and $\mathfrak{P}$ are subsets of $2^{\mathbf{P}}$ (thus this notions are non-elementary, so structures examined are non-elementary as well), **B** and **D** are, respectively, ternary and quaternary relation in **P**. Elements of $\mathfrak{L}$ and $\mathfrak{P}$ are called, respectively, *lines* and *planes*, **B** is called *betweenness relation* and **D** *equidistance relation.* Next, we put specific axioms on **P**, $\mathfrak{L}$, $\mathfrak{P}$, **B** and **D**, and in this way we obtain a system of geometry that would probably satisfy Euclid and his contemporaries.

We can modify the above approach to start with structures $\langle \mathbf{P}, \mathbf{B}, \mathbf{D}\rangle$ and subsequently take such a collection of axioms that $\mathfrak{L}$ and $\mathfrak{P}$ will be definable be means of **B**. The set of lines can be defined in the following way

$$\begin{aligned} X \in \mathfrak{L} \overset{\text{df}}{\iff} &\exists_{p,q\in\mathbf{P}}(p \neq q \wedge \\ &X = \{r \in \mathbf{P} \mid \langle r,p,q\rangle \in \mathbf{B} \vee \langle p,r,q\rangle \in \mathbf{B} \vee \langle p,q,r\rangle \in \mathbf{B}\} \cup \{p,q\}), \end{aligned}$$

where the condition '$\langle r,p,q\rangle \in \mathbf{B}$' says that *point* p *is between points* q *and* r.

To define $\mathfrak{P}$, first we introduce a new relation $\mathbf{L} \subseteq \mathbf{P}^3$, so called *relation of collinearity of points*

$$\langle p,q,r\rangle \in \mathbf{L} \overset{\text{df}}{\iff} \exists_{X\in\mathfrak{L}}(p \in X \wedge q \in X \wedge r \in X).$$

Subsequently we define *a triangle*, whose cones are located in three points p, q, r (in symbols 'tr(pqr)') that are not collinear

$$\begin{aligned} \mathrm{tr}(pqr) := \{a \in \mathbf{P} \mid \neg\,\mathbf{L}(p,q,r) \wedge (a = p \vee a = q \vee a = r \vee \\ \langle p,a,q\rangle \in \mathbf{B} \vee \langle p,a,r\rangle \in \mathbf{B} \vee \langle q,a,r\rangle \in \mathbf{B})\}. \end{aligned}$$

Now we define

$$\begin{aligned} X \in \mathfrak{P} \overset{\text{df}}{\iff} &\exists_{p,q,r\in\mathbf{P}}\Big[\neg\,\mathbf{L}(p,q,r) \wedge X = \{c \in \mathbf{P} \mid \\ &\exists_{a,b\in\mathbf{P}}[a \neq b \wedge a,b \in \mathrm{tr}(pqr) \wedge \langle c,a,b\rangle \in \mathbf{B} \vee \langle a,c,b\rangle \in \mathbf{B}]\}\Big]. \end{aligned}$$

Thus, in light of the above constructions, we conclude that to construct Euclidean geometry one can do with just three primitive notions: of *point*, of *betweenness relation* and of *equidistance relation.*

However, we will briefly describe one more solution of the problem which is especially important for our presentation of point-free geometry. To construct a system of Euclidean geometry one actually needs only two primitive notions, as it was proved by Italian mathematician Mario Pieri in [17] (an English translation in [16]). These notions are that of *point* and that of *equidistance relation*, which in Pieri's system case is a ternary relation among points. Denoting this relation by means of 'Δ' we can say that while doing geometry in Pieri's manner we analyze elementary structures $\langle \mathbf{P}, \Delta \rangle$, since $\Delta \subseteq \mathbf{P}^3$. Now we of course have to choose axioms to define $\mathfrak{L}$, $\mathfrak{P}$, $\mathbf{B}$ and $\mathbf{D}$ in such a way to be able to prove that this approach is definitionally equivalent to Hilbert's one. A heuristic argument for feasibility of such a proof together with a presentation of Pieri's constructions can be found in [8]. The following theorem is crucial for Pieri's structures.

Theorem 7.1 ([8]). *All Pieri's structures are isomorphic, for any Pieri's structure $\langle \mathbf{P}, \Delta \rangle$ is isomorphic to $\langle \mathbb{R}^3, \Delta^{\mathbb{R}^3} \rangle$, where $\Delta^{\mathbb{R}^3}$ is introduced by means of the following definition*

$$\bar{x}\bar{y}\ \Delta^{\mathbb{R}^3}\ \bar{z} \overset{\mathrm{df}}{\iff} \varrho(\bar{x}, \bar{z}) = \varrho(\bar{y}, \bar{z})\,, \qquad (\mathrm{def}\ \Delta^{\mathbb{R}^3})$$

where $\bar{x}, \bar{y}, \bar{z} \in \mathbb{R}^3$ and $\varrho\colon \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ is the standard Euclidean metric.

8. Points

Space of Euclidean geometry and physics (for which such geometry is a model of the surrounding world) is a distributive set of points, which are its simplest constituents. However, as Russell noticed it in [21, p. 119] "no one has ever seen or touched a point". Points are absent from sensually accessible objects, and things which we encounter in the perspective space seemed not to be built out of them. It is also hard to imagine any kind of experiment which could «prove» that points are basic constituents of the space that embraces us. Thus it seems quite reasonable to assume that points cannot be found in the world. As we read in [3, s. 431] „nature does not provide objects without dimensions (a property that geometry ascribes to points)". One could even say more than this—nature does not provide objects that would have less than three dimensions. Apart from points, in nature we will not encounter either one-dimensional lines and segments or two dimensional planes and surfaces. On the contrary, space of geometry is «full» of objects which are even stranger than those mentioned, of course from everyday life point of view.

So what are space of geometry and objects that exists within it? First, this space is a distributive set, that is something that is abstract and is much closer to Platonic realms than to the perspective space. Similar thing can be said about its «pieces», that is subsets of the set of all points. Thus figures, as sets of points, are abstract objects. Space which is an object of analysis in geometry and physics is infested with abstract objects. On the other hand, the perspective space does not share this property—we are rather inclined to say that it is very concrete.

According to the Euclid's definition, a point is an object that does not have dimensions. So it is something essentially different from objects occupying the perspective space like trees and grains of sand and so on. Thus according to our intuitions taken from everyday life on one side and from natural sciences on the other, something that is built out of dimensionless things should be deprived of dimensions as well. Objects that we sensually encounter have dimensions, so this fact testifies against points being constituents of the perspective space.

Remark 8.1. Spaces which are analyzed in geometry and physics are built from infinitely many dimensionless points, and these spaces have dimensions and, in a way, «parts». Actually they can have an arbitrary (in case of geometry even infinite) number of dimensions (of course under a specific meaning of the term 'dimension'). They also have (uncountably) infinitely many «parts», if we treat the relational notion of *being a subset* as an analogon of the relational notion of *being a part*. The problem in question does not concern mathematical theories as such, since they handle very well the transition from points to objects that have «parts» and dimensions, but this kind of transition is totally unnatural in the world of which those theories are models. Even if we treat the space in Leśniewski's manner, as a mereological sum (but not a distributive set) of all points (see p. 167), it is still hard to understand how the three-dimensional perspective space can be constructed from dimensionless objects. ⊣

The above objections against classical point-based geometry can be briefly summed up in the following way

(i) points, from which in a geometrical or a physical model space is built, are neither sensually experienced nor its existence can be derived from data (both by some experiment or some kind of reasoning); moreover we cannot point to objects in the real world, that could be «natural» counterparts of points;

(ii) the space of geometry and its «parts» as distributive sets are abstract and as such they cannot be experienced empirically; the perspective space and its parts are concrete (sensually experienced);

(iii) all objects that exist in the perspective space have dimensions and parts, so points cannot be elements of this space.

The problems described above were a stimulus to search for some other, different from point-based one, approach to geometry. Those approaches are usually named *point-free* or *pointless*. It must be stressed here that those geometries do not either aim at replacing classical geometry with some other formal science or question usefulness of the notion of *point*. It will not be an exaggeration if we say that introduction of this notion to science by the ancients was ingenious and enabled really impressive development of both mathematics and physics. The names 'point-free' and 'pointless' are a bit misguiding here, since those are not theories which do not have any notion of *point* whatsoever. The crucial difference between point-based and point-free geometry lies in the fact that in the latter we will not find the notion of *point* among its primitive notions but it is defined by means of other primitive notions which intuitive interpretation is less problematic. In light of what has been said so far—point-free geometry looks for such foundations for classical, point-based geometry which are most satisfying from a point of view of our intuitions and representations concerning the perspective space. Point-free geometry still talks about points but the difference is that these are abstract objects constructed from objects that can be found in the perspective space. Points as such objects are still to behave like those in classical geometry and standard geometrical relations are to hold among them. Let us notice as well that points as constructed from spatial objects does not have dimensions in this sense like the perspective space and its parts have, since they are not spatial at all. Therefore they satisfy, in a way, the Euclid's definition.

9. Basic assumptions of point-free geometry

Let us begin this section with the following words of Russell's [21, p. 119]

> It is customary to think of points as simple and infinitely small, but geometry in no way demands that we should think of them in this way. All that is necessary for geometry is that they should have mutual relations possessing certain enumerated abstract properties, and it may be that an assemblage of data of sensation will serve this purpose.

It thus can be said that the task of point-free geometry is to construct such mathematical objects among which there hold the same relations as among «ordinary» points and which fulfill the following requirements

(i) their ontological status will be less problematic than in case of Euclidean points;

(ii) its «building material», out of which they will be constructed, could be naturally and intuitively interpreted in the perspective space.

An important feature of point-free geometry is the fact, that its space and figures are not distributive sets of points. This is a natural consequence of eliminating the notion of *point* from primitive notions of such theories. Both space and figures, which are its parts, are mereological sets in case of the approach presented in his paper.

It was Leśniewski himself who in [12] proposed one of the first approaches to geometry in which space and figures are not distributive sets. He assumed however that among figures there are less than three-dimensional objects and that dimensionless points are parts of space. Segments, lines, planes and other figures were, at the same time, parts of space and mereological sets of points. Space itself was the mereological sum (fusion) of all points. Therefore his approach to geometry was still point-based. The difference between his system and classical geometry lied in the fact that he used mereological tools instead of set theoretical ones.[4]

From a formal point of view Leśniewski's approach to geometry can be characterized as follows. Let us assume that $\mathtt{s}$ is space, $\sqsubset$ is a relation of *being a proper part* and that Pt, F, L and P are, respectively, distributive sets of all points, figures, lines and planes.[5] Then we have that

(i) $\mathtt{s} \neq Pt$ (space is not the set of all points);

(ii) $\mathtt{s} \in F$ (space is one of figures);

(iii) $x \in F$ and $x \neq \mathtt{s}$ iff $x \sqsubset \mathtt{s}$ (every figure which is different from space is its part and conversely, every part of space is a figure);

(iv) $Pt, L, P \subseteq F$ (all points, lines and planes are figures, therefore they are parts of space).

From an ontological point of view Leśniewski's approach has actually the same faults as classical point-based geometries. To tell the truth neither space nor figures are any longer identified with distributive sets of points,

[4] In contemporary systems of point-free geometry mereology is used *next to* set theory. Leśniewski did not recognize the existence of distributive sets, so in his case purely mereological approach was a consequence of his ontology.

[5] Here we use different symbols for these sets than earlier, since these are *different* sets indeed. First, the set of all points is no longer space. Second, in classical geometry figures, lines and planes are distributive sets of points. On the other hand, in the case considered this are fusions of points. Clearly, this approach is incompatible with views of the creator of mereology, since we use set theoretical tools.

but still space is «infested with» less than three-dimensional objects whose counterparts are not present in the perspective space.

In systems of point-free geometry we entirely accept (i). Instead of the set F from (ii) we have the set of *solids*, which are also called *regions* or *spatial bodies*. Contrary to F, the set of solids does not contain objects like points, lines or planes. Its elements are only three-dimensional and «regular» parts of space. By «regular» we mean such parts to which nothing having less dimensions that whole space was either «added» or «subtracted».[6] Let us assume that the set of solids will be denoted by '$\mathbb{S}$'. Now, points will be distributive sets of solids or sets of sets of solids. Let us denote the set of such points by 'Π'. Except for the set of solids we also have the set of all «abstract» figures understood, in a traditional way, as non-empty sets of points. Denoting this set by '$\mathfrak{F}$' we have that $\mathfrak{F} := 2^{\Pi} \setminus \{\emptyset\}$. It follows from the above remarks that $\Pi \cap \mathbb{S} = \emptyset = \Pi \cap \mathfrak{F}$, that is points are neither solids nor abstract figures. Of course, the set of all abstract points is a figure, since $\Pi \in 2^{\Pi} \setminus \{\emptyset\}$. Lines and planes, similarly as in classical geometry, are distributive sets of points. It is the case since after having generated points further geometrical constructions are the same as those within point-based geometries. Therefore now it is more reasonable to denote lines and planes by '$\mathfrak{L}$' and '$\mathfrak{P}$', than by 'P' and 'L', since elements of the latter sets are mereological sets.

We notice as well that while in point-based geometries $\mathfrak{F}$ has the type $(*)$ in a hierarchy of types over a base set, in point-free approach it has either the type $((*))$ or $(((*)))$.

In light of what has been just said we have

(i′) $\mathsf{s} \neq \Pi$;

(ii′) $\mathsf{s} \in \mathbb{S}$ and $\mathsf{s} \notin \mathfrak{F}$ (space is one of solids and is not an «abstract» figure, that is it is not a distributive set of points);

(iii′) $x \in \mathbb{S}$ and $x \neq \mathsf{s}$ iff $x \sqsubset \mathsf{s}$ (every solid which is different from space is its part and conversely, every part of space is a solid)[7];

(iv′) $\Pi \subseteq 2^{\mathbb{S}}$ or $\Pi \subseteq 2^{2^{\mathbb{S}}}$ and $\mathfrak{L}, \mathfrak{P} \subseteq \mathfrak{F}$ (all points are sets whose elements are solids or sets of solids; all lines and planes are abstract figures, but they are not parts of s).

In light of the above remarks we can say that the conditions (iii′)–(iv′) are natural assumptions of point-free geometry.

[6] Some authors put other constraints on solids but those will not be relevant for the approach presented.

[7] Let us notice that from both (iii′) and transitivity of the parthood relation we have that every part of a solid is a solid.

10. From regions of space to points

Now we will explain how we can construct objects that can take the role of points. But first let us remind that what we want to do is to build a mathematical structure that will share all properties of the Euclidean space, so in particular, such a structure must share all topological properties with the Euclidean space. Moreover, taking mereology as a model of the perspective space was, so to say, a first-level abstraction in our construction. Points and space that they form will be on a second level—now what we aim to do is to abstract points from elements of the mereological model.

Intuitively points can be treated as locations in space, but not all of them. By a location we may understand for example some region of space, in a sense that we speak about a localization of some object somewhere in space. But, if we say for example that the Eiffel Tower is in France, this is a very vague description—France is a very large country and compared to it the Eiffel Tower is small and occupies a very small area of the French territory. Thus, on the one side some localizations are too large to be points, on the other they are not abstract objects, as we required them to be in Section 9. But let us imagine that we try to explain to somebody where exactly in France the Eiffel tower is localized. First we can say that it is in the northern part of the country. Then, more exactly, that it is somewhere in Île-de-France region, and still more precisely that it is in Paris, finally pointing to the Champ de Mars as the place where the famous construction is erected. In this process of explanation we take smaller and smaller regions of space that eventually shrink so much to finally be (almost) exactly this place which is occupied by the Eiffel Tower.

We can of course imagine continuing this process to converge to even smaller regions of the space, and in extreme and limit case to the most exact location in the space which cannot be precised any more. Of course this idea resembles the process of convergence known very well in topology. There when we, roughly speaking, lack some information about the power of some subsets of a topological space[8], we express the idea of limit by means of filters.[9] And usually construction of points from regions exploits this idea.

So, the intuition behind the whole idea is that a point is that around which regions of the space «shrink». If the space is infinitely divisible, then

[8] Precisely speaking, when we do not know whether given space has a dense countable subset.

[9] Of course, the equivalent way to do this is via so called *nets*.

this process of shrinking does not terminate and at its limit we obtain that location of the space that we can call a point.

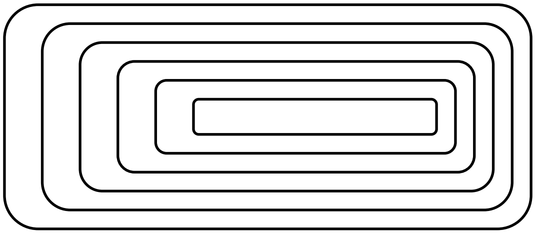

Figure 5. A set of «shrinking» regions. We can imagine those regions to be smaller and smaller, finally giving us a point.

Of course a point in this sense is a result of our ability to abstract some property of the space and is not its part. If we decide to model the perspective space by means of mereology, then points must be constructed by means of tools we have at hand. Since points are not elements of a domain, then a natural idea is to take as points those sets of regions that «shrink» around some location in the space. Thus our process of constructing points includes the following two steps

(i) first, we distinguish some sets of regions that converge to some unique location in the perspective space;

(ii) second, we take as point filters generated by those sets of regions (see Section 4.4), by which we assure the uniqueness of points.

We may limit ourselves to only (i) in case our construction assures the uniqueness of points (see the definition of the set Π_{T} on p. 181). However, we have to be rather careful while deciding to solve the problem in question in this way. First, it may happen that two different sets of regions can descend similarly, that is shrink towards the same location in the space, as it can be seen in Figure 6.

As a result, we cannot take both of these sets to be points since we would have undesirable situation in which two different sets would be the same point. Of course the natural solution is to take as a point the filter generated by each of those two sets. So, it may be said that a general idea of constructing points reduces to defining some sets which will shrink to one location in the space and then to take filters generated by those sets.

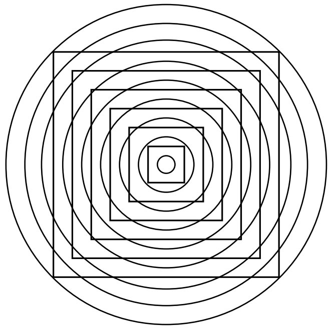

Figure 6. Two sets of regions shrink to the same location in the space.

The above is not the only obstacle to be eliminated. Let us examine the other ones. We of course have to exclude from our construction sets of regions that look more or less like the one in Figure 7.

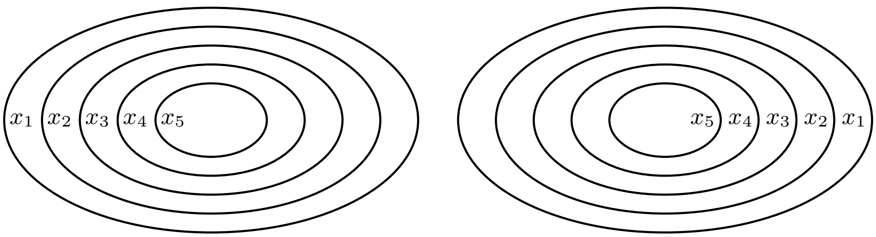

Figure 7. Descending regions x_1, x_2, x_3, x_4, x_5 represent two different locations in the space.

To analyze this situation we introduce notions of *coherent* and *super-coherent* region.[10]

Definition 10.1. A region x is *coherent* iff

$$\forall_{y,z \in M}(x = y \sqcup z \implies y \mathbb{C} z)\,.$$

Definition 10.2. A region x is *super-coherent* iff

$$\forall_{y,z \in M}(x = y \sqcup z \implies \exists_{u \in M}(u \ll x \wedge u \bigcirc y \wedge u \bigcirc z \wedge u \sqcap y \mathbb{C}\, u \sqcap z))\,.$$

[10] Both notions are used by Roeper in [20, p. 255]. However, he uses the term 'convex' instead of 'super-coherent'. We decide to use the latter since convexity is generally stronger property than coherence.

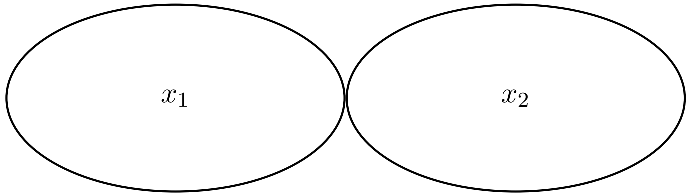

Figure 8. Both x_1 and x_2 are super-coherent regions, $x_1 \sqcup x_2$ is a coherent but not a super-coherent region.

Thus super-coherent region is such that allow us to «go» from anywhere to anywhere within this region without touching its complement.

Coherent regions are not enough to assure the uniqueness of points—Figure 9 represents the situation in which three different sets of descending regions shrink to the same location in space. Of course, the third one would be the set of regions consisted of coherent regions which parts are equally divided between X_1 and X_2.

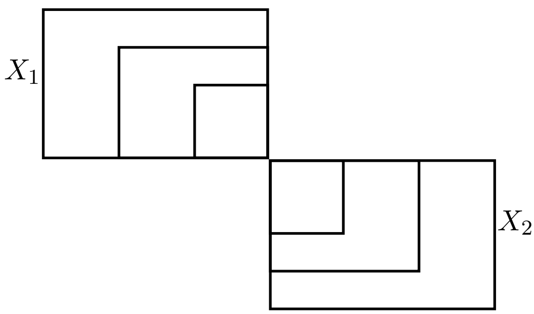

Figure 9. Three sets of regions descending to the same location in the space. The third one consists of mereological sums of regions from X_1 and X_2; that is, it is the set $\{x \sqcup y \mid x \in X_1 \wedge y \in X_2\}$.

Super-coherence let us avoid this difficulty since excludes the case depicted above. However, we do not have to require that those sets that are to represent points have to consist of only super-coherent regions, but that only super-coherent parts of those regions are relevant in construction of points. For example in Figure 10 we can see incoherent regions descending to some unique location in the space. This uniqueness is guaranteed by the fact that the process of descending takes place only within super-coherent «leftmost» parts of incoherent regions x, y and z. The situation bears slight resemblance to a construction of a part of the Cantor set of reals.

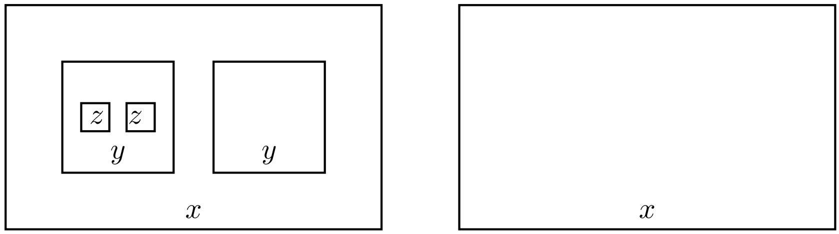

Figure 10. Only super-coherent «leftmost» parts of incoherent regions x, y, z are relevant for determining some unique location in the space.

Thus we should require something like this: if a set X of regions represents a location in the space, then there is a subset of X which consists of super-coherent parts of regions from X and represents the same location in the space. This should not be difficult to notice that excluding sets of regions such as the one in Figure 7 is a step towards Hausdorff (or T2) property of a space of points we try to construct. But this does not fully eliminate the problem of representing the same point by two different sets of regions. X_1 and X_2 both consist of super-coherent regions and they represent the same location in the space. We may exclude this case by requiring that every set X of regions that is to represent a point has to fulfil the following condition

$$\forall_{x,y \in X}(x \neq y \Longrightarrow x \ll y \vee y \ll x)\,.$$

Thus, since both X_1 and X_2 consist of regions that are connected with their complements, they cannot serve as good candidates to represent points.

Remark 10.1. We could avoid this difficulty by taking as points some equivalence classes of filters. We say that two filters F_1 and F_2 *are connected* iff $x \mathrel{\mathbb{C}} y$, for all $x \in F_1$ and $y \in F_2$. Now in the set $\mathscr{U}$ of all ultrafilters (see Section 4.4) we distinguish some special class $\mathscr{U}'$ of ultrafilters and prove that connectedness relation on filters is transitive in $\mathscr{U}' \times \mathscr{U}'$. Then we define a point to be an equivalence class under the relation of connectedness between ultrafilters in $\mathscr{U}' \times \mathscr{U}'$. The reader interested in details of the above construction should consult [20].

It is worth noticing that when we choose the above solution, then points are sets of sets of regions, that is they have the type $((*))$ in the hierarchy of types above a base set. When we decide to take as points simply sets of regions, then points have the type $(*)$. ⊣

Another threat that is looming behind constructions above is that we still have not eliminated those sets of regions that may represent points in infinity, that is those sets of regions that have a structure of a set presented in Figure 11. According to what we said above, we want our space of points to be Euclidean. Allowing such sets of regions to represent points we actually allow this space to be compact, which is not a property of Euclidean spaces. Yet another problem is presented in Figure 12, where regions descend to perpendicular lines rather than to a point. As it is seen the problem here is also unboundedness of regions. In case cross-like regions were bounded, then together with the requirement that they should be ordered by non-tangential inclusion it would guarantee that they were shrinking in all directions and thus to a point.

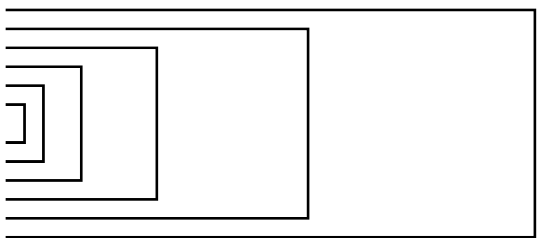

Figure 11. A set of regions shrinking to some location in infinity.

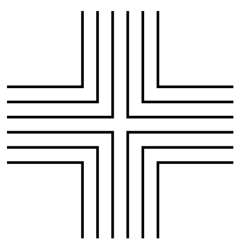

Figure 12. A descending set of regions that does not represent any point.

A natural solution that comes to mind is introducing another notion of *bounded* or *limited* region, and require that representatives of points contain such regions (this is actually Roeper's solution in [20]). As it seems that the notion of *bounded region* is very hard (if possible at all) to be defined properly solely by means of parthood and connection relations, the natural choice is to enrich $\langle M, \sqsubseteq, \mathbb{C}\rangle$ with yet another primitive notion and try to axiomatize it. If we agree that $B \subseteq M$ is a family of regions that we will call bounded ones, then examples of natural axioms for B could be

(B1) $\mathbb{1} \notin B$ – the space is not bounded,

(B2) if $x \in B$ and $y \sqsubseteq x$, then $y \in B$ – every subregion of a bounded region is bounded,

(B3) if $x, y \in B$ then $x \sqcup y \in B$ – the mereological sum of bounded regions is bounded.

So as we can see B is an ideal in $\langle M, \sqsubseteq\rangle$ (see Section 4.4). This actually goes along the intuitions that bounded regions should be small parts of the space, while the space itself is something large. What else could be said about bounded regions? It seems natural to assume that every region contains a bounded subregion. Actually, following Roeper [20, p. 256] we may even require something else that may help us locate bounded regions in the space by saying that bounded subregions of given regions are certainly in those places of the space where regions are connected with each other

$$\forall_{x,y\in M}(x \mathrel{\mathbb{C}} y \Longrightarrow \exists_{z\in B}(z \sqsubseteq x \wedge z \mathrel{\mathbb{C}} y)\,. \tag{B4}$$

Due to reflexivity of $\mathbb{C}$, (B4) is of course stronger than simple assertion of existence of bounded regions. From this and the axiom above we immediately obtain that every region has a bounded subregion. This should be clear that we do not have to require that all regions in a representative of a point are bounded—it is enough to assure that every such a representative contain at least one bounded region. In such case those regions that are below this bounded region are also bounded and only they have any importance in determining a unique location in the space (see Figure 13).

Let us now summarize what properties sets of regions should have to be representatives of points. So, $X \subseteq M$ is a representative of a point iff

(r1) the relation of non-tangential inclusion is connected in X (in an ordinary sense), that is

$$\forall_{x,y\in X}(x \neq y \Longrightarrow x \ll y \vee y \ll x)\,,$$

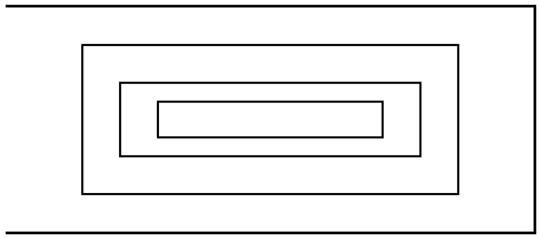

Figure 13. A set of regions is a representative of a point although it contains a region that is unbounded.

(r2) every region in X has a non-tangentially included subregion that is also in X

$$\forall_{x\in X}\exists_{y\in X}\, y \ll x\,,$$

(r3) X contains at least one bounded region,

(r4) if X consists of regions that are not super-coherent, then there exists a set X' of super-coherent subregions of elements of X satisfying (r1)–(r3).

Notice that from some moment the set X contains only bounded regions, such are all these below the one whose existence is postulated.

By *a pre-point* we will call any set (in a mereological structure) satisfying (r1)–(r4) and denote the set of all pre-points by '$\mathbb{Q}$'. By (r1) and (5.6), all pre-points have finite intersection property (see Section 4.4), so points can be defined as follows.

Definition 10.3. *A point* is any filter in $\langle M, \sqsubset\rangle$ which is generated by some element of $\mathbb{Q}$, that is

$$\alpha \in \Pi \overset{\text{df}}{\iff} \exists_{X\in\mathbb{Q}}\, \alpha = \{y \in M \mid \exists_{x\in X}\, x \sqsubseteq y\}\,. \qquad \text{(def }\Pi\text{)}$$

As we noticed it above points have the type $(*)$, which is different from point-based systems of geometry where points, as elements of a domain, have the type $*$. Solids are simply regions of the space, so they have the type $*$. The space, which is the unity of $\langle M, \sqsubset\rangle$, is different from the set of all points, $\mathbb{1} \neq \Pi$, but is one of solids. Figures, as sets of points, are sets of sets of regions, so they have the type $((*))$. Clearly no solid is a figure, and in particular $\mathbb{1}$ is not a figure. Thus we can say that a method

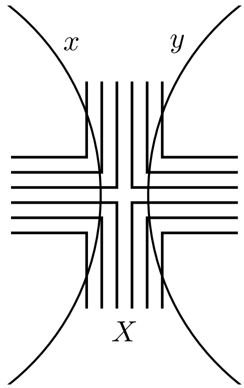

Figure 14. A descending set X of regions is not a G-representative of a point, since x and y overlap all regions in X but are not connected with each other.

of constructing points presented above preserves all the basic assumptions of point-free systems of geometry.

11. Grzegorczyk's construction

To present some particular construction of points we will shortly analyze Grzegorczyk's system of point-free topology from [10]. Let then $\langle M, \sqsubseteq, \mathbb{C}\rangle$ such that $\langle M, \sqsubseteq\rangle$ is a mereological structure and $\mathbb{C}$ satisfies axioms (C1)–(C3). The remaining axioms for $\mathbb{C}$ are consequences of the ones mentioned and the only specific Grzegorczyk's axiom (G).

Before we introduce (G) let us first introduce the family Γ of elements of 2^M that we will call *G-representative of points.*

DEFINITION 11.1. By *a G-representative of a point* we will understand any non-empty set of regions X such that satisfies (r1), (r2) and the following condition

$$\forall_{x,y\in M}(\forall_{z\in X}(z \bigcirc x \wedge z \bigcirc y) \Longrightarrow x \,\mathbb{C}\, y). \qquad \text{(g1)}$$

That is

$$\Gamma := \{X \in 2^M \setminus \{\emptyset\} \mid X \text{ satisfies (r1), (r2) and (g1)}\}. \qquad (\text{def}\,\Gamma)$$

A moment of reflection shows that (g1) grasps both (r3) and (r4). If examined more closely, it indeed excludes the situations depicted in figures 7, 12 and 11. However, the answer to a question whether this categorically describes such situations is negative and we will briefly explain why it is the case. First, Grzegorczyk's axiom goes as follows.

AXIOM G. For any $x, y \in M$ that are connected, i.e. $x \mathbb{C} y$, there exists a set $X \in \Gamma$ such that[11]

$$\forall_{z \in X}(z \bigcirc x \wedge z \bigcirc y), \tag{g2}$$

$$x \bigcirc y \Longrightarrow \exists_{z \in X}\, z \sqsubseteq x \sqcap y\,. \tag{g3}$$

By *G-structure* we will understand any triple $\langle M, \sqsubset, \mathbb{C}\rangle$ satisfying axioms of mereology, (C1)–(C3) and (G). It is provable that all G-structures are connection structures, i.e. they satisfy (C4). Moreover, we obtain that $\sqsubseteq$ is definable by means of $\mathbb{C}$ (and thus also by $\sqsubset$)

$$\begin{aligned} x \sqsubseteq y &\iff \forall_z(z \mathbb{C}\, x \;\Rightarrow\; z \mathbb{C}\, y) \\ &\iff \forall_z(z \,)(\, y \;\Rightarrow\; z \,)(\, x). \end{aligned}$$

A point in any G-structure is a filter generated by some element of Γ (by (r1) and (5.6) all members of Γ have finite intersection property). Thus in this case we have

$$\alpha \in \Pi_{\mathrm{G}} \overset{\mathrm{df}}{\iff} \exists_{X \in \Gamma}\, \alpha = \{y \in M \mid \exists_{x \in X}\, x \sqsubseteq y\}\,. \tag{def Π_{G}}$$

Existence of elements of Π_{G} follows from (C1) and (G).

Let us now explain what is a role of the conditions (g2) and (g3). So far we have defined points but we actually know nothing about the properties of Π_{G}. The conditions in question are responsible for imposing some topological structure on Π_{G}. To see that, first we define the set of internal points of a given region x by means of the following definition

$$\mathrm{Irl}(x) := \{\, \alpha \in \Pi_{\mathrm{G}} \mid x \in \alpha \,\}\,. \tag{def Irl}$$

We leave it to the reader to check that the family $\{\mathrm{Irl}(x) \mid x \in M\}$ is a basis (in topological sense). Thus $\langle \Pi_{\mathrm{G}}, \mathscr{O}\rangle$, where $\mathscr{O}$ is the set of set-theoretical sums of elements from the basis, is a topological space.

As it was noticed in [10] for any x, the set $\mathrm{Irl}(x)$ is non-empty and regular open in $\langle \Pi_{\mathrm{G}}, \mathscr{O}\rangle$, i.e. $\mathrm{Irl}(x) = \mathrm{INT}\,\mathrm{CL}\,\mathrm{Irl}(x)$, where INT and CL are, respectively, operations of interior and of closure in $\langle \Pi_{\mathrm{G}}, \mathscr{O}\rangle$. Moreover,

$$x \sqsubseteq y \iff \mathrm{Irl}(x) \subseteq \mathrm{Irl}(y)\,, \tag{11.1}$$

[11] In [10] the condition (g3) has the following form: $\exists_{z \in M}(z \in X \wedge (x \sqsubseteq y \Rightarrow z \sqsubseteq x)$. In our case $X \neq \emptyset$, since $X \in \Gamma$. Moreover, one can prove that replacing both forms of (g3) with one another we will obtain equivalent version of the axiom (G).

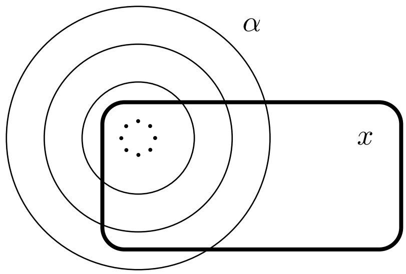

Figure 15. The point α is an internal point of the region x (the dotted region is a member of α and is an ingrediens of x).

$$x \mathbin{\mathbb{C}} y \iff \mathrm{CL}\,\mathrm{Irl}(x) \cap \mathrm{CL}\,\mathrm{Irl}(y) \neq \emptyset\,, \tag{11.2}$$

$$\mathrm{CL}\,\mathrm{Irl}(x) = \{\, \alpha \in \Pi_{\mathrm{G}} \mid \forall_{y \in \alpha}\, x \bigcirc y \,\}\,. \tag{11.3}$$

We can prove that $\langle \Pi_{\mathrm{G}}, \mathscr{O} \rangle$ is a Haussdorff (or T2) space, i.e. for any two points α and β there are two disjoint open sets containing, respectively, α and β. This is due to the fact that two points α and β have to contain disjoint regions $x \in \alpha$ and $y \in \beta$, that is

$$\alpha \neq \beta \Longrightarrow \exists_{x \in \alpha} \exists_{y \in \beta}\, x \wr y\,.$$

The latest fact is obtained by means of (g2) and (g3). Of course, $\alpha \in \mathrm{Irl}(x)$ and $\beta \in \mathrm{Irl}(y)$. Moreover, from definitions of filter and Irl, by (4.6), we obtain $\mathrm{Irl}(x) \cap \mathrm{Irl}(y) = \emptyset$. So as disjoint open sets containing, respectively, α and β it is enough to take $\mathrm{Irl}(x)$ and $\mathrm{Irl}(y)$.

Unfortunately, if we do not assume any other specific axioms except for (G) we cannot prove much more about $\langle \Pi_{\mathrm{G}}, \mathscr{O} \rangle$ as a topological space. For example, if the relation of external tangency is empty, then what we obtain is a totally disconnected space. This is due to the fact that in such case $\bigcirc = \mathbb{C}$ (in consequence $\wr =)($), so every region being a part of the space is separated from its complement, owing to ($\mathrm{irr}_{\wr}$). In consequence (C6) is not satisfied in such structures (except for the trivial one element mereological structure). Moreover, one can prove that Grzegorczyk's axioms allows for, either finite or infinite, atomic structures. In case of such structures there are such (probably among others) that are discrete (finite case) and such that are one-point compactifications of a discrete space. So such spaces are quite remote from the Euclidean space that we aim to grasp via some structures put upon the space and its parts. Moreover, this is very hard to see how we could construct any geometrical relation between points that would allow us to speak about geometry.[12] There is another problem here as well. Since among G-structures there are such that consist of finitely or countably many elements, topological spaces built upon them will consist of finitely or countably many points. In consequence we actually cannot speak about any interesting geometry, since to produce such we need to have at our disposal continuum of points.

Of course, we do not want to say that the above facts are weak points of Grzegorczyk's theory. His theory should rather be considered as one determining quite a large class of topological spaces. This class can be narrowed by adding additional axioms to the original set of postulates of Grzegorczyk's.

12. Tarski's geometry of solids

Now we will show that due to Tarski's ingenious ideas and constructions from [24] we can indeed construct a point-free system of Euclidean geometry.

Again, let $\langle M, \sqsubset \rangle$ be a mereological structure. Tarski's idea was to enrich such a structure with an additional primitive notion that he called a notion of (*mereological*) *ball*. We will use letter '$\mathbb{B}$' to denote a set of balls and we will briefly analyze relational structures of the form $\langle M, \mathbb{B}, \sqsubset \rangle$.

First, by means of $\mathbb{B}$ and $\sqsubseteq$ we define the notion of *concentricity of balls*. To define it Tarski introduces a series of quite complicated definitions of *external tangency*, *internal tangency*, *external diametrical tangency* and *internal diametrical tangency*. All this relations hold in B, the first two of them are binary and the second two ternary. To give the reader a feeling for those definitions we will present a definition of *external tangency of balls*. Detailed presentation is beyond the scope of this article but all the details can be found in [9].

Definition 12.1. A ball a is *externally tangent* to a ball b iff (i) a is external to b and (ii) for any balls x and y such that a is ingrediens of both x and y, while x and y are external to b, it is the case that x is an ingrediens of y or y is an ingrediens of x (see Figure 16[13]). Formally

[12] However there is such a possibility. In [7] Gerla presented a theory of pointless metric spaces. One could check whether any of topological spaces obtained within G-structures is metrizable in the sense of [7] and what are the properties of metrics defined.

[13] The figures 16 and 17 originally published in [9, pp. 491 and 495]. Used by permission of Association for Symbolic Logic.

$$a \, \mathrm{ET} \, b \overset{\mathrm{df}}{\iff} a, b \in \mathbb{B} \wedge$$
$$a \wr b \wedge \forall_{x,y \in B}(a \sqsubseteq x \wr b \wedge a \sqsubseteq y \wr b \Rightarrow x \sqsubseteq y \vee y \sqsubseteq x).$$

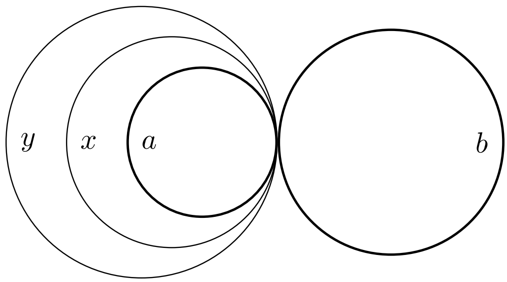

Figure 16. a is externally tangent to b.

Tarski's proceeds in a similar way to define other relations listed above to finally define a key binary relation of *concentricity of balls*. Intuitively, two balls are concentric if they have the same middle point. However, we cannot speak about points of course, since we have none at our disposal. But we can overcome this difficulty by saying that a is concentric with b in the case either $a = b$ or $a \sqsubset b$ and there is a ball c such that touches a from the outside (is externally tangent to it) and b from the inside (is internally tangent to it) and can go all the way round a touching a and b at every single moment of «movement». Of course the third possibility is that this is b which is a part of a, everything else being similar. The reader interested in details is asked one more time to consult both description and analysis of this relation in [9].

Let us denote the concentricity relation of balls by means of '$\circledcirc$'. Now our gate to a definition of point is wide open, since as a point we can simply take the set of all balls that are concentric with some given ball, that is

$$\alpha \in \Pi_{\mathrm{T}} \overset{\mathrm{df}}{\iff} \exists_{b \in \mathbb{B}} \, \alpha = \{x \in \mathbb{B} \mid x \circledcirc b\} \,. \qquad (\mathrm{def}\ \Pi_{\mathrm{T}})$$

We could of course treat any such set as above as a representative of a point and define points as filters generated by such sets but there would be no gains resulting from such a definition. There is no problem with individuation of points in case of Tarski's theory. There are of course other sets descending to exactly the same location as sets of concentric balls (sets of cubes, cuboids etc.) but they are excluded from the set Π_{T} by its very definition. Of course, we still know very little about mereological balls and at this stage we cannot exclude that they are indistinguishable from cubes for example. But the remaining axioms to be presented further assure that balls are exactly those entities we expect them to be.

The difference between Tarski's approach and, for example, Grzegorczyk's one is that in the former we are able to reconstruct a very important ternary geometric relation, the one of *equidistance of points*. And this lets us (as we pointed it in Section 7, see especially Theorem 7.1) speak about Euclidean geometry. Tarski's definition of equidistance relation is actually very simple and intuitive. We say that points α and β are equally distant from a point γ iff they are all equal to each other or there is a ball b in γ such that no ball from either α or β is an ingrediens of b or is external to b. Formally we define a relation $\Delta \subseteq \Pi_{\mathrm{T}}^3$

$$\alpha\beta \,\Delta\, \gamma \overset{\mathrm{df}}{\iff} \alpha = \beta = \gamma \vee \exists_{c\in\gamma} \neg \exists_{a\in\alpha\cup\beta} (a \sqsubseteq c \vee a \wr c)\,. \tag{12.1}$$

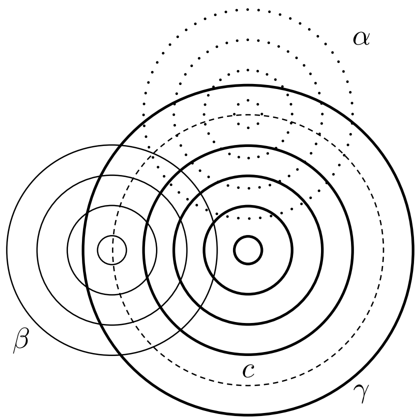

Figure 17. α and β are equidistant from γ.

In our opinion Tarski's greatest merit lies in the fact that he noticed that the triangle relation is inherent in our intuitions rooted in the notion of *ball*. And thanks to the relation just defined we are able to talk about geometry, in particular about Euclidean geometry. This is what actually Tarski does in one of the specific axioms of his theory, an axiom which is very powerful and which actually forces the whole three-dimensional Euclidean geometry to be a part of the system, as it simply states that $\langle \Pi_{\mathrm{T}}, \Delta \rangle$ is a Pieri's structure, that is $\langle \Pi_{\mathrm{T}}, \Delta \rangle$ satisfies all the axioms (expressed in set-theoretical language) of Pieri's geometry

$$\langle \Pi_{\mathrm{T}}, \Delta \rangle \text{ is a Pieri's structure.} \tag{P1}$$

An important observation is such that if we have the whole Euclidean geometry at our disposal we can speak about topological properties of the Euclidean space. In particular among the subsets of Π_{T} we can distinguish those that are regular open in standard Euclidean topology. Let us remind that regular open are those sets which, roughly speaking, have no «cracks», «holes», «threads» etc., which would have less dimensions than the whole space, that is less than three dimensions in our case. Formally, regular open are those sets which are equal to the interior of their closure. Let $\mathbf{RO}_{\Pi_{\mathrm{T}}}$ be the set of all regular open sets in Π_{T} as topological space with Euclidean topology and let $\mathbf{RO}^{+}_{\Pi_{\mathrm{T}}} := \mathbf{RO}_{\Pi_{\mathrm{T}}} \setminus \{\emptyset\}$. Moreover, let $\mathbf{BO}_{\Pi_{\mathrm{T}}}$ be the set of all open balls in $\langle \Pi_{\mathrm{T}}, \Delta \rangle$.

To introduce remaining specific axioms of Tarski's theory we define the notion of *internal point* of a region in a similar way to the one presented earlier, i.e.

$$\alpha \in \mathrm{Irl}(x) \overset{\mathrm{df}}{\iff} \exists_{b \in \mathbb{B}} (b \in \alpha \wedge b \sqsubseteq x)\,,$$

which means simply exactly as much that α is an interior point of a region x iff some ball from α is an ingrediens of x.

In Tarski's sense a *solid* is an arbitrary sum of balls. Let $\mathbb{S}$ be the set of all solids, i.e.

$$\mathbb{S} := \{x \in M \mid \exists_{Y \in 2^{\mathbb{B}}}\ x \ \mathsf{sum}\ Y\}.$$

This lets us state the second postulate of his theory: if x is a solid then the set of interior points of x is a non-empty regular open set, i.e.

$$\forall_{x \in \mathbb{S}}\ \mathrm{Irl}(x) \in \mathbf{RO}^{+}_{\Pi_{\mathrm{T}}}\,. \tag{P2}$$

The third Tarski's postulate says: if a set X of points is a non-empty regular open set, then there exists a solid x such that X is the set of of all interior points of x; i.e.

$$\forall_{X \in \mathbf{RO}^{+}_{\Pi_{\mathrm{T}}}} \exists_{x \in \mathbb{S}}\ X = \mathrm{Irl}(x)\,. \tag{P3}$$

As it can be easily noticed this axioms are to ensure that regions are modeled by regular open sets in Euclidean space. These sets often are argued to be a decent counterparts of parts of the perspective space. However, some authors maintain that the class of these sets is too large. This is due to the fact, that even among regular open sets such can be found that are too peculiar to be counterparts of any parts of the perspective space.

In [24] Tarski assumed one more postulate which says the following: if x and b are solids and all interior points of x are at the same time interior points of y, then x is ingrediens of y; i.e.

$$\forall_{x,y\in\mathbb{S}}(\mathrm{Irl}(x) \subseteq \mathrm{Irl}(y) \Longrightarrow x \sqsubseteq y)\,. \tag{P4}$$

Unfortunately—as it was shown in [9]—in Tarski's theory it is not provable that *every region is a solid*

$$\forall_{x\in M}\, x \in \mathbb{S}\,, \qquad \text{i.e. } M = \mathbb{S}\,. \tag{$\star$}$$

We must accept ($\star$) (or some equivalent condition) as a new axiom.

As we prove in [9, p. 496], in mereological structures the condition ($\star$) is equivalent to one saying that *every region has a ball as its ingrediens*, i.e.

$$\forall_{x\in M}\exists_{b\in\mathbb{B}}\, b \sqsubseteq x\,, \tag{$\star\star$}$$

Indeed, ($\star$) entails ($\star\star$), by (def sum) and the definition of $\mathbb{S}$. Moreover, ($\star\star$) says that the set $\mathbb{B}$ is *dense* in $\langle M, \sqsubset\rangle$. In the standard way, from density of $\mathbb{B}$ we can prove that every region is the mereological sum of set of balls which are its ingredienses. So every region is a solid.

We also show in [9, pp. 516–517] that every structure $\langle M, \mathbb{B}, \sqsubset\rangle$ satisfying ($\star$) fulfils (P4). Thus our theory has only the following axioms: (P1)–(P3) and ($\star$) (or ($\star\star$)).

Theorem 12.1 ([9]). *Let $\langle M, \mathbb{B}, \sqsubset\rangle$ satisfy axioms of mereology and* (P1)–(P3), ($\star$). *Then the mapping* Irl *is an isomorphism from $\langle M, \mathbb{B}, \sqsubset\rangle$ onto* $\langle \mathbf{RO}_{\Pi_{\mathrm{T}}}, \mathbf{BO}_{\Pi_{\mathrm{T}}}, \subset\rangle$.

Setting aside the problem of adequacy of the class of regular open sets as models of regions of the space, we can prove that Tarski's geometry of solids is a very good construction of a point-free system of Euclidean geometry. Crucial metamathematical properties of his system are embodied in the following theorem, where $\mathbf{RO}^{+}_{\mathbb{R}^3}$ and $\mathbf{BO}_{\mathbb{R}^3}$ are, respectively, the sets of all no-empty regular open sets and open balls in three-dimensional Cartesian space.

Theorem 12.2 ([9]). *All structures $\langle M, \mathbb{B}, \sqsubset\rangle$ satisfying axioms of mereology,* (P1)–(P3) *and* ($\star$) *are isomorphic, since any such structure is isomorphic to* $\langle \mathbf{RO}^{+}_{\mathbb{R}^3}, \mathbf{BO}_{\mathbb{R}^3}, \subset\rangle$.

The most important steps in a proof of the above theorem are

(a) the fact that for any topological space $\langle X, \mathscr{O}\rangle$, the pair $\langle \mathbf{RO}^{+}_{\mathscr{O}}, \subset\rangle$ is a mereological structure;

(b) demonstration of the fact that for any ball $b \in \mathbb{B}$, $\mathrm{Irl}(b) \in \mathbf{BO}_{\Pi_{\mathrm{T}}}$, which is definitely the hardest part of the task.

Actually, what (b) shows is that our intuitions about balls as parts of the perspective space (or rather as idealizations of some parts of the perspective space) embodied in definitions and axioms of Tarski's system are correct.

It is worth noticing as well that Tarski's geometry of solids satisfies all the basic assumptions of point-free systems of geometry. Moreover, elements of Π_{T} consists of only such sets of regions that satisfy conditions (r1)–(r4) (boundedness of balls is assured by Theorem 12.2).

To conclude this section, let us notice that thanks to Tarski's definitions of tangency relation between balls, we could actually prove that the connection relation $\mathbb{C}$ is definable within Tarski's structures. To see how this could be done it is enough to notice that by means of ET we can define the relation of external tangency of regions. Let us remind that every region is the mereological sum of some non-empty set of balls. Thus we can define the relation $\mathbb{ET}$, of external tangency between regions, by te following formula

$$x \,\mathbb{ET}\, y \overset{\mathrm{df}}{\iff} x \wr y \wedge \exists_{a\in B}\exists_{b\in B}(a \sqsubseteq x \wedge b \sqsubseteq y \wedge a \,\mathrm{ET}\, b)\,.$$

Now, $\mathbb{C}$ can be of course defined as follows

$$x \,\mathbb{C}\, y \overset{\mathrm{df}}{\iff} x \bigcirc y \vee x \,\mathbb{ET}\, y\,,$$

and in light of Theorem 12.2 we can see that it satisfies (C1)–(C6). On the other hand we have the conditions (11.2) and (11.3), where now CL is the closure in the Euclidean topology in $\langle \Pi_{\mathrm{T}}, \Delta\rangle$ and the set $\mathrm{Irl}(x)$ consists of points from Π_{T}. In language of Tarski's theory conditions (11.2) and (11.3) can be expressed as follows

$$x \,\mathbb{C}\, y \iff \exists_{a\in\mathbb{B}}\forall_{b\in\mathbb{B}}(a \circledcirc b \Longrightarrow (b \bigcirc x \wedge b \bigcirc y)).$$

Remark 12.1. (i) It is provable that if $\langle M, \mathbb{B}, \sqsubset\rangle$ is a model of geometry of solids, then it can be definitionally extended to a structure $\langle M, \mathbb{B}, \sqsubset, \mathbb{C}\rangle$ which satisfies axioms (C1)–(C3) and (G) (hence also all the axioms of connection structures). Π_{T} takes the role of Γ, that is points from Π_{T} are

G-representatives of points (in sense of Grzegorczyk). As the set of points we can take either the set of all filters generated by elements of Π_{T} or simply Π_{T}.

(ii) Within Tarski's theory points could as well be defined as filters generated by sets of concentric balls, that is as sets $\{y \in M \mid \exists_{b\in\alpha}\, b \sqsubseteq y\}$, for $\alpha \in \Pi_{\mathrm{T}}$. Let β be a filter generated by $\alpha \in \Pi_{\mathrm{T}}$. Then for any $x \in M$: $x \in \beta$ iff $\exists_{b\in\alpha}\, b \sqsubseteq x$ iff $\alpha \in \mathrm{Irl}(x)$, and the operation Irl could be defined by means of the same condition as in Grzegorczyk's theory. $\dashv$

13. Conclusion

As we wrote it in the abstract, our intention was not to present mathematically developed theory of region-based geometry but rather show the reader that it is possible to establish a transition between the sensual world and Platonic realm of pure geometry. We hope that we managed to obtain our goal and that those that are interested in the problem will be able to transform some intuitions presented above in yet another, different from constructed so far, mathematical systems of point-free geometry and topology. That this is task which is worth being undertaken is beyond doubt for us.

References

[1] Bennett, B., A. G. Cohn, P. Torrini and S. M. Hazarika, "Region-Based Qualitative Geometry", University of Leeds, School of Computer Studies, Research Report Series, Report 2000.07.

[2] Bennet, B., and I. Düntsch, "Axioms, algebras and topology", pp. 99–159 in: *Handbook of Spatial Logics*, M. Aiello, I. Pratt-Hartmann, and J. Van Benthem (eds.), Springer, 2007.

[3] Biacino, L., and G. Gerla, "Connection structures: Grzegorczyk's and Whitehead's definitions of point", *Notre Dame Journal of Formal Logic* 37, 3 (1996): 431–439.

[4] Borgo, S., N. Guarino, and C. Masolo, "A pointless theory of space based on strong connection and congruence", in: *Principles of Knowledge Representation and Reasoning, Proceedings of the 5th International Conference KR96*, L. C. Aiello and J. Doyle (eds.), Morgan Kaufmann, 1996.

[5] Borsuk, K., and W. Szmielew, *Foundations of Geometry: Euclidean and Bolyai-Lobachevskian Geometry, Projective Geometry*, North Holand Publishing Company, Amsterdam, 1960.

[6] Euclid, *Project Gutenberg's first six books of the Elements of Euclid*, http://www.gutenberg.org/etext/21076, 2007.

[7] Gerla, G., "Pointless metric spaces", *Journal of Symbolic Logic* 55, 1 (1990): 207–219.

[8] Gruszczyński, R., and A. Pietruszczak, "Pieri's structures", *Fundamenta Informaticae* 81, 1–3 (2007): 139–154.

[9] Gruszczyński, R., and A. Pietruszczak, "Full development of Tarski's geometry of solids", *The Bulletin of Symbolic Logic* 14, 4 (2008): 481–540.

[10] Grzegorczyk, A., "Axiomatizability of geometry without points", *Synthese* 12, 2–3 (1960): 228–235.

[11] Hilbert, D., *Foundations of Geometry*, The Open Court Publishing Company, La Salle, Illinois, 1950.

[12] Leśniewski, S., "O podstawach matematyki I", *Przegląd Filozoficzny*, XXX (1927): 164–206.

[13] Leśniewski, S., "O podstawach matematyki V", *Przegląd Filozoficzny* XXXIV (1931): 142–170.

[14] Leśniewski, S., "On the foundations of mathematics", pp. 174–382 in: *Collected Works*, vol. I, S. J. Surma *et al.* (eds.), Nijhoff International Philosophy Series, no. 44, Kluwer Academic Publishers, Dordrecht, 1991.

[15] Leonard, H. S., and N. Goodman, "The calculus of individuals and its uses", *Journal of Symbolic Logic* 5 (1940): 45–55.

[16] Marchisotto, E. A., and J. T. Smith (eds.), *The Legacy of Mario Pieri in Geometry and Arithmetic*, Birkhäuser, Boston-Basel-Berlin, 2007.

[17] Pieri, P., "La geometria elementare istituita sulle nozioni 'punto' é 'sfera' ", *Matematica e di Fisica della Società Italiana delle Scienze* 15 (1908): 345–450.

[18] Pietruszczak, A., *Metamereologia*, Wydawnictwo Uniwersytetu Mikołaja Kopernika, Toruń, 2000.

[19] Pietruszczak, A., "Pieces of mereology", *Logic and Logical Philosophy* 14, 2 (2005): 211–234. DOI: 10.12775/LLP.2005.014

[20] Roeper, P., "Region-based topology", *Journal of Philosophical Logic* 26, 3 (1997): 251–309.

[21] Russell, B., *Our Knowledge of the External World*, George Allen and Unwin LTD, London, 1914. http://www.archive.org/details/ourknowledgeofth005200mbp

[22] Simons, P., *Parts. A Study in Ontology*, Clarendon Press, Oxford, 1987.

[23] Steen, L. A., and J. A. Seebach, *Counterexamples in Topology*, Dover Publications, Inc., New York, 1995.

[24] Tarski, A., "Foundations of the geometry of solids", pp. 24–29 in: *Logic, Semantics, Metamathematics. Papers from 1923 to 1938*, Oxford, 1956.

[25] Varzi A. C., "Spatial reasoning and ontology: parts, wholes and locations", pp. 945–1038 in: *Handbook of Spatial Logics*, M. Aiello, I. Pratt-Hartmann, and J. Van Benthem (eds.), Springer, 2007.

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