Logic and Logical Philosophy
Volume 14 (2005), 253–263
DOI: 10.12775/LLP.2005.016

Krystyna Mruczek-Nasieniewska

P-COMPATIBLE ABELIAN GROUPS

Abstract. Let $\tau\colon F \to \mathcal{N}$ be a type of a variety V. Every partition P of the set F determines a so-called P-compatible variety. We consider the varieties $\mathcal{G}_P^n$ defined by so-called P-compatible identities of Abelian groups with exponent n. Besides, we study a connection between the lattice of all partitions of the set F and the lattice of all subvarieties of the variety defined by some kind of P-compatible identities — externally compatible identities satisfied in the class of all Abelian groups with exponent n.

Keywords: Abelian groups, P-compatible identity, variety, partition of a set.

2000 *Mathematics Subject Classification*: 03C05, 20A05

1. Preliminaries

Let $Id(\tau)$ be the set of all identities of the type $\tau : F \to \mathcal{N}$, where F is a set of fundamental operation symbols and $\mathcal{N}$ is the set of non-negative integers.

For a set $\Sigma \subseteq Id(\tau)$ we denote by $Cn(\Sigma)$ the deductive closure of Σ, i.e. $Cn(\Sigma)$ is the smallest subset of $Id(\tau)$ containing Σ such that:

1. $x \approx x \in Cn(\Sigma)$ for every variable x;
2. if $p \approx q \in Cn(\Sigma)$, then $q \approx p \in Cn(\Sigma)$;
3. if $p \approx q, q \approx r \in Cn(\Sigma)$, then $p \approx r \in Cn(\Sigma)$;
4. $Cn(\Sigma)$ is closed under replacement, i.e., given any $p \approx q \in Cn(\Sigma)$ and any term r of the type τ, if p occurs as a subterm of r then, by letting s be the result of replacing that occurence of p by q, we have $r \approx s \in Cn(\Sigma)$;

Received October 28, 2005

ISSN: 1425-3305

5. $Cn(\Sigma)$ is closed under substitution, which means that for each $p \approx q \in Cn(\Sigma)$ and each term r of the type τ, if we replace every ocurrence of a given variable x in $p \approx q$ by r, then resulting identity belongs to $Cn(\Sigma)$.

If $\Sigma = Cn(\Sigma)$, then Σ is called equational theory.

The concept of a P-compatible identity is related to the special structure of terms occurring in the identity. This structure is preserved by the operator Cn, so we can consider equational theories determined by P-compatible identities. Let P be a partition of F. By $[f]_P$ we will denote the block of P containing f. An identity $p \approx q$ of the type τ is P-*compatible* iff it is of the form $x \approx x$ or the outermost operation symbols in p and q belong to the same block of P.

The notion of a P-compatible identity was introduced by J. Płonka in [9] and was a generalization of both an externally compatible identity introduced by W. Chromik in [1] and a normal identity defined independently by J. Płonka [8] and I. I. Melnik [5].

An identity $p \approx q$ of the type τ is *externally compatible* iff it is P-compatible, where P contains singletons only.

An identity $p \approx q$ of the type τ is *normal* iff it is of the form $x \approx x$ or neither p nor q is a variable.

By $Mod(\Sigma)$ we denote the class of all models of Σ, that is the class of all algebras of the type τ satisfying the identities from Σ.

We will use the following notations:

$P(\tau)$ — the set of all P-compatible identities of the type τ,
$Ex(\tau)$ — the set of all externally compatible identities of the type τ,
$Id(V)$ — the set of all identities satisfied in V,
$P(V)$ — the set of all P-compatible identities satisfied in V,
$Ex(V)$ — the set of all externally compatible identities satisfied in V,
V_P — the variety $Mod(P(V))$,
V_{Ex} — the variety $Mod(Ex(V))$.

It is known that the set of all subvarieties of a given variety is a lattice ordered by set inclusion. The lattices of varieties were studied in many works (see [2], [3], [6], [10]). Let $\mathcal{L}(V) = (L(V); \subseteq)$ denotes the lattice of all subvarieties of the variety V. For any partition P we have $Ex(V) \subseteq P(V)$. Thus, the class $Mod(P(V))$ is a subvariety of $Mod(Ex(V))$. In the next section we will describe the equational bases of varieties defined by P-compatible identities of Abelian groups with exponent n of the type $\tau_1 : \{\cdot, ^{-1}, e\} \longrightarrow \mathcal{N}$, where $\tau_1(\cdot) = 2, \tau_1(^{-1}) = 1, \tau_1(e) = 0$ and in the last section we will study connections between the lattice of the partitions of the set $\{\cdot,^{-1},e\}$ and the lattice of all subvarieties of the variety defined by externally compatible identities satisfied in the class of Abelian groups with exponent n.

Let us consider algebras $\mathcal{A} = (A; F^{\mathcal{A}})$ and $\mathcal{I} = (I; F^{\mathcal{I}})$ of the type τ and a partition P of the set F. The algebra $\mathcal{A}$ is a *P-dispersion of $\mathcal{I}$* (see [4], [7]) iff there exists a partition $\{A_i\}_{i\in I}$ of A and there exists a family $\{c_{[f]_P}\}_{f\in F}$ of mappings $c_{[f]_P} : I \to A$ satisfying the following conditions:

i. *for each $i \in I$ $c_{[f]_P}(i) \in A_i$;*

ii. *for each $f \in F$ and for each $a_i \in A_{k_i}$, $i = 0, ..., \tau(f) - 1$,*

$$f^{\mathcal{A}}(a_0, ..., a_{\tau(f)-1}) = c_{[f]_P}(f^{\mathcal{I}}(k_0, ..., k_{\tau(f)-1}));$$

iii. *if $f \in [g]_P$ then for each $i \in I$ $c_{[f]_P}(i) = c_{[g]_P}(i)$.*

In the case if the partition P contains singletons only, P-dispersion is called a *dispersion*.

In [9] it was proved that

(1.1) *Every P-dispersion of the algebra $\mathcal{I}$ satisfies all P-compatible identities satisfied in $\mathcal{I}$.*

2. Equational bases of P-compatible Abelian groups

Let us fix the type $\tau_1 : \{\cdot,^{-1},e\} \longrightarrow \mathcal{N}$,where $\tau_1(\cdot) = 2, \tau_1(^{-1}) = 1, \tau_1(e) = 0$. Let $\mathcal{G}^n$ denotes the variety of Abelian groups of the type τ_1 satisfying identity $x^n \approx x{\cdot}x^{-1}$, where $n \in \mathcal{N}$. In the present section we will construct equational bases of the varieties $\mathcal{G}^n_P$ defined by P-compatible identities satisfied in $\mathcal{G}^n$. We restrict ourselves to the classes $\mathcal{G}_P$ defined by P-compatible identities of Abelian groups, because it is clear that the identity $x^n \approx x \cdot x^{-1}$ together with equational bases of $\mathcal{G}_P$ form equational bases of $\mathcal{G}^n_P$.

First we describe the construction of finding an equational bases presented in [9]. Let $\tau : F \to \mathcal{N}$ be a type of algebras. Let P be a partition of F. A block $[f]_P$ of a partition P will be called *nullary* iff $\tau(g) = 0$ for each $g \in [f]_P$. Let V be a variety of the type τ satisfying the following three conditions:

(2.1) *There exists a non-trivial unary term $q(x)$ such that for each $f \in F$, the identity*

$$q(f(x_0, ..., x_{\tau(f)-1})) \approx q(f(q(x_0), ..., q(x_{\tau(f)-1}))) \tag{1}$$

belongs to $Id(V)$.

From now on, let $q(x)$ be a fixed term whose existence is guaranteed by the above condition.

(2.2) *If* $[f]_P$ *is a non-nullary block of* P *and* $g, h \in [f]_P$, *then there exists a non-trivial unary term* $q_{g,h}(x)$ *such that the most external fundamental operation symbol in term* $q_{g,h}(x)$ *belongs to* $[f]_P$ *and the identities*

$$g(x_0, ..., x_{\tau(g)-1}) \approx q_{g,h}(q(g(x_0, ..., x_{\tau(g)-1}))), \tag{2}$$

$$h(x_0, ..., x_{\tau(h)-1}) \approx q_{g,h}(q(h(x_0, ..., x_{\tau(h)-1}))) \tag{3}$$

belong to $Id(V)$.

From now on, for any $[f]_P$ being a non-nullary block of P and $g, h \in [f]_P$, let $q_{g,h}(x)$ be a fixed term satisfying the above condition. The existence of $q_{g,h}(x)$ is guaranteed.

(2.3) *If* $[f]_P$ *is a nullary block of* P, *then for each* $g \in [f]_P$ *the identity*

$$f \approx g \tag{4}$$

belongs to $Id(V)$.

Let B be an equational base of V. We define a set B^* of identities of the type τ satisfying the following three conditions:

(2.4) *The identities:*

> (1), *for any* $f \in F$,
> (2), *for any* $[f]_P$ — *a non-nullary block of* P *and* $g, h \in [f]_P$,
> *and* (4), *for any* $[f]_P$ — *a nullary block of* P *and any* $g \in [f]_P$,

belong to B^*.

(2.5) *If* $\phi \approx \psi$ *belongs to* B, *then the identity* $q(\phi) \approx q(\psi)$ *belongs to* B^*.

(2.6) B^* *is the smallest set satisfying* (2.4) *and* (2.5).

In [9] J. Płonka proved the following

THEOREM 2.1. *If B is an equational base of V, then B^* is an equational base V_P.*

It is a well known fact that the following identities of the type τ_1:

$$\begin{array}{rl} (2.\mathrm{i}) & (x\cdot y)\cdot z \approx x\cdot (y\cdot z),\\ (2.\mathrm{ii}) & x\cdot y\approx y\cdot x,\\ (2.\mathrm{iii}) & x\cdot x^{-1}\approx e,\\ (2.\mathrm{iv}) & x\cdot e\approx x \end{array}$$

form an equational base of the variety $\mathcal{G}$ of Abelian groups of the type τ_1.

We have only the following partitions of the set $\{\cdot, ^{-1}, e\}$:

$P_0 = \{\{\cdot\}, \{^{-1}\}, \{e\}\}$,

$P_1 = \{\{\cdot, ^{-1}\}, \{e\}\}$,

$P_2 = \{\{\cdot, e\}, \{^{-1}\}\}$,

$P_3 = \{\{^{-1}, e\}, \{\cdot\}\}$,

$P_4 = \{\{\cdot, ^{-1}, e\}\}$.

The partition P_0 we will traditionally denote by Ex, and the partition P_4 by N.

It is easy to check that the variety $\mathcal{G}$ of the type τ_1 fulfills the assumptions of the above J. Płonka's theorem concerning the construction of the equational bases. Putting $q(x) \approx x\cdot e$, $q_{\cdot}(x) \approx x\cdot e$, $q_{-1}(x) \approx (x^{-1})^{-1}$ we receive the following

LEMMA 2.1. *The following identities of the type τ_1:*

$$\begin{array}{rl} (2.\mathrm{v}) & (x\cdot y)\cdot e \approx ((x\cdot e)\cdot (y\cdot e))\cdot e,\\ (2.\mathrm{vi}) & x^{-1}\cdot e\approx (x\cdot e)^{-1}\cdot e,\\ (2.\mathrm{vii}) & x\cdot y\approx ((x\cdot y)\cdot e)\cdot e,\\ (2.\mathrm{viii}) & x^{-1}\approx ((x^{-1}\cdot e)^{-1})^{-1},\\ (2.\mathrm{ix}) & ((x\cdot y)\cdot z)\cdot e\approx (x\cdot (y\cdot z))\cdot e,\\ (2.\mathrm{x}) & (x\cdot y)\cdot e\approx (y\cdot x)\cdot e,\\ (2.\mathrm{xi}) & (x\cdot x^{-1})\cdot e\approx e\cdot e,\\ (2.\mathrm{xii}) & (x\cdot e)\cdot e\approx x\cdot e \end{array}$$

form the equational base of the variety $\mathcal{G}_{Ex}$.

In the next theorem we present simplified set of identities axiomatizing the same variety $\mathcal{G}_{Ex}$.

THEOREM 2.2. *Identities (2.i), (2.ii) of the type τ_1 and additionally identities:*

(2.xiii) $x \cdot y \cdot e \approx x \cdot y$,

(2.xiv) $(x \cdot e)^{-1} \approx x^{-1}$,

(2.xv) $x \cdot x^{-1} \approx e \cdot e$

form an equational base of the class $\mathcal{G}_{Ex}$.

Proof. Let $B^{**}_{\mathcal{G}_{Ex}}$ denotes the set of identities (2.i),(2.ii) and (2.xiii)–(2.xv), while by $B^{*}_{\mathcal{G}_{Ex}}$ we denote the set (2.v)–(2.xii). One can show that $Cn(B^{**}_{\mathcal{G}_{Ex}}) = Ex(\mathcal{G})$. From the lemma (2.1) follows that $Cn(B^{*}_{\mathcal{G}_{Ex}}) = Ex(\mathcal{G})$. Thus, to prove the lemma it is enough to show that the sets $Cn(B^{**}_{\mathcal{G}_{Ex}})$ and $Cn(B^{*}_{\mathcal{G}_{Ex}})$ are equal. Since every base identity of the set $Cn(B^{**}_{\mathcal{G}_{Ex}})$ is an externally compatible identity fulfilled in the variety of Abelian groups, thus it is true that $Cn(B^{**}_{\mathcal{G}_{Ex}}) \subseteq Cn(B^{*}_{\mathcal{G}_{Ex}})$. We will prove the reverse inclusion. Let us notice that from the identity (2.i) follows the identity (2.ix), while from (2.ii) directly follows (2.x). From (2.xiii) and (2.xv) we have the identity (2.xi). By substitution of the term e for a variable y in (2.xiii) and by the associativity of the operation '$\cdot$', we get that (2.xii) belongs to the set $Cn(B^{**}_{\mathcal{G}_{Ex}})$. By substitution of the term $y \cdot (e \cdot e)$ for a variable y in (2.xiii) and by the associativity and commutativity of the operation '$\cdot$', we obtain that $x \cdot y \cdot e \cdot e \approx x \cdot e \cdot y \cdot e \cdot e$ belongs to the set $Cn(B^{**}_{\mathcal{G}_{Ex}})$. From that, by the fact that the identity (2.xiii) belongs to $Cn(B^{**}_{\mathcal{G}_{Ex}})$, follows that (2.v) belongs to $Cn(B^{**}_{\mathcal{G}_{Ex}})$. Moreover, directly from the identity (2.xiv) follows that the identity (2.vi) belongs to the set $Cn(B^{**}_{\mathcal{G}_{Ex}})$. From the identity (2.xiii) and the identity (2.i) follows that the identity (2.vii) is deducible from the set $Cn(B^{**}_{\mathcal{G}_{Ex}})$. Now, let us substitute in (2.xiv) a term $(x^{-1})^{-1}$ for the variable x. We receive $((x^{-1})^{-1})^{-1} \approx ((x^{-1})^{-1} \cdot e)^{-1} \in Cn(B^{**}_{\mathcal{G}_{Ex}})$. Using the identity (2.xiii) we get $((x^{-1})^{-1} \cdot e)^{-1} \approx ((x^{-1})^{-1} \cdot e \cdot e)^{-1} \in Cn(B^{**}_{\mathcal{G}_{Ex}})$. Further, by (2.xv), commutativity and associativity of the operation '$\cdot$' we have the following the identities: $((x^{-1})^{-1} \cdot e \cdot e)^{-1} \approx ((x^{-1})^{-1} \cdot x \cdot x^{-1})^{-1} \approx (x \cdot e \cdot e)^{-1}$. From (2.xiii) follows that $(x \cdot e \cdot e)^{-1} \approx (x \cdot e)^{-1} \in Cn(B^{**}_{\mathcal{G}_{Ex}})$. Thus, we have that $((x^{-1})^{-1})^{-1} \approx (x \cdot e)^{-1}$ belongs to the set $Cn(B^{**}_{\mathcal{G}_{Ex}})$. By the last identity and (2.xiv), we have the identity $((x^{-1})^{-1})^{-1} \approx x^{-1}$. By the substitution of the term x^{-1} for a variable x in (2.xiv), we see that $((x^{-1} \cdot e)^{-1}) \approx ((x^{-1})^{-1}) \in Cn(B^{**}_{\mathcal{G}_{Ex}})$. By the last statement we obtain the identity $((x^{-1} \cdot e)^{-1})^{-1} \approx ((x^{-1})^{-1})^{-1}$. Therefore by the fact that $x^{-1} \approx ((x^{-1})^{-1})^{-1} \in Cn(B^{**}_{\mathcal{G}_{Ex}})$, it follows that (2.viii) belongs to $Cn(B^{**}_{\mathcal{G}_{Ex}})$. □

Analogously one can prove the next theorems:

THEOREM 2.3. *The identities: (2.i), (2.ii), (2.xiii), (2.xv) and*

(2.xvi) $x^{-1} \cdot e \approx x^{-1}$.

form an equational base of $\mathcal{G}_{P_1}$.

THEOREM 2.4. *The identities: (2.i), (2.ii), (2.xiii) and*

(2.xvii) $e \approx x^{-1} \cdot x$,

(2.xviii) $x^{-1} \approx (x \cdot e)^{-1}$

form an equational base of $\mathcal{G}_{P_2}$.

THEOREM 2.5. *The identities: (2.i), (2.ii), (2.xiii), (2.xviii) and*

(2.xix) $e \approx (x^{-1} \cdot x)^{-1}$

form an equational base of $\mathcal{G}_{P_3}$.

THEOREM 2.6. *The identities: (2.i), (2.ii), (2.xiii), (2.xvi), (2.xvii), (2.xix) form an equational base of* $\mathcal{G}_N$.

From the theorem (2.4) follows

COROLLARY 2.1. $\mathcal{G}_{P_2} = Mod(Cn(Ex(\mathcal{G}) \cup \{e \approx e \cdot e\}))$.

From theorem (2.5) we obtain

COROLLARY 2.2. $\mathcal{G}_{P_3} = Mod(Cn(Ex(\mathcal{G}) \cup \{e \approx e^{-1}\}))$.

3. The lattice of partitions and the lattice of varieties

Let Π_F be the family of all partitions of the set F. It is known fact that ordered pair $(\Pi_F; \leq)$, where $\leq$ is the relation on Π_F defined in the following way:

(3.1) *$P_1 \leq P_2$ iff for every block $A \in P_1$ there exists block $B \in P_2$, such that $A \subseteq B$,*

is a lattice. From now on, by $\mathbf{1}$ we denote some fixed object which does not belong to Π_F. Let us define the relation $\leq_1$ on the set $\Pi_F \cup \{\mathbf{1}\}$ in the following way:

(3.2) *$P_1 \leq_1 P_2$ iff $P_1 \leq P_2$ for every $P_1, P_2 \in \Pi_F$,*

(3.3) *for every $P \in \Pi_F \cup \{\mathbf{1}\}$ we have $P \leq_1 \mathbf{1}$.*

One can see that $(\Pi_F \cup \{\mathbf{1}\}; \leq_1)$ is a lattice. Let $(\Pi_F + \mathbf{1})$ denotes this lattice. Let $(\Pi_F + \mathbf{1})^d = ((\Pi_F \cup \{\mathbf{1}\})^d; \leq_1^d)$ be the dual lattice to the lattice $(\Pi_F + \mathbf{1})$. As usually, $\Pi_F \cup \{\mathbf{1}\} = (\Pi_F \cup \{\mathbf{1}\})^d$. In [4] it was proved that:

(3.4) *The lattice* $(\Pi_F + \mathbf{1})^d$ *is isomorphic to the lattice of all subvarieties of the variety determined by the theory* $Ex(\tau)$, *where the function* h *stating the isomorphism is defined as follows:* $h(\mathbf{1}) = Mod(Id(\tau))$, $h(P) = Mod(P(\tau))$, *for* $P \in \Pi_F$.

In the present section we study a connection of the lattice $(\Pi_F + \mathbf{1})^d$ with the lattice $\mathcal{L}(\mathcal{G}^n_{Ex})$, where $\mathcal{G}^n_{Ex}$ is the variety of the type τ_1. First, let us notice that $Mod(P(\tau)) = \mathcal{G}^1_P$ and $Mod(Id(\tau)) = \mathcal{G}^1$. So, from the lemma (3.4) we obtain the following result

Lemma 3.1. *The lattice of all subvarieties of the variety* $\mathcal{G}^1_{Ex}$ *has the following diagram:*

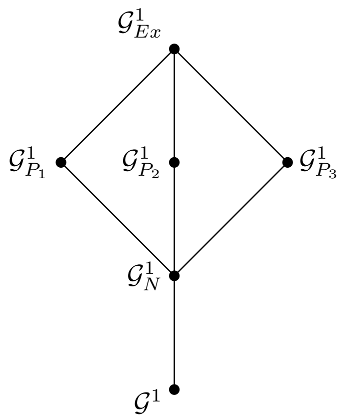

Figure 1. The lattice $\mathcal{L}(\mathcal{G}^1_{Ex})$

However there are other connections between the lattice $(\Pi_F + \mathbf{1})^d$ and the lattice $\mathcal{L}(\mathcal{G}^n_{Ex})$.

We have the following theorem:

Theorem 3.1. *The function* $h : (\Pi_F \cup \{\mathbf{1}\})^d \to L(\mathcal{G}^n_{Ex})$ *defined as follows:*

$$h(1) = \mathcal{G}^n$$

$$h(P_i) = \mathcal{G}^n_{P_i} \text{ for } i \in \{0, 1, \ldots, 4\}$$

is order-preserving and injective.

PROOF. Let $P_i, P_j \in \Pi_F$, where $i, j \in \{0, \ldots, 4\}$ and $P_i < P_j$. Then $P_i(\mathcal{G}^n) \subseteq P_j(\mathcal{G}^n)$ and $\mathcal{G}^n_{P_j} \subseteq \mathcal{G}^n_{P_i}$. Obviously $\mathbf{1} > P_i$ for $i \in \{0, \ldots, 4\}$ and $P_i(\mathcal{G}^n) \subseteq Id(\mathcal{G}^n)$. This implies $\mathcal{G}^n \subseteq \mathcal{G}^n_{P_i}$. So, h is order-preserving.

If $P_i \neq P_j$ and $i, j \in \{1, 2, 3\}$ then there exists an identity, which is P_i-compatible and it is not P_j-compatible. Thus we see that $\mathcal{G}^n_{P_i} \neq \mathcal{G}^n_{P_j}$ for $i, j \in \{1, 2, 3\}$ and $i \neq j$. The partition P_0 is different from the partitions P_i for $i \in \{1, 2, 3, 4\}$ and it is easy to see that there exist identities which are P_i-compatible and are not externally compatible for any $i \in \{1, 2, 3, 4\}$ (see theorems 2.3, 2.4, 2.5, 2.6). So the variety $\mathcal{G}^n_{Ex}$ is different from the varieties $\mathcal{G}^n_{P_i}$ for $i \in \{1, 2, 3, 4\}$. Similarly we prove that the variety $\mathcal{G}^n_N$ is different from the varieties $\mathcal{G}^n_{P_i}$ for $i \in \{1, 2, 3\}$. So we have that h is injection. $\square$

Let us observe that $P_2 \vee P_3 = P_N$, and it is obvious that $\mathcal{G}^n_N \subseteq \mathcal{G}^n_{P_2} \wedge \mathcal{G}^n_{P_3}$ (this is a consequence of the theorem 3.1). We will prove that $\mathcal{G}^n_{P_2} \wedge \mathcal{G}^n_{P_3} \neq \mathcal{G}^n_N$. We will construct an algebra which belongs to the variety $\mathcal{G}^n_{P_2} \wedge \mathcal{G}^n_{P_3}$ and does not belong to the class $\mathcal{G}^n_N$. Let us consider a dispersion $\mathcal{R}$ of the group $\mathcal{Z}_n = (\{0, 1, \ldots, n-1\}; +_n, -_n, 0)$ which is defined in the following way: $\mathcal{R} = (\{0, 1, \ldots, n-1^-, n-1^+\}; +_n, -_n, 0)$, where:

$$R_i = \{i\} \text{ } dla \text{ } i \in \{0, 1, \ldots, n-2\},$$
$$R_{n-1} = \{n-1^+, n-1^-\},$$
$$i = c_{+_n}(i) = c_{-_n}(i) = c_0(i) \text{ } dla \text{ } i \in \{0, 1, \ldots, n-2\},$$
$$n-1^- = c_{-_n}(n-1),\ n-1^+ = c_{+_n}(n-1) = c_0(n-1).$$

From (1.1) we have that all externally compatible identities satisfied in $\mathcal{Z}_n$ are satisfied in $\mathcal{R}$ and it is easy to see that $0 = 0 +_n 0$, $0 = -_n 0$ in $\mathcal{R}$. Thus the algebra $\mathcal{R}$ belongs to the class $\mathcal{G}^n_{P_2} \wedge \mathcal{G}^n_{P_3}$. Let us consider identity $e \cdot x^{-1} \approx x^{-1}$, which belongs to the set $Id(\mathcal{G}^n_N)$. This identity is not satisfied in variety $\mathcal{G}^n_N$ because $-_n 1 = n - 1^-$ whereas $0 +_n (-_n 1) = n - 1^+$.

THEOREM 3.2. *The function* $h : (\Pi_F \cup \{\mathbf{1}\})^d \to L(\mathcal{G}^n_{Ex})$ *defined as follows:*

$$h(\mathbf{1}) = \mathcal{G}^n$$
$$h(P) = Mod(Ex(\mathcal{G}^n) \cup E_P),$$

where $E_P = \{f(e, \ldots, e) \approx g(e, \ldots, e) : f, g \in \{\cdot, ^{-1}, e\}, g \in [f]_P\}$ *is a lattice embedding.*

PROOF. Let us note first that instead of considering all identities in the sets E_P it is enough to consider only some of them. Namely, it is easy to see that the following equalities are true:

$$Cn(Ex(\mathcal{G}^n) \cup E_{Ex}) = Cn(Ex(\mathcal{G}^n)),$$
$$Cn(Ex(\mathcal{G}^n) \cup E_{P_1}) = Cn(Ex(\mathcal{G}^n) \cup \{e \cdot e \approx e^{-1}\}),$$
$$Cn(Ex(\mathcal{G}^n) \cup E_{P_2}) = Cn(Ex(\mathcal{G}^n) \cup \{e \approx e \cdot e\}),$$
$$Cn(Ex(\mathcal{G}^n) \cup E_{P_3}) = Cn(Ex(\mathcal{G}^n) \cup \{e \approx e^{-1}\}),$$
$$Cn(Ex(\mathcal{G}^n) \cup E_N) = Cn(Ex(\mathcal{G}^n) \cup \{e \approx e \cdot e, e \approx e^{-1}\}).$$

As an immediate consequence of the first equality is: $Mod(Cn(Ex(\mathcal{G}^n) \cup E_{Ex})) = \mathcal{G}^n_{Ex}$. From Corollary 2.1 and 2.2 we have $Mod(Cn(Ex(\mathcal{G}^n) \cup E_{P_2})) = \mathcal{G}^n_{P_2}$ and $Mod(Cn(Ex(\mathcal{G}^n) \cup E_{P_3})) = \mathcal{G}^n_{P_3}$. Let C^n, C^n_N denote classes $Mod(Cn(Ex\mathcal{G}^n \cup E_{P_1}))$ and $Mod(Cn(Ex\mathcal{G}^n \cup E_N))$ respectively. It is clear that h is injective and order preserving. Thus it is enough to prove only that:

$$C^n \wedge \mathcal{G}^n_{P_2} = C^n_N,$$
$$C^n \wedge \mathcal{G}^n_{P_3} = C^n_N,$$
$$\mathcal{G}^n_{P_2} \wedge \mathcal{G}^n_{P_3} = C^n_N.$$

Obviously $C^n \wedge \mathcal{G}^n_{P_2} = Mod(Cn((Ex(\mathcal{G}^n) \cup \{e \cdot e \approx e^{-1}\}) \cup (Ex(\mathcal{G}^n) \cup \{e \approx e^{-1}\})))$. Since $Mod(Cn((Ex(\mathcal{G}^n) \cup \{e \cdot e \approx e^{-1}\}) \cup (Ex(\mathcal{G}^n) \cup \{e \approx e^{-1}\}))) = Mod(Cn(Ex(\mathcal{G}^n) \cup \{e \cdot e \approx e^{-1}, e \approx e^{-1}\}))$, so $C^n \wedge \mathcal{G}^n_{P_2} = C^n_N$. We act similarly in the remain cases. $\square$

References

[1] Chromik, W., "On externally compatible identities of algebras", *Demonstratio Mathematica* 23(2): 345–355, 1990.

[2] Gajewska-Kurdziel, K., "On the lattices of some warieties defined by externally comapatible identities". In: K. Denecke, (ed.), *General ALgebra and Discrete Mathematics*, Heldermann-Verlag, Berlin, (1995), pp. 107–110.

[3] Hałkowska, K., "On free algebras in varieties defined by externally compatible identities". In: K. Hałkowska and B. Stawski, (eds.), *Universal and applied algebra, Proceedings of the V Universal Algebra Symposium*, World Scientific, Singapore, 1988, pp. 143–148.

[4] Hałkowska, K., "Lattice of equational theories of P-compatible varieties". In: E. Orłowska, (ed.), *Logic at Work. Essays dedicated to the memory of Helena Rasiowa*, Heidelberg, New York, Springer, 1998, pp. 587–595.

[5] Mel'nik, I. J., "Nilpotent shifts of varieties" (in Russian), *Mat. Zametki* 14, (5) (1973). English translation: *Math. Notes* 14: 962–966, 1973.

[6] Mruczek, K., "On some lattice of varieties related to changes of the type". In: K. Denecke and H.-J. Vogel, (eds.), *General Algebra and Applications, Proceedings of the 59th Workshop on General Algebra*, Shaker Verlag, Aachen, 2000, pp. 147–153.

[7] Płonka, J., "On equational classes of abstract algebras defined by regular equations", *Fund. Math.* 64: 241–147, 1969.

[8] Płonka, J., "On the subdirect product of some equational classes of algebras", *Math. Nachr.* 63: 303–305, 1974.

[9] Płonka, J., "*P*-compatible identities and their applications to classical algebras", *Math. Slovaca* 40(1): 21–30, 1990.

[10] Płonka, J., "On varieties of algebras defined by identities of some special forms", *Houston Jurnal of Math.* 14: 253–263, 1988.

Krystyna Mruczek-Nasieniewska
Department of Logic,
Nicolaus Copernicus University,
ul. Asnyka 2b,
PL 87–100 Toruń, Poland
`mruczek@uni.torun.pl`