x,y) \text{ is finite iff } x \text{ and } y \text{ are both bounded.}$$

Proof. Trivially, $\Sigma(x,y) - \sigma(x,y) \geqslant 0$. To prove the second inequality, note that, by $\mathbf{A_4}$ and Proposition 2.2, there are z and r such that $x \mathrel{\mathsf{O}} z$, $y \mathrel{\mathsf{O}} z$, $x \leq r$, $y \leq r$, $z \leq r$ and $\delta(r) \leqslant \delta(x) + \delta(y) + \delta(z)$. Therefore $\Sigma(x,y) \leqslant \delta(r) \leqslant \delta(x) + \delta(y) + \delta(z)$. Thus,

$$\begin{aligned}\Sigma(x,y) \leqslant \delta(x) + \delta(y) + \inf\{\delta(z) \mid x \mathrel{\mathsf{O}} z \;\&\; y \mathrel{\mathsf{O}} z\}& \\ = \delta(x) + \delta(y) + \sigma(x,y).&\end{aligned}$$

The remaining part of the theorem we obtain by $\mathbf{A_1}$ and Lemma 2.4. ⊣

3. Regions we can consider approximately as a point

In accordance with common intuition, the small regions can be approximately regarded as points.

Definition 3.1. Let $(\mathfrak{R}, \leq, \delta)$ be a diametric poset. Given $\varepsilon \geqslant 0$, we call *ε-point* a region whose diameter is less than or equal to ε. We denote by P_ε the class of all ε-points.

Then, we can reformulate $\mathbf{A_3}$, by saying that however we fix $\varepsilon > 0$, each region includes at least an ε-point. Assume that ε is so small as to be negligible. Then, in the class of ε-points, the property (iii) in Theorem 2.7 becomes, approximately, the triangular inequality and therefore (P_ε, σ) is, approximately, a pseudometric space. Analogously, in accordance with condition (ii) of Theorem 2.9, (P_ε, Σ) is, approximately, a pseudometric space. Moreover in accordance with Theorem 2.10, if ε is negligible then σ and Σ are approximately equal. In the case $\varepsilon = 0$ we obtain the following obvious results.

Proposition 3.2. *Suppose that the set P_0 of regions with zero diameter is nonempty. Then in P_0 the function σ coincides with Σ and $(P_0, \sigma) = (P_0, \Sigma)$ is a pseudometric space.*

Furthermore, the set of all atoms At of $(\mathfrak{R}, \leq)$ is included in P_0 and therefore, if $At \neq \emptyset$ then (At, σ) is a subspace of (P_0, σ).

PROOF. We prove only that $At \subseteq P_0$. Indeed, if a is an atom and $\delta(a) \neq 0$ then, by $\mathbf{A_3}$, there is $r \leq a$ such that $\delta(r) \leqslant \delta(a)/2 < \delta(a)$. As a consequence $z \lneq a$, and so we obtain a contradiction. ⊣

Observe that $P_0 \neq At$, in general. For example, consider the diametric poset defined by a measure in the Euclidean plane. In such a case each set of zero measure with more than one point has diameter zero although it is not an atom.

PROPOSITION 3.3. 1. *For every a, a', b, $b' \in P_0$: if $a' \leq a$ and $b' \leq b$, then $\sigma(a, b) = \sigma(a', b')$.*

2. *If (P_0, σ) is a metric space, then $P_0 = At$.*

PROOF. 1. Since σ is order-reversing, $\sigma(a, b) \leqslant \sigma(a', b')$, and since Σ is order-preserving, $\Sigma(a, b) \geqslant \Sigma(a', b')$. Finally, by Proposition 3.2, σ and Σ coincide in P_0, so $\sigma(a, b) = \sigma(a', b')$.

2. By Proposition 3.2, $At \subseteq P_0$. Assume that (P_0, σ) is a metric space and $a \in P_0$. If $x \leq a$, since $x \bigcirc a$, then $\sigma(x, a) = 0$. Therefore $x = a$, by the assumption. Hence $a \in At$. ⊣

Example. There is a diametric poset for which $At = P_0 \neq \emptyset$ and (P_0, σ) is not a metric space.

For any $a, b, c, d \in \mathbb{R}$ such that $a < b$ and $c < d$ let R_{ab}^{cd} be the open rectangle in $\mathbb{R}^2$ with four vertices (a, c), (b, c), (b, d) and (a, c). Let $\mathbf{R}$ be the set of all such rectangles. We put $\mathfrak{R} := \mathbf{R} \cup$ the set of all singletons in $\mathbb{R}^2$, $\leq := \subseteq$ and for any $x \in \mathfrak{R}$

$$\delta(x) := \begin{cases} 0 & \text{if } x \text{ is a singleton} \\ |d - c| & \text{if } x = \mathrm{R}_{ab}^{cd} \text{ for some } a, , b, c, d \in \mathbb{R} \end{cases}$$

Of course, $(\mathfrak{R}, \leq, \delta)$ is a diametric poset and $At =$ the set of all singletons in $\mathbb{R}^2 = P_0$. Note that for any $a, b, c \in \mathbb{R}$, if $x = \{(a, b)\}$ and $y = \{(c, b)\}$, then $\sigma(x, y) = 0$. Thus, (P_0, σ) is not a metric space.

4. Whitehead's abstraction processes to define the points

One of the main task in pointfree geometry is to give a good definition of point. Now, although the atoms and the 0-points are good candidates to represent the points, the spirit of pointfree geometry is

– to start from structures in which no point exists,
– to build up the points through suitable "abstraction processes".

The notion of abstraction process was defined by Whitehead as suitable classes of regions. Indeed an *abstractive class* is a set R of regions which is totally ordered and such that there is no region included in all the regions in R. Such a condition is putted in order to avoid abstractive classes representing a region (Whitehead's idea is that an abstractive class represents an "abstract" geometrical entity). We modify slightly such a definition by skipping out such a condition and confining ourselves only to enumerable classes.

DEFINITION 4.1. Given a diametric poset $(\mathfrak{R}, \leq, \delta)$, a sequence $R = (r_n)_{n\in\mathbb{N}}$ of regions such that $r_{i+1} \leq r_i$ is called an *abstractive class*. By $\boldsymbol{AC}$ we denote the class of all abstractive classes.

From an intuitive point of view, an abstractive class $R = (r_n)_{n\in\mathbb{N}}$ represents a geometrical entity which is "*the limit*" of the sequence $(r_n)_{n\in\mathbb{N}}$. Obviously, it is possible that two different abstractive classes represent the same entity. Then Whitehead introduces the *covering* relation $\leq_{\mathrm{c}}$ by setting, for any $R = (r_n)_{n\in\mathbb{N}}$ and $S = (s_n)_{n\in\mathbb{N}}$ in $\boldsymbol{AC}$,

$$R \leq_{\mathrm{c}} S \iff \text{for each } n \in \mathbb{N} \text{ there exists } m \in \mathbb{N} \text{ such that } r_m \leq s_n.$$

The covering relation $\leq_{\mathrm{c}}$ is a pre-order and therefore it is associated with the equivalence relation $\equiv_{\mathrm{c}}$ defined by setting

$$R \equiv_{\mathrm{c}} S \iff R \leq_{\mathrm{c}} S \text{ and } S \leq_{\mathrm{c}} R\,.$$

Also, $\leq_{\mathrm{c}}$ is extended to the quotient $\boldsymbol{AC}/_{\equiv_{\mathrm{c}}}$ by setting

$$[R] \leq_{\mathrm{c}} [T] \iff R \leq_{\mathrm{c}} T\,.$$

A *geometrical element* is any element of the quotient $\boldsymbol{AC}/_{\equiv_{\mathrm{c}}}$, a *point* is a geometrical element which is minimal in $(\boldsymbol{AC}/_{\equiv_{\mathrm{c}}}, \leq_{\mathrm{c}})$.

Unfortunately this definition of point is not satisfactory. To show this, we will consider the following example.

Example. Consider in $\mathbb{R}^2$ the abstractive class G defined by the sequence B_n of closed balls with center in the origin $(0,0)$ and radius $r_n = \frac{1}{n}$. From an intuitive point of view such an abstractive class represents a point. Now, consider the classes G_1 and G_2 defined by the sequences of closed balls with center in $(\frac{-1}{n}, 0)$ and $(\frac{1}{n}, 0)$, respectively, and radius r_n (see Fig. 1). Then G covers both the classes G_1 and G_2, but it is not equivalent with these classes. Thus $[G]$ is not minimal and it cannot represent a point. In the other words, since G, G_1 and G_2 are not equivalent the Euclidean point $P = (0, 0)$ is split in three different "geometrical elements" P^-, P, P^+ and this is far from the planned aim of Whitehead and from the traditional approach to geometry. On the other hand we are not convinced at all that in the "natural" models of the theory of Whitehead a point exists. Perhaps Whitehead's passage from the inclusion-based approach to the connection-based approach was done to avoid such a counterintuitive behaviour.

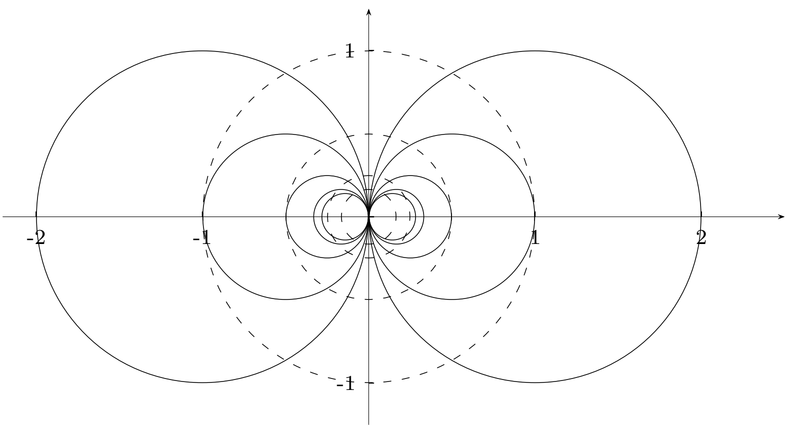

Figure 1. The classes G, G_1 and G_2

The metrical notions of diameter and distance enables us to adopt a different strategy.

Definition 4.2. Given an abstractive class $R = (r_n)_{n\in\mathbb{N}}$ we call *diameter* of R the number

$$\delta(R) := \lim_{n\to\infty} \delta(r_n).$$

Given two abstractive classes $R = (r_n)_{n\in\mathbb{N}}$ and $S = (s_n)_{n\in\mathbb{N}}$, the *lower distance* $\sigma(R, S)$ and the *upper distance* $\Sigma(R, S)$ are defined by setting

$$\sigma(R, S) := \lim_{n\to\infty} \sigma(r_n, s_n),$$

$$\Sigma(R,S) := \lim_{n\to\infty} \Sigma(r_n, s_n).$$

The above limits exist, since $\delta\colon \mathfrak{R} \to [0,+\infty]$ and $\Sigma\colon \mathfrak{R}\times\mathfrak{R} \to [0,+\infty]$ are order-preserving and $\sigma\colon \mathfrak{R}\times\mathfrak{R} \to [0,+\infty)$ is order-reversing. Moreover, for each abstractive classes $R = (r_n)_{n\in\mathbb{N}}$ and $S = (s_n)_{n\in\mathbb{N}}$ we obtain,

$$\delta(R) = \inf\{\delta(r_i) \mid i \in \mathbb{N}\}, \tag{4.2}$$

$$\sigma(R,S) = \sup\{\sigma(r_i, s_j) \mid i,j \in \mathbb{N}\}, \tag{4.3}$$

$$\Sigma(R,S) = \inf\{\Sigma(r_i, s_j) \mid i,j \in \mathbb{N}\}. \tag{4.4}$$

Observe that we can identify a region r with the abstractive class $(r_n)_{n\in\mathbb{N}}$ such that $r_n = r$, for every $n \in \mathbb{N}$. Then the just given definitions of diameter, lower distance and upper distance extend the analogous ones for regions.

Definition 4.3. We call *infinitesimal* an abstractive class R whose diameter is 0.

Denote by $\boldsymbol{IAC}$ the class of infinitesimal abstractive classes. The proof of the following theorem is immediate.

Theorem 4.4. *The functions σ and Σ coincide in $\boldsymbol{IAC}$ and $(\boldsymbol{IAC}, \sigma)$ is a pseudometric space.*

As usual, we can associate $(\boldsymbol{IAC}, \sigma)$ with a metric space. To do this, we call *equiconvergent* two infinitesimal abstractive classes R and S such that $\sigma(R,S) = 0$ and we denote by $\equiv$ such a relation, i.e. for each $R, S \in \boldsymbol{IAC}$,

$$R \equiv S \overset{\text{df}}{\iff} \sigma(R,S) = 0.$$

Definition 4.5. The metric space $(\boldsymbol{IAC}/_{\equiv}, \sigma)$ obtained as a quotient of $(\boldsymbol{IAC}, \sigma)$ modulo the equiconvergence is called *metric space associated with a diametric poset*. Any element of $\boldsymbol{IAC}/_{\equiv}$ is called a *point*.

Observe that while Whitehead's equivalence based on the notion of covering entails the equiconvergence. The following proposition, whose proof is immediate, shows that the converse implication is false.

Proposition 4.6. *The three abstractive classes in the above considered example are equiconvergent and therefore they represent the same point. This in spite of the fact that they are not equivalent in Whitehead's sense.*

5. Points by semifilters and filters

In accordance with the tradition of the representation theorems for distributive lattices, it is possible to define the points by the notion of filter.

DEFINITION 5.1. Given a poset $(X, \leq)$, a nonempty subset H of X is a *semifilter* iff H is downward directed (i.e., for every $x, y \in H$ there is $z \in H$ such that $z \leq x$ and $z \leq y$). A *filter* is a semifilter which is an upper-set (i.e., for every $x \in H$ and $y \in X$, if $x \leq y$ then $y \in H$).

If X is a filter, then we say that X is the *improper* filter.

Given a semifilter H, by setting

$$\underline{H} := \{x \in \mathfrak{R} \mid x \text{ includes an element } h \in H\}$$

one obtains the *filter generated by* H, i.e., the smallest filter including H. Given $x \in X$, the smallest semifilter containing x as an element is the singleton $\{x\}$, and the smallest filter containing x is the *principal filter*

$$F_x = \{z \in \mathfrak{R} \mid x \leq z\}.$$

Notice that the intersection of two filters is not necessarily a semifilter.

Given a diametric poset $(\mathfrak{R}, \leq, \delta)$, we denote by $\boldsymbol{SF}$ and $\boldsymbol{F}$ the class of semifilters and the class of filters, respectively.

DEFINITION 5.2. The *diameter* of a semifilter H is the number

$$\delta(H) := \inf\{\delta(h) \mid h \in H\}.$$

The *lower distance* between two proper semifilters H and K is the number

$$\sigma(H, K) := \sup\{\sigma(h, k) \mid h \in H \text{ and } k \in K\}.$$

The *upper distance* is the number

$$\Sigma(H, K) := \inf\{\Sigma(h, k) \mid h \in H \text{ and } k \in K\}.$$

These functions extend the corresponding ones early defined for the abstractive classes. This is evident provided we identify every region x with the semifilter $\{x\}$. The same holds true if we identify x with the principal filter $F_x = \{z \in \mathfrak{R} \mid x \leq z\}$ generated by x.

PROPOSITION 5.3. *The lower distance $\sigma(H, K)$ between two semifilters H and K is not necessarily finite. Nevertheless, if H and K have finite diameters, then $\sigma(H, K)$ is finite.*

PROOF. Let us consider the diametric poset defined by the open sets in $\mathbb{R}^2$ and two semifilters $H := \{\mathbb{D}_c \mid c > 0\}$ and $K := \{\mathbb{S}_c : c < 0\}$, where $\mathbb{D}_c$ is right half-plane $\{(x, y) \in \mathbb{R}^2 \mid x > c\}$ and $\mathbb{S}_c$ is the left half-plane $\{(x, y) \in \mathbb{R}^2 \mid x < c\}$. Then, $\sigma(H, K) = +\infty$.

Let us suppose that $\delta(H)$ and $\delta(K)$ are finite. Then $h_0 \in H$ and $k_0 \in K$ exist such that $\delta(h_0)$ and $\delta(k_0)$ are finite. By Proposition 2.3 there are z_0 and r_0 such that $\delta(z_0) < +\infty$, $z_0 \leq r_0$, $h_0 \leq r_0$, $k_0 \leq r_0$ and $\delta(r_0) \leqslant \delta(h_0) + \delta(k_0) + \delta(z_0) < +\infty$. Since H and K are downward directed, so for every every $h \in H$ and $k \in K$ we have $h \mathrel{\mathsf{O}} h_0$ and $k \mathrel{\mathsf{O}} k_0$. Therefore also $h \mathrel{\mathsf{O}} r_0$ and $k \mathrel{\mathsf{O}} r_0$. Hence $\sigma(h, k) \leqslant \delta(r_0)$. Also, it turns out $\sigma(H, K) \leqslant \delta(r_0)$, and so $\sigma(H, K)$ is finite. ⊣

The proof of next proposition is obvious.

PROPOSITION 5.4. *For each semifilters H and K*

$$\Sigma(H, K) = \inf\{\delta(r) \mid h \leq r \text{ and } k \leq r, \text{ for some } h \in H, k \in K\}.$$

In particular, if F and G are filters, then

$$\Sigma(F, G) = \inf\{\delta(r) \mid r \in F \cap G\}.$$

Intuitively we look at a semifilter H as a way to represent an limit-object viewed as "the formal intersection" of the regions in H. This leads to consider the class $\boldsymbol{SF}$ ordered by the dual $\leq$ of the inclusion order, i.e., to set $H \leq K \iff H \supseteq K$.

THEOREM 5.5. *The diameter δ is order-preserving in $(\boldsymbol{SF}, \leq)$. The lower distance σ is order-reversing and for all proper semifilters X, Y and Z,*

(i) $\sigma(X, Y) = \sigma(Y, X)$,

(ii) $\sigma(X, X) = 0$,

(iii) $\sigma(X, Y) \leqslant \sigma(X, Z) + \sigma(Z, Y) + \delta(Z)$.

The upper distance Σ is order-preserving and for all proper semifilters X, Y and Z,

(j) $\Sigma(X,Y) = \Sigma(Y,X)$,

(jj) $\Sigma(X,X) = \delta(X)$,

(jjj) $\Sigma(X,Y) \leqslant \Sigma(X,Z) + \Sigma(Z,Y)$.

Finally,

$$0 \leqslant \Sigma(X,Y) - \sigma(X,Y) \leqslant \delta(X) + \delta(Y).$$

PROOF. It is immediate that δ is order-preserving, that Σ is order-reversing and that (i), (ii) hold true. To prove (iii), asumme that x, y and z are regions in proper semifilters X, Y and Z, respectively. Then

$$\sigma(x,y) \leqslant \sigma(x,z) + \sigma(z,y) + \delta(z) \leqslant \sigma(X,Z) + \sigma(Z,Y) + \delta(z)$$

and therefore,

$$\begin{aligned}\sigma(x,y) \leqslant \sigma(X,Z) + \sigma(Z,Y) + \inf\{\delta(z) \mid z \in Z\}\\ = \sigma(X,Z) + \sigma(Z,Y) + \delta(Z).\end{aligned}$$

So $\sigma(X,Y) \leqslant \sigma(X,Z) + \sigma(Z,Y) + \delta(Z)$.

Likewise, it is immediate that Σ is order-preserving and that (j), (jj) hold true. To prove (jjj), assume that $x \in X$, $y \in Y$ and $z \in Z$. By Proposition 2.3, let r_1 and r_2 be any two regions such that $x \leq r_1$, $z \leq r_1$ and $z \leq r_2$, $y \leq r_2$. Then, since $r_1 \mathsf{O} r_2$, by $\mathbf{A_2}$, there is r_0 such that $r_1 \leq r_0$, $r_2 \leq r_0$, $\delta(r_0) \leqslant \delta(r_1) + \delta(r_2)$ and therefore, by Proposition 5.4,

$$\begin{aligned}\Sigma(X,Y) = \inf\{\delta(r) \mid x \leq r \text{ and } y \leq r, \text{ for some } x \in X, y \in Y\}\\ \leqslant \delta(r_0) \leqslant \delta(r_1) + \delta(r_2).\end{aligned}$$

This entails

$$\begin{aligned}\Sigma(X,Y) \leqslant \inf\{\delta(r) \mid x \leq r \text{ and } z \leq r, \text{ for some } x \in X, z \in Z\} +\\ + \inf\{\delta(r) \mid z \leq r \text{ and } y \leq r, \text{ for some } z \in Z, y \in Y\}\\ = \Sigma(X,Z) + \Sigma(Z,Y).\end{aligned}$$

Finally, to prove that $\sigma(H,K) \leqslant \Sigma(H,K)$, by Proposition 5.4, it is enough to observe that $\sigma(H,K) \leqslant \delta(r)$ for every r such that for some $h \in H$ and $k \in K$ we have that $h \leq r$ and $k \leq r$.

To prove that $\Sigma(H,K) - \sigma(H,K) \leqslant \delta(H) + \delta(K)$ notice that, by Theorem 2.10, for every $h \in H$, $k \in K$,

$$\Sigma(h,k) \leqslant \delta(h) + \delta(k) + \sigma(h,k) \leqslant \delta(h) + \delta(k) + \sigma(H,K).$$

Thus,

$$\begin{aligned}\Sigma(H,K) := \inf\{\Sigma(x,y) \mid x \in H, y \in K\} &\leqslant \Sigma(h,k) \\ &\leqslant \delta(h) + \delta(k) + \sigma(H,K).\end{aligned}$$

Hence

$$\begin{aligned}\Sigma(H,K) &\leqslant \inf\{\delta(x) \mid x \in H\} + \inf\{\delta(y) \mid y \in K\} + \sigma(H,K) \\ &= \delta(H) + \delta(K) + \sigma(H,K). \quad \dashv\end{aligned}$$

A natural way to reach the concept of *point* is by the notion of *infinitesimal semifilter.*

Definition 5.6. A proper semifilter is called *infinitesimal* if its diameter is null. We denote by $\boldsymbol{ISF}$ the class of the infinitesimal semifilter and by $\boldsymbol{IF}$ the class of the infinitesimal filter.

The proofs of the following proposition and theorem are evident.

Proposition 5.7. *Given* $R = (r_n)_{n\in\mathbb{N}} \in \boldsymbol{AC}$ *we put* $R^\star := \{r_n \mid n \in \mathbb{N}\}$.

(i) $R^\star \in \boldsymbol{SF}$.

(ii) $\delta(R) = \delta(R^\star)$; *consequently,* $R \in \boldsymbol{IAC}$ *iff* $R^\star \in \boldsymbol{ISF}$.

(iii) $\sigma(R,S) = \sigma(R^\star, S^\star)$.

Theorem 5.8. *In the class* $\boldsymbol{ISF}$ *both the functions* σ *and* Σ *coincide and they define a pseudometric space* $(\boldsymbol{ISF}, \sigma) = (\boldsymbol{ISF}, \Sigma)$.

As usual the equivalence $\equiv$ associated with σ define a metric space, where for each $X, Y \in \boldsymbol{ISF}$,

$$X \equiv Y \overset{\text{df}}{\iff} \sigma(X,Y) = 0\,.$$

Definition 5.9. We call *metric space associated* with the diametric poset $(\mathfrak{R}, \leq, \delta)$, the quotient $(\boldsymbol{ISF}/_{\equiv}, \sigma)$. We call *point* every element in $\boldsymbol{ISF}/_{\equiv}$.

Alternatively, we can start from the pseudometric space $(\boldsymbol{IF}, \sigma)$ of infinitesimal filters and so to associate the quotient $(\boldsymbol{IF}/_{\equiv}, \sigma)$. The following theorem show that the two metric spaces coincide and that both coincide with the one defined by the infinitesimal abstractive classes.

THEOREM 5.10. *Metric spaces* $(\boldsymbol{ISF}/_{\equiv}, \sigma)$, $(\boldsymbol{IF}/_{\equiv}, \sigma)$ *and* $(\boldsymbol{IAC}/_{\equiv}, \sigma)$ *are isometric.*

PROOF. Let h be the map associating each semifilter H with the filter $h(H)$ generated by H. Then

$$\sigma(H, K) \leqslant \sigma(H, h(H)) + \sigma(h(H), h(K)) + \sigma(h(K), K) = \sigma(h(H), h(K)).$$

Moreover,

$$\sigma(h(H), h(K)) \leqslant \sigma(h(H), H) + \sigma(H, K) + \sigma(K, h(K)) = \sigma(H, K)$$

and therefore $\sigma(H, K) = \sigma(h(H), h(K))$. It turns out that by setting $h^*([H]) = [h(H)]$ we obtain an isometry h^* between $(\boldsymbol{ISF}/_{\equiv}, \sigma)$ and $(\boldsymbol{IF}/_{\equiv}, \sigma)$.

Let R be an infinitesimal abstractive class. Then by $[R]_{\boldsymbol{IAC}}$ and $[R^\star]_{\boldsymbol{ISF}}$ we indicate the related classes of equivalence in $\boldsymbol{IAC}$ and $\boldsymbol{ISF}$, respectively (see Proposition 5.7). We consider the function $f\colon \boldsymbol{IAC}/_{\equiv} \to \boldsymbol{ISF}/_{\equiv}$, where $f([R]_{\boldsymbol{IAC}}) := [R^\star]_{\boldsymbol{ISF}}$, for every $R \in \boldsymbol{IAC}$. It is evident that F is well-defined. Moreover, by (4.3), f is an isometry. To prove that f is surjective it is sufficient to show that if $H \in \boldsymbol{ISF}$, then there exists $R \in \boldsymbol{IAC}$ such that the semifilter $R^\star$ is equivalent to H, i.e. $f([R]_{\boldsymbol{IAC}}) := [R^\star]_{\boldsymbol{ISF}} = [H]_{\boldsymbol{ISF}}$. Indeed, we can set R equal to the sequence $(r_n)_{n\in\mathbb{N}}$ of regions of H defined recursively by setting r_1 equal to any region in H with diameter less than 1 and r_n be any region in H included in r_{n-1} and with diameter less than $1/n$. Such a region exists. In fact, for some $s \in H$ we have that $\delta(s) \leqslant 1/n$, because $H \in \boldsymbol{ISF}$. Moreover, since H is downward directed, there is $r \in H$ such that $r \leq s$ and $r \leq r_{n-1}$. By $\mathbf{A_1}$, $\delta(r) \leqslant \delta(s) \leqslant 1/n$. Obviously, $\sigma(H, R^\star) = 0$, since all elements of H overlap and $R^\star \subseteq H$. ⊣

6. A comparison with the literature

In this section we will show that the notion of diametric poset proposed in this note is an extension of all the metrical approaches to pointfree geometry, in a sense. As an example, in Previale [9], the following definition was proposed:

DEFINITION 6.1. By *abstractive metric lattice* we mean a structure $(L, \leq, \mathrm{o}, \delta)$, where $(L, \leq, \mathrm{o})$ is a lattice with the minimum o and $\delta\colon L \to [0, +\infty]$ is a function such that:

B₁ $\delta(\mathrm{o}) = 0$,

B₂ for each $x, y \in L$: $x \leq y \implies \delta(x) \leqslant \delta(y)$,

B₃ for each $x, y \in L$: $x \mathrel{\mathsf{O}} y \implies \delta(x \vee y) \leqslant \delta(x) + \delta(y)$,

B₄ every maximal filter is infinitesimal,[1]

B₅ for each $x \neq \mathrm{o}$ and $y \neq \mathrm{o}$ there is z such that $\delta(z) < +\infty$, $x \mathrel{\mathsf{O}} z$ and $z \mathrel{\mathsf{O}} y$.

The theory of diametric posets extends the one of abstractive metric lattices.

PROPOSITION 6.2. *Given an abstractive metric lattice* $(L, \leq, \mathrm{o}, \delta)$, *the structure* $(L \setminus \{\mathrm{o}\}, \leq, \delta)$ *is a diametric poset. Nevertheless there are diametric posets which are not abstractive metric lattices.*

PROOF. $\mathbf{A_1}$ and $\mathbf{A_4}$ coincide with $\mathbf{B_2}$ and $\mathbf{B_5}$, respectively. $\mathbf{A_2}$ follows from $\mathbf{B_3}$. To prove $\mathbf{A_3}$, given $x \in L \setminus \{\mathrm{o}\}$ and $\varepsilon > 0$, let F be a maximal filter containing x (see Footnote 1). Then, by $\mathbf{B_4}$, F is infinitesimal and an ε-point y in F exists, i.e. $\delta(y) \leqslant \varepsilon$. Of course, $y \wedge x \neq \mathrm{o}$, $y \wedge x \leq x$ and $y \wedge x \leq y$. Hence $\delta(y \wedge x) \leqslant \varepsilon$, by $\mathbf{B_2}$.

To prove the second part of the proposition, let FC be the class of finite or cofinite subsets of $[0, 1]$ and let $(\mathrm{FC}, \subseteq)$ be the finite-cofinite Boolean algebra over $[0, 1]$. Then the class U of all cofinite subsets of $[0, 1]$ is a maximal filter in this algebra. Let $\delta\colon \mathrm{FC} \to [0, +\infty]$ be defined by (2.1). It is evident that $(\mathrm{FC} \setminus \{\emptyset\}, \subseteq, \delta)$ is a canonical diametric poset (see p. 291). Since the diameter of every cofinite subset is equal to 1, so also $\delta(U) = 1$. ⊣

[1]Notice that for each proper filter F of $(L, \leq, \mathrm{o})$ (see Definition 5.1), $\mathrm{o} \notin F$ and for all $x, y \in L$: $x, y \in F$ iff $\mathrm{o} \neq x \wedge y \in F$. A set F is a *maximal filter* iff F is proper filter and there is no proper filter having F as a proper subset. By Kuratowski-Zorn Lemma we obtain that every proper filter is included in some maximal filter. Hence every member of $L \setminus \{\mathrm{o}\}$ belongs to some maximal filter.

Successively Previale in [10] proposed a slightly different system of axioms by referring to a *generalized Boolean algebra*, i.e., a distributive, relatively complemented lattice $\mathfrak{L} = (L, \leq, \mathrm{o})$ with the minimum o. We recall that a lattice $\mathfrak{L}$ is *relatively complemented* iff for every nonempty interval $I = [b, c]$ of $\mathfrak{L}$ and for every $a \in I$ there is an element $z \in I$ such that $a \wedge z = b$ and $a \vee z = c$. In a sense the notion of *generalized Boolean algebra* is obtained by skipping out from the axioms for a Boolean algebra the existence of a unit and by defining in the place of the complement operation the *difference operation*. This was done in order to admit as a prototycal model the set of bounded regions of the Euclidean space.

DEFINITION 6.3. An *abstractive lattice* is a structure $(L, \leq, \mathrm{o}, \delta)$, where $(L, \leq, \mathrm{o})$ is a *generalized Boolean algebra*, $\delta\colon L \to [0, +\infty)$ is a function assuming only finite values and satisfying the axioms $\mathbf{B_1}$–$\mathbf{B_3}$ and

$\mathbf{B_{4,1}}$ given $x \neq 0$, for every $\varepsilon > 0$ there is $y \neq \mathrm{o}$ such that $y \leq x$ and $\delta(y) \leqslant \varepsilon$.

The next proposition, whose proof is immediate, shows that the theory of diametric posets extends the theory of abstractive lattices.

PROPOSITION 6.4. *The abstractive lattices coincide with the diametric posets which are generalized Boolean algebras with a finite diameter.*

A different approach has been proposed by B. Banaschewski and A. Pultr in the framework of pointfree topology ([1, 2, 11, 12]). Recall that a *frame* is a complete lattice $(L, \leq)$ such that for every $x \in L$ and every family $(y_i)_{i \in I}$ of elements of L

$$x \wedge \Big(\bigvee_{i \in I} y_i\Big) = \bigvee_{i \in I} (x \wedge y_i).$$

A frame $(L, \leq)$, as all complete lattices, has both the greatest element $1 := \bigvee L$ and the least element $\mathrm{o} := \bigwedge L$. A typical example of frame is the lattice of open subsets of a topological space.

DEFINITION 6.5. A *frame with a diameter* is a structure $(L, \leq, \delta)$, where $(L, \leq)$ is a frame and $\delta\colon L \to [0, +\infty]$ is a function satisfying the axioms $\mathbf{B_1}$–$\mathbf{B_3}$ and

$\mathbf{B_{4,2}}$ for each $\varepsilon > 0$, $\bigvee P_\varepsilon = 1$.

The axioms $\mathbf{B_2}$ and $\mathbf{B_{4.2}}$ entail that every $x \in L$ is the union of its ε-points. In fact, $x = x \wedge \bigvee P_\varepsilon = \bigvee\{x \wedge r \mid r \in P_\varepsilon\} = \bigvee\{r \in P_\varepsilon \mid r \leq x\}$. In particular we obtain $\mathbf{A_3}$ and thus the next proposition is established.

PROPOSITION 6.6. *If $(L, \leq, \delta)$ is a frame with a finite diameter, then $(L \setminus \{0\}, \leq, \delta)$ is a diametric poset.*

Another metric approach to pointfree geometry is based on the ideas of interval analysis. Indeed in [4] C. Coppola and T. Pacelli argue that in several cases while it is impossible to determinate the precise value of a distance, it is possible to individuate an interval containing such a value. This leads to consider an "approximate distance" function whose values are intervals and to propose the following definition where the functions σ and Σ represent the lower bounds and the upper bounds of the intervals.

DEFINITION 6.7. An *interval semimetric space* is a structure $(\mathfrak{R}, \leq, \sigma, \Sigma)$, where $(\mathfrak{R}, \leq)$ is a poset, $\sigma\colon \mathfrak{R}\times\mathfrak{R} \to [0, \infty)$ is an order-reversing mapping, $\Sigma\colon \mathfrak{R}\times\mathfrak{R} \to [0, \infty)$ is order-preserving mapping and for every $x, y, z \in \mathfrak{R}$,

$\mathbf{P_1}$ $\sigma(x, x) = 0$,

$\mathbf{P_2}$ $\sigma(x, y) = \sigma(y, x)$, $\Sigma(x, y) = \Sigma(y, x)$,

$\mathbf{P_3}$ $\sigma(x, y) \leqslant \sigma(x, z) + \sigma(z, y) + \delta(z)$,

$\mathbf{P_4}$ $\Sigma(x, y) \leqslant \Sigma(x, z) + \Sigma(z, y)$,

$\mathbf{P_5}$ $0 \leqslant \Sigma(x, y) - \sigma(x, y) \leqslant \delta(x) + \delta(y)$,

where for each $x \in \mathfrak{R}$, $\delta(x) := \Sigma(x, x)$.

PROPOSITION 6.8. *Let $(\mathfrak{R}, \leq, \delta)$ be a diametric poset such that δ assumes finite values. Then if σ and Σ are the upper and the lower distances, it turns out that $(\mathfrak{R}, \leq, \sigma, \Sigma)$ is an interval semimetric space such that for each $x \in \mathfrak{R}$, $\delta(x) := \Sigma(x, x)$.*

PROOF. See the list of properties of σ and Σ proved in Section 2. ⊣

We conclude with a comparison with the definition in [5] and [8] in which the notions of *diameter* and of *distance between regions* are both assumed as primitives.

DEFINITION 6.9. A *pointless pseudo-metric space* is a structure $(\mathfrak{R}, \leq, \sigma, \delta)$, where $(\mathfrak{R}, \leq)$ is a poset, $\sigma\colon \mathfrak{R} \times \mathfrak{R} \to [0, +\infty)$ is an order-reversing mapping, $\delta\colon \mathfrak{R} \to [0, +\infty]$ is an order-preserving mapping and for every $x, y, z \in \mathfrak{R}$,

(i) $\sigma(x, y) = \sigma(y, x)$,

(ii) $\sigma(x, x) = 0$,

(iii) $\sigma(x, y) \leqslant \sigma(x, z) + \sigma(z, y) + \delta(z)$.

In accordance with Theorem 2.7, it is immediate that every diametric poset defines a pointless pseudo-metric space.

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