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ON UNIVOCAL CONNECTIVES

Abstract. We pay attention to the concept of univocal connective. Considering the corresponding definition in the context of the sequent calculus a problem arises in a paper by Belnap. We provide an explanation by Belnap and finally give some examples and non-examples of univocal connectives.

Keywords: univocal connectives, non-classical logics.

1. Introduction

The concept of univocal connective appears naturally in certain contexts, e.g. when considering non-classical logics (see e.g. [5] and [2]). In a natural deduction setting we may tentatively define a unary connective k to be univocal if $k\varphi \dashv\vdash k'\varphi$, where k' is a *new* connective (that is, a symbol that does not appear in the list of the symbols for the connectives already present in the logic under consideration) having analogous rules to the rules of k. With this definition we may easily prove that conjunction, disjunction, conditional and negation are all univocal in intuitionistic logic with the usual natural deduction rules. For a non-example consider negation in the context of Johansson's minimal logic (for details see Section 4 in this note).

In his influential paper "Tonk, Plonk and Plink" Belnap defines the concept of univocal connective (he uses the word 'uniqueness') in the context of a concept of derivability (he uses the word 'deducibility') characterized in the following way:

Axiom (rx): $A \vdash A$.

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Rules:

Weakening (W)

from $A_1, \ldots, A_n \vdash C$ to infer $A_1, \ldots, A_n, B \vdash C$.

Permutation (Per)

from $A_1, \ldots, A_i, A_{i+1}, \ldots, A_n \vdash B$ to infer $A_1, \ldots, A_{i+1}, A_i, \ldots, A_n \vdash B$.

Contraction (Con)

from $A_1, \ldots, A_n, A_n \vdash B$ to infer $A_1, \ldots, A_n \vdash B$.

Transitivity (Cut)

from $A_1, ..., A_m \vdash B$ and $C_1, ..., C_n, B \vdash D$ to infer $A_1, ..., A_m, C_1, ..., C_n \vdash D$.

These are a variant of the usual left structural rules of Gentzen.

If the system is extended with connectives, Belnap requires that they will have to satisfy two conditions, to wit, conservativeness and uniqueness. Note that these properties have also been suggested in papers such as [5] and [2].

Now Belnap asks "What do we mean by uniqueness ...?" (see [1], p. 133). He himself answers:

"Clearly that at most *one* inferential role is permitted by the characterization of *plonk*; i.e., that there cannot be two connectives which share the characterization given to *plonk* but which otherwise sometimes play different roles. Formally put, uniqueness means that if exactly the same properties are ascribed to some other connective, say *plink*, then A-*plink*-B will play exactly the same role in inference as A-*plonk*-B, both as premiss and as conclusion. To say that *plonk* (characterized thus and so) describes a unique way of combining A and B is to say that if *plink* is given a characterization formally identical to that of *plonk*, then

(1) $A_1, \ldots, B\text{-}plonk\text{-}C, \ldots, A_n \vdash D$ iff $A_1, \ldots, B\text{-}plink\text{-}C, \ldots, A_n \vdash D$

and

(2) $A_1, \ldots, A_n \vdash B\text{-}plonk\text{-}C$ iff $A_1, \ldots, A_n \vdash B\text{-}plink\text{-}C$.

Whether or not we can prove this will depend, of course, not only on the properties ascribed to the connectives, but also on the properties ascribed to deducibility. Given the characterization of deducibility above, it is sufficient and necessary that $B\text{-}plonk\text{-}C \vdash B\text{-}plink\text{-}C$ and conversely." ([1], p. 134).

For the following section we label the just given "sufficient and necessary" conditions:

(3) $B\textit{-plonk-}C \vdash B\textit{-plink-}C$,

(4) $B\textit{-plink-}C \vdash B\textit{-plonk-}C$.

2. A problem

Now we come to the following problem. In his paper Belnap did not write down his proof of the mentioned fact, i.e. that conditions (1) and (2) are equivalent to conditions (3) and (4), but this is not difficult to do:

$$\text{(1) and (2)} \Longrightarrow \text{(3) and (4)} \qquad (\dagger)$$

1. $B\textit{-plink-}C \vdash B\textit{-plink-}C$ (rx)
2. $B\textit{-plonk-}C \vdash B\textit{-plink-}C$ (1), 1

and we analogously get (4).

$$\text{(3) and (4)} \Longrightarrow \text{(1) and (2)} \qquad (\ddagger)$$

In the proof of (1) we use sets of formulas Γ to the left and Σ to the right instead of Belnap's notation:

1. $\Gamma, B\textit{-plonk-}C, \Sigma \vdash A$ Sup
2. $B\textit{-plink-}C \vdash B\textit{-plonk-}C$ (4)
3. $\Gamma, B\textit{-plink-}C, \Sigma \vdash A$ (Cut), (Per), 1, 2

1. $\Gamma \vdash B\textit{-plonk-}C$ Sup
2. $B\textit{-plonk-}C \vdash B\textit{-plink-}C$ (3)
3. $\Gamma \vdash B\textit{-plink-}C$ (Cut), 1, 2

In the proof (†) one finds, surprisingly enough, that either (1) or (2) is enough. As a matter of fact, they are both equivalent as we easily see in the next

Proposition. (1) $\Longleftrightarrow$ (2).

Proof. "$\Rightarrow$"
1. $A_1, \ldots, A_n \vdash B\textit{-plonk-}C$ Sup
2. $B\textit{-plink-}C \vdash B\textit{-plink-}C$ (rx)

3. $A_1, \ldots, B\text{-}plink\text{-}C, \ldots, A_n \vdash B\text{-}plink\text{-}C$ (W), (Per), 2
4. $A_1, \ldots, B\text{-}plonk\text{-}C, \ldots, A_n \vdash B\text{-}plink\text{-}C$ (1), 3
5. $A_1, \ldots, A_n \vdash B\text{-}plink\text{-}C$ (Cut), (Con), (Per), 1, 4

and the other conditional is analogous.

"$\Leftarrow$"

1. $A_1, \ldots, B\text{-}plonk\text{-}C, \ldots, A_n \vdash D$ Sup
2. $B\text{-}plink\text{-}C \vdash B\text{-}plink\text{-}C$ (rx)
3. $A_1, \ldots, B\text{-}plink\text{-}C, \ldots, A_n \vdash B\text{-}plink\text{-}C$ (W), (Per), 2
4. $A_1, \ldots, B\text{-}plink\text{-}C, \ldots, A_n \vdash B\text{-}plonk\text{-}C$ (2), 3
5. $A_1, \ldots, B\text{-}plink\text{-}C, \ldots, A_n \vdash D$ (Cut), (Con), (Per), 1, 4

and the other conditional is analogous. $\dashv$

This is surprising because conditions (1) and (2) *seem* to be saying different things. Also, why did Belnap include both if one is enough?

3. Belnap's explanation

Fortunately, we have got an answer provided by a personal communication by Belnap, for which we are grateful:

"I have an explanation for laying down both of the conditions (1) and (2), but not exactly an excuse. In fact I should have noted that in the particular context at hand, either one would suffice.

The explanation is that even at that time I had done a lot of work on "Gentzen systems," most of which appears in volume I and volume II of *Entailment* (vol. 1 by Alan Ross Anderson and myself, and vol. 2 including Michael Dunn as a third co-author). The setting in Gentzen systems is this: one postulates rules for each connective on the left of the turnstile, and separately postulates rules for those connectives (whichever ones are at issue) on the right of the turnstile. Then the major mathematical fact, the result that Gentzen called his "Hauptsatz," is that transitivity (even when generalized) is redundant. You have said all there is to say about a connective when you establish its properties on the left, and on the right. Furthermore, every rule, whether on the left or on the right, only introduces a connective, and never eliminates it. This is one of the principal results of proof theory.

So I was thinking of something that I did not mention, namely, a version of the calculus with those separate rules instead of transitivity. And in that case, (1) and (2) stand as separate properties until when and if transitivity is proved.

That is as close as I can come to an explanation."

We consider that Belnap's explanation answers our question and that the problem has been useful to throw light on the role of transitivity in the concept of univocal connective.

4. Some examples and non-examples

It is easily seen that the usual conjunction, disjunction and conditional are univocal in positive logic. Consequently, they are also univocal in Johansson's minimal logic **M**, intuitionistic logic **I** and classical logic **C**. In **I** also negation is univocal (consequently also in **C**). But negation is *not* univocal in **M**. To see that a connective is univocal one derivation is enough. To see that a connective is *not* univocal in a given logic, we will use the following procedure: we will consider a logical calculus of the given logic and provide two different semantic characterizations of the connective under consideration. Then we will prove that the calculus is sound with respect to both semantic characterizations. Finally, we will find a formula containing the connective and a valuation such that by one semantic characterization of the connective the formula takes a designated value and by the other characterization it takes a non-designated value.

Now, let us consider the case of '$\neg$' in **M**, which may be given by an axiomatic system with *modus ponens* (MP) as only rule and the following axioms:

(A1) $\varphi \to (\psi \to \varphi)$,

(A2) $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$,

(A3) $(\varphi \to \psi) \to ((\varphi \to \chi) \to (\varphi \to (\psi \wedge \chi)))$,

(A4) $(\varphi \wedge \psi) \to \varphi$,

(A5) $(\varphi \wedge \psi) \to \psi$,

(A6) $\varphi \to (\varphi \vee \psi)$,

(A7) $\psi \to (\varphi \vee \psi)$,

(A8) $(\varphi \to \chi) \to ((\psi \to \chi) \to ((\varphi \vee \psi) \to \chi))$,

(A9) $(\varphi \to \psi) \to ((\varphi \to \neg\psi) \to \neg\varphi)$.

Now, let us consider a semantics with values 0 and 1, the last as only designated value. Let us define a valuation in the usual way in classical logic for '$\wedge$', '$\vee$' and '$\rightarrow$'. Now, in the case of negation we give the following two characterizations: (a) $V(\neg\varphi) = 1$ iff $V(\varphi) = 0$; (b) $V(\neg\varphi) = 1$, for all φ. It is easily seen that the given axiomatic system is sound with respect to both characterizations. But, consider the formula '$\neg p$' and a valuation V such that $V(p) = 1$. Then, in the first characterization, $V(\neg p) = 0$, but in the second, $V(\neg p) = 1$. This proves that '$\neg$' is not univocal in **M**.

Also, '$\neg$' is not univocal in the logics **D** and **E**, where $\mathbf{D} := \mathbf{M} + \varphi \vee \neg\varphi$ and $\mathbf{E} := \mathbf{M} + ((\varphi \rightarrow \psi) \rightarrow \varphi) \rightarrow \varphi$ (see e.g. [3, Chap. 6, p. 261]). To see that '$\neg$' is not univocal in the logics **D** and **E**, first note that **E** is stronger than **D** (this may be seen substituting in the axiom $((\varphi \rightarrow \psi) \rightarrow \varphi) \rightarrow \varphi$ the formula $\varphi \vee \neg\varphi$ for φ and the formula $\varphi \wedge \neg\varphi$ for ψ). It follows that if '$\neg$' is not univocal in **E**, then it also cannot be univocal in **D**. To see that '$\neg$' is not univocal in **E**, we may proceed exactly as we did in the case of **M** (the new axiom in **E**, that is, Peirce's Law, does not include '$\neg$' as one of its symbols).

Negation is also not univocal in paraconsistent logic $\mathbf{P^1}$ (see [7]). In the just cited place this logic appears in an axiomatic setting with language $\{\rightarrow, \sim\}$, the usual intuitionistic axioms for '$\rightarrow$' (that is, (A1) and (A2)), the following axioms for the negation '$\sim$':

(S1) $\quad (\sim\varphi \rightarrow \sim\psi) \rightarrow ((\sim\varphi \rightarrow \sim\sim\psi) \rightarrow \varphi)$,

(S2) $\quad (\varphi \rightarrow \psi) \rightarrow \sim\sim(\varphi \rightarrow \psi)$,

(there is one more but redundant axiom) and MP as only rule. It is also proven that the given calculus is sound and complete (with designated values T_0 and T_1) relatively to the semantics given by the following valuations:

$\rightarrow$	T_0	T_1	F
T_0	T_0	T_0	F
T_1	T_0	T_0	F
F	T_0	T_0	T_0

$\sim$	
T_0	F
T_1	T_0
F	T_0

Now, let us consider the semantic characterization given by the following table:

$\sim$	
T_0	F
T_1	F
F	T_0

It is easily seen that the given calculus is sound also with respect to this characterization. For this, we just need to check that the given axioms for '$\sim$'

have one of the designated values in any valuation. In the case of axiom (S1), suppose a valuation V with $V(\mathrm{S1}) = \mathrm{F}$. Then, (i) $V({\sim}\varphi \to {\sim}\psi) = \mathrm{T}_0$ and $V(({\sim}\varphi \to {\sim}{\sim}\psi) \to \varphi) = \mathrm{F}$. Then, (ii) $V({\sim}\varphi \to {\sim}{\sim}\psi) = \mathrm{T}_0$ and $V(\varphi) = \mathrm{F}$. So, $V({\sim}\varphi) = \mathrm{T}_0$. Then, using (i) it follows that $V({\sim}\psi) \neq F$. Then, $V({\sim}{\sim}\psi) = \mathrm{F}$. But then $V({\sim}\varphi \to {\sim}{\sim}\psi) = \mathrm{F}$, a contradiction with (ii). The case of (S2) is similar. Now, consider the formula ${\sim}p$ and a valuation V such that $V(p) = \mathrm{T}_1$. Then, in the first characterization, $V({\sim}p) = \mathrm{T}_0$, but in the second, $V({\sim}p) = \mathrm{F}$. This proves that '$\sim$' is not univocal in $\mathbf{P^1}$.

Ending this note, we provide two more examples of univocal connectives: negations 'α' and '$\hbar$'. The connective 'α' may be defined adding to an axiomatic system of $\mathbf{I}$ the formulas of the form $\alpha\varphi$, the following axiom schema

$$(\alpha 1) \qquad (\alpha\varphi \to \varphi) \to \varphi$$

and the rule

$$(\alpha 2) \qquad \frac{(\psi \to \varphi) \to \varphi}{\alpha\varphi \to \psi},$$

obtaining an axiomatic system of the logic $\mathbf{I}_\alpha$ (see [6] for the original non-axiomatic formulation and [4] for an alternative axiomatization). Moreover, if we add another connective 'α'' satisfying (α1) and (α2), then we obtain the logic $\mathbf{I}_\alpha^{\alpha'}$. The fact of derivability of φ from formulas in Γ and axioms of $\mathbf{I}_\alpha^{\alpha'}$ using both (MP) and (α2) is denoted by $\Gamma \vdash_{\mathbf{I}_\alpha^{\alpha'}} \varphi$ (we often say that φ is *globally derivable* from Γ in $\mathbf{I}_\alpha^{\alpha'}$). We have the following global derivation:

1. $\alpha\varphi$	Sup
2. $(\alpha'\varphi \to \varphi) \to \varphi$	Axiom (α1) for 'α''
3. $\alpha\varphi \to \alpha'\varphi$	Rule (α2) for α, 2
4. $\alpha'\varphi$	(MP), 1, 3

As we can similarly obtain a global derivation of $\alpha\varphi$ from $\alpha'\varphi$ in $\mathbf{I}_\alpha^{\alpha'}$, we may deduce both

$$\alpha\varphi \vdash_{\mathbf{I}_\alpha^{\alpha'}} \alpha'\varphi \qquad \text{and} \qquad \alpha'\varphi \vdash_{\mathbf{I}_\alpha^{\alpha'}} \alpha\varphi\,.$$

Therefore, 'α' is univocal.

Remark. Notice that 'α' is also univocal for another concept of derivability in $\mathbf{I}_\alpha^{\alpha'}$ which is often called *deducibility*.

We say that a formula φ is *deducible* from a set of formulas Γ in a logic $\boldsymbol{L}$, we write $\Gamma \Vdash_{\boldsymbol{L}} \varphi$, iff there is a derivation of φ from formulas in Γ and theses of $\boldsymbol{L}$ with the help of only *modus ponens*. Because the set of theses of $\boldsymbol{L}$ is closed under (MP): $\emptyset \Vdash_{\boldsymbol{L}} \varphi$ iff $\emptyset \vdash_{\boldsymbol{L}} \varphi$, i.e. φ is a thesis of $\boldsymbol{L}$, where $\vdash_{\boldsymbol{L}}$ is a relation of *global derivability* with the help of all axioms and all rules for $\boldsymbol{L}$.[1] If (A2) and $\varphi \to \varphi$ are theses of $\boldsymbol{L}$, then the standard deduction theorem — $\Gamma, \psi \Vdash_{\boldsymbol{L}} \varphi$ iff $\Gamma \Vdash_{\boldsymbol{L}} \psi \to \varphi$ — holds.[2] Hence, $\psi \Vdash_{\boldsymbol{L}} \varphi$ iff $\vdash_{\boldsymbol{L}} \psi \to \varphi$.

By the steps 2 and 3 in the above global derivation, the logic $\mathbf{I}_{\alpha}^{\alpha'}$ has the following theses: $\alpha\,\varphi \to \alpha'\,\varphi$ and $\alpha'\,\varphi \to \alpha\,\varphi$. Thus, we may deduce both: $\alpha\,\varphi \Vdash_{\mathbf{I}_{\alpha}^{\alpha'}} \alpha'\,\varphi$ and $\alpha'\,\varphi \Vdash_{\mathbf{I}_{\alpha}^{\alpha'}} \alpha\,\varphi$. $\dashv$

The connective 'h' may be defined adding to an axiomatic system of $\mathbf{I}$ the formulas of the form $h\,\varphi$ and the following axiom schemas

(h1) $\quad \neg\varphi \to \neg\neg\, h\,\varphi$,

(h2) $\quad h\,\varphi \to (\psi \vee (\neg\varphi \wedge \neg\psi))$,

obtaining an axiomatic system of the logic $\mathbf{I}_h$ (for more on this sort of extension of $\mathbf{I}$ see [4]). It may be seen that $h\,\varphi$ is equivalent to $\gamma \wedge \neg\varphi$, where γ is the constant with the axiom schemas $\neg\neg\gamma$ and $\gamma \to (\varphi \vee \neg\varphi)$ added to $\mathbf{I}$ (see [8] or [9]).

Suppose that we add another connective 'h′' satisfying (h1) and (h2). Then we have the following derivation in $\mathbf{I}_h^{h'}$:

1. $h\,\varphi$	Sup
2. $h\,\varphi \to (h'\,\varphi \vee (\neg\varphi \wedge \neg\, h'\,\varphi))$	Axiom (h2) for 'h'
3. $h'\,\varphi \vee (\neg\varphi \wedge \neg\, h'\,\varphi)$	(MP), 1, 2
4. $\neg\varphi \to \neg\neg\, h'\,\varphi$	Axiom (h1) for 'h′'
5. $h'\,\varphi \vee (\neg\neg\, h'\,\varphi \wedge \neg\, h'\,\varphi)$	theses of $\mathbf{I}$, 3, 4
6. $h'\,\varphi$	theses of $\mathbf{I}$, 5

Thus, it follows that $h\,\varphi \vdash_{\mathbf{I}_h^{h'}} h'\,\varphi$ (notice that $\vdash_{\mathbf{I}_h^{h'}} = \Vdash_{\mathbf{I}_h^{h'}}$; see Remark). We may analogously get the reciprocal. Hence, the connective 'h' is univocal.

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[1] If we consider an axiomatic system of $\boldsymbol{L}$ with *modus ponens* as only rule, then $\Vdash_{\boldsymbol{L}} = \vdash_{\boldsymbol{L}}$.

[2] Notice that the deduction theorem does not hold for $\vdash_{\mathbf{I}_\alpha}$ and $\vdash_{\mathbf{I}_\alpha^{\alpha'}}$.

References

[1] Belnap, N. D., "Tonk, plonk and plink", *Analysis* 22 (1962): 130–134.

[2] Caicedo, X., and R. Cignoli, "An algebraic approach to intuitionistic connectives", *The Journal of Symbolic Logic* 66 (2001): 1620–1636.

[3] Curry, H. B., *Foundations of Mathematical Logic*, McGraw-Hill, New York, 1963.

[4] Ertola, R., "On some operations using the min operator", accepted for publication in *The Many Sides of Logic, Studies in Logic Series*, College Publications, London, 2009.

[5] Gabbay, D. M., "On some new intuitionistic propositional connectives. I", *Studia Logica* 36 (1977): 127–139.

[6] Humberstone, L., "The pleasures of anticipation: Enriching Intuitionistic Logic", *Journal of Philosophical Logic* 30 (2001): 395–438.

[7] Sette, A., "On the propositional calculus P^1", *Mathematica Japonicae* 16 (1973): 173–180.

[8] Smetanich, Y. S., "On the completeness of a propositional calculus with a supplementary operation in one variable", *Tr. Mosk. Mat. Obsch.* 9 (1960): 357–371.

[9] Yashin, A. D., "The Smetanich logic T^Φ and two definitions of a new intuitionistic connective", *Mathematical Notes* 56 (1994): 745–750.

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