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Andrzej Pietruszczak

PIECES OF MEREOLOGY*

Abstract. In this paper† we will treat mereology as a theory of some structures that are not axiomatizable in an elementary language (one of the axioms will contain the predicate 'belong' ('$\in$') and we will use a variable ranging over the power set of the universe of the structure). A mereological structure is an ordered pair $\mathfrak{M} = \langle M, \sqsubseteq \rangle$, where M is a non-empty set and $\sqsubseteq$ is a binary relation in M, i.e., $\sqsubseteq$ is a subset of $M \times M$. The relation $\sqsubseteq$ is a relation of *being a mereological part* (instead of '$\langle x, y\rangle \in \sqsubseteq$' we will write '$x \sqsubseteq y$' which will be read as "x is a part of y"). We formulate an axiomatization of mereological structures, different from Tarski's axiomatization as presented in [10] (Tarski simplified Leśniewski's axiomatization from [6]; cf. Remark 3). We prove that these axiomatizations are equivalent (see Theorem 1). Of course, these axiomatizations are definitionally equivalent to the very first axiomatization of mereology from [5], where the relation of *being a proper part* $\sqsubset$ is a primitive one.

Moreover, we will show that Simons' "Classical Extensional Mereology" from [9] is essentially weaker than Leśniewski's mereology (cf. Remark 5).

Keywords: mereology, mereological structures, axioms of mereology, collective sets, mereological sets, mereological fusions, mereological parts.

1. Relation $\sqsubseteq$ is a partial order

Let M be a non-empty set. We assume (among others) that the relation $\sqsubseteq$ partially orders the set M, that is it is reflexive, transitive and antisymmetric

*This is a modified version of the Polish paper [7]. The main changes involve transforming some of the footnotes into remarks and facts.

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(A1) $$\bigwedge_{x\in M} x \sqsubseteq x\,,$$

(A2) $$\bigwedge_{x,y,z\in M} (x \sqsubseteq y \wedge y \sqsubseteq z \implies x \sqsubseteq z),$$

(A3) $$\bigwedge_{x,y\in M} (x \sqsubseteq y \wedge y \sqsubseteq x \implies x = y).$$

Structures which satisfy (A1) and (A2) are called *quasi-partial orders*. The conjunction of (A1) and (A2) is logically equivalent to the theorem

(1) $$\bigwedge_{x,y\in M} \Big(x \sqsubseteq y \iff \bigwedge_{z\in M} (z \sqsubseteq x \Rightarrow z \sqsubseteq y)\Big).$$

Moreover, the conjunction of (A1) and (A3) is logically equivalent to the thesis

(2) $$\bigwedge_{x,y\in M} (x = y \iff (x \sqsubseteq y \wedge y \sqsubseteq x)).$$

From (1) and (A3) for any $a, b \in M$ we obtain

(ext$_{\sqsubseteq}$) $$\bigwedge_{x\in M} (x \sqsubseteq a \Leftrightarrow x \sqsubseteq b) \implies a = b\,.$$

We will introduce an auxiliary relation of *having a common part* $\between \subseteq M \times M$

(def $\between$) $$a \between b \iff \bigvee_{x\in M} (x \sqsubseteq a \wedge x \sqsubseteq b).$$

Directly from the definition we have for $a, b \in M$

(3) $$a \between b \implies b \between a\,.$$

From the axiom (A1) and (def $\between$) it follows that for $a, b \in M$

(4) $$a \between a\,,$$

(5) $$a \sqsubseteq b \vee b \sqsubseteq a \implies a \between b\,.$$

And from (A2) for all $a, b, c \in M$ we have

(6) $$c \sqsubseteq a \wedge b \between c \implies b \between a\,.$$

Fact 1. (i) *By* (A2) *and* (def $\between$), *for any* $a, b \in M$, *the condition*

(a) $$\bigwedge_{x\in M} (x \sqsubseteq a \Rightarrow x \between b)$$

entails the condition

$$\text{(b)} \qquad \bigwedge_{x \in M} (x \between a \;\Rightarrow\; x \between b).$$

(ii) *By* (A1) *and* (def $\between$), *the condition* (b) *entails the condition* (a). *So in all quasi-partial orders the conditions* (a) *and* (b) *are equivalent.*

PROOF. (i) Let us take an arbitrary $x \in M$ such that $x \between a$. So there is a y such that $y \sqsubseteq x$ and $y \sqsubseteq a$. Hence, by (a), $y \between b$, i.e., for some z: $z \sqsubseteq y$ and $z \sqsubseteq b$. Hence, by (A2), $z \sqsubseteq x$. So, $x \between b$, by (def $\between$).

(ii) If $x \sqsubseteq a$ then $x \between a$, by (A1) and (def $\between$). So $x \between b$, by (b). □

2. Relation $\mathcal{S}$ of being a supremum of a distributive set

Let $\mathcal{S}$ and $\mathcal{I}$ be respectively relations of being supremum and infimum in a structure $\mathfrak{M}$, i.e. $\mathcal{S}, \mathcal{I} \subseteq M \times 2^M$ and they are defined by means of the following conditions

$$\text{(def } \mathcal{S}) \qquad a \,\mathcal{S}\, X \iff \bigwedge_{x \in X} x \sqsubseteq a \wedge \bigwedge_{y \in M} \Big(\bigwedge_{x \in X} x \sqsubseteq y \;\Rightarrow\; a \sqsubseteq y \Big),$$

$$\text{(def } \mathcal{I}) \qquad a \,\mathcal{I}\, X \iff \bigwedge_{x \in X} a \sqsubseteq x \wedge \bigwedge_{y \in M} \Big(\bigwedge_{x \in X} y \sqsubseteq x \;\Rightarrow\; y \sqsubseteq a \Big).$$

Relations $\mathcal{S}$ and $\mathcal{I}$ are mutually (logically) definable

$$\text{(7)} \qquad a \,\mathcal{I}\, X \iff a \,\mathcal{S}\, \{z : \bigwedge_{x \in X} z \sqsubseteq x\},$$

$$\text{(8)} \qquad a \,\mathcal{S}\, X \iff a \,\mathcal{I}\, \{z : \bigwedge_{x \in X} x \sqsubseteq z\}.$$

Indeed, let $a \,\mathcal{I}\, X$ and $Z := \{z : \bigwedge_{x \in X} z \sqsubseteq x\}$. Since $\bigwedge_y (\bigwedge_{x \in X} y \sqsubseteq x \Rightarrow y \sqsubseteq a)$, thus $\bigwedge_{z \in Z} z \sqsubseteq a$. Moreover let us take an arbitrary $y \in M$ such that $\bigwedge_{z \in Z} z \sqsubseteq y$. Since $\bigwedge_{x \in X} a \sqsubseteq x$, so $a \in Z$. Hence $a \sqsubseteq y$, that is $a \,\mathcal{S}\, Z$. Conversely, if $a \,\mathcal{S}\, Z$ then we have $\bigwedge_{z \in Z} z \sqsubseteq a$, i.e. $\bigwedge_{z \in M} (\bigwedge_{x \in X} z \sqsubseteq x \Rightarrow z \sqsubseteq a)$. Moreover, let as take an arbitrary $x \in X$. Since $\bigwedge_{z \in Z} z \sqsubseteq x$ and $a \,\mathcal{S}\, Z$, so $a \sqsubseteq x$. Therefore $a \,\mathcal{I}\, X$. We prove (8) in a similar way.

In structures satisfying the axioms (A1) and (A2), definitions (def $\mathcal{S}$) and (def $\mathcal{I}$) are respectively equivalent to the following conditions

$$\text{(9)} \qquad a \,\mathcal{S}\, X \iff \bigwedge_{y \in M} \Big(a \sqsubseteq y \Leftrightarrow \bigwedge_{x \in X} x \sqsubseteq y\Big),$$

$$a \,\mathcal{I}\, X \iff \bigwedge_{y\in M} \big(y \sqsubseteq a \Leftrightarrow \bigwedge_{x\in X} y \sqsubseteq x\big). \tag{10}$$

Indeed, if $a \,\mathcal{S}\, x$ and $a \sqsubseteq y$, then $\bigwedge_{x\in X} x \sqsubseteq y$, by (A2) and assumption $\bigwedge_{x\in X} x \sqsubseteq a$. Conversely, if $a \sqsubseteq y \Leftrightarrow \bigwedge_{x\in X} x \sqsubseteq y$, then $\bigwedge_{x\in X} x \sqsubseteq a$, since $a \sqsubseteq a$, by (A1). Therefore $a \,\mathcal{S}\, X$. The proof of (10) is analogous.

It follows from (A3) that relations $\mathcal{S}$ and $\mathcal{I}$ are functions of the second argument, i.e.

$$a \,\mathcal{S}\, X \,\wedge\, b \,\mathcal{S}\, X \implies a = b\,, \tag{11}$$

$$a \,\mathcal{I}\, X \,\wedge\, b \,\mathcal{I}\, X \implies a = b\,. \tag{12}$$

Indeed, for (11): $\bigwedge_{x\in X} x \sqsubseteq a$, $\bigwedge_{x\in X} x \sqsubseteq b$, $\bigwedge_z(\bigwedge_{x\in X} x \sqsubseteq z \Rightarrow a \sqsubseteq z)$ and $\bigwedge_z(\bigwedge_{x\in X} x \sqsubseteq z \Rightarrow a \sqsubseteq z)$. Hence $a \sqsubseteq b$ and $b \sqsubseteq a$. Therefore, by (A3), we have $a = b$. We prove (12) in a similar way.

3. Relation $\mathcal{F}$ of being a fusion of a distributive set

An additional notion of the theory of mereological structures is a *«fusion»* of a distributive set whose elements are elements of the universe. Let us then define the relation of *being a mereological fusion* of a distributive set $\mathcal{F} \subseteq M \times 2^M$

$$a \,\mathcal{F}\, X \iff \bigwedge_{x\in X} x \sqsubseteq a \,\wedge\, \bigwedge_{y\in M} \big(y \sqsubseteq a \Rightarrow \bigvee_{x\in X} x \between y\big). \tag{def $\mathcal{F}$}$$

In Leśniewski's mereology the notion of *being a collective class* was defined in a way that can be paraphrased by means of the following schema[1]

M is a class of **S**-es iff there exists exactly one **M**
and every **S** is a part of **M**
and every part of **M** has a common part with some **S**.

[1] Cf. Definition II given in [5] on p. 264 (English version p. 230). Leśniewski, the creator of mereology, formulated it using his own logical system which had its peculiar language. Leśniewski did not recognize the existence of distributive sets. Therefore definition couched as (def $\mathcal{F}$) would have been unacceptable for him. Since sets are foundation of mathematics, thus—according to Leśniewski—these are collective sets, since no others exist. This is probably the source of Leśniewski's work title:"On foundations of mathematics".

In the above schema letters ‘**M**’ and ‘**S**’ represent arbitrary names, while the word ‘part’ has been used in a way that allows improper parts.

Assuming that a phrase ‘object **P**’ stands for an arbitrary definite name, the schema can be simplified as follows

> an object **P** is a class of **S**-es iff every **S** is a part of an object **P** and every part of an object **P** has a common part with some **S**.

Leśniewski used to assume that if a name represented by ‘**S**’ is not empty, then exists exactly one object which is a class of **S**-es.[2]

Let $\mathfrak{M} = \langle M, \sqsubseteq \rangle$ be a mereological structure and let ‘**S**’ stand for an arbitrary non-empty name, whose range (a distributive set of its designates) is a subset of M. Then—comparing the above schemas with (def $\mathfrak{F}$)—we can say that a class of **S**-es (in Leśniewski’s sense) is equal to a fusion of a distributive set $\{x \in M : x \text{ is an } \mathbf{S}\}$. Therefore if $a \in M$ is a fusion of a distributive set $X \subseteq M$, then we say that object a is a collective set of distributive elements of the set X.

Directly from (def $\mathfrak{F}$) and (A1) we have theorems[3]

$$\neg \bigvee_{x \in M} x \,\mathfrak{F}\, \emptyset \,, \tag{13}$$

$$a \,\mathfrak{F}\, \{a\} \,, \tag{14}$$

$$a \,\mathfrak{F}\, \{x : x \sqsubseteq a\} \,. \tag{15}$$

Nevertheless, without taking additional assumptions concerning a mereological fusion, we cannot even prove that it is a function of the second argument. For example, in a partial order that is depicted in the following diagram, in which $M := \{1, 2\}$ and $\sqsubseteq\, := \{\langle 1, 1\rangle, \langle 1, 2\rangle, \langle 2, 2\rangle\}$, we have $1 \,\mathfrak{F}\, \{1\}$ and $2 \,\mathfrak{F}\, \{1\}$.

[2]In set theoretical setting to this assumption corresponds either a pair of formulae (fun $\mathfrak{F}$) and (A5), or a formula (A5&fun$\mathfrak{F}$) that is equivalent to their conjunction (cf. Remark 2 and Theorem 1).

[3]The sentence (13), saying that there is no fusion of the empty set, is some version of a statement saying that there is no empty collective set.

Remark 1. In mereology a different notion of *being a collective set* is often accepted, i.e., authors use a different definition of the relation of *being a mereological fusion*. Let $\mathfrak{F}_\star \subseteq M \times 2^M$, where

$$\text{(def } \mathfrak{F}_\star\text{)} \qquad a\ \mathfrak{F}_\star\ X \iff \bigwedge_{y \in M} \big(y \between a \Leftrightarrow \bigvee_{x \in X} x \between y\big).$$

The above definition was used by Leśniewski and, after him, by Leonard and Goodman [4].[4] Breitkopf [1], Eberle [2], and Simons [9] defined the relation of being a fusion in the same way. □

FACT 2. *From* (A2) *and* (def $\between$) *we obtain* $\mathfrak{F} \subseteq \mathfrak{F}_\star$.

PROOF. Indeed, let $a\ \mathfrak{F}\ X$. Let as take an arbitrary $y \in M$. "$\Rightarrow$" We suppose $y \between a$, i.e., for some z: $z \sqsubseteq y$ and $z \sqsubseteq a$. Since $a\ \mathfrak{F}\ X$ and $z \sqsubseteq a$, by (def $\mathfrak{F}$), there is an $x \in X$ such that $x \between z$. Hence $x \between y$, by (6). "$\Leftarrow$" We suppose that there is an $x \in X$ such that $x \between y$, i.e., for some z: $z \sqsubseteq x$ and $z \sqsubseteq y$. Since $a\ \mathfrak{F}\ X$ and $x \in X$, by (def $\mathfrak{F}$), we have $x \sqsubseteq a$. Hence, by (A2) we obtain $z \sqsubseteq a$. So $y \between a$. □

In Remark 5 we will construct a partial order in which $\mathfrak{F} \neq \mathfrak{F}_\star$. In all mereological structures we have $\mathfrak{F} = \mathfrak{F}_\star$ (see Corollary 1, p. 219).

4. Partial order $\sqsubseteq$ is separative

As we shall prove further in the text (cf. thesis (fun $\mathfrak{F}$)) the relation of *being a fusion* will be the function of the second argument in case if we assume that mereological structures contain only those partial orders that are *separative*, i.e., in case if the following condition is satisfied

$$\text{(A4)} \qquad a \not\sqsubseteq b \implies \bigvee_{x \in M} (x \sqsubseteq a \wedge \neg\, x \between b).$$

Structures which satisfy (A1), (A2) and (A4) we may call *separative quasi-partial orders*. In all separative quasi-partial orders the axiom (A1) is dependent from the other ones.

[4] Actually these authors define the relation $\mathfrak{F}_\star$ by the following condition

$$a\ \mathfrak{F}_\star\ X \iff \bigwedge_{y \in M} \big(\neg\, y \between a \Leftrightarrow \bigwedge_{x \in X} \neg\, x \between y\big).$$

Of course, the above formula is logically equivalent to (def $\mathfrak{F}_\star$).

FACT 3 (cf. [8, Lemma IV.5.4]). *Let M be an arbitrary non-empty set and let $\sqsubseteq$ and $\between$ be binary relations in M which satisfy conditions* (A2), (A4) *and* (def $\between$). *Then the relation $\sqsubseteq$ is reflexive, i.e., it satisfies condition* (A1).

PROOF. Assume towards contradiction that for some a we have $a \not\sqsubseteq a$. Then, by (A4), for some x we have $x \sqsubseteq a$ and $\neg\, x \between a$. Hence, by (def $\between$), $\bigwedge_{y\in M}(y \sqsubseteq x \;\Rightarrow\; y \not\sqsubseteq a)$. Hence, by (A2), we have $\neg \bigvee_{z\in M} z \sqsubseteq x$. Hence, $x \not\sqsubseteq x$ and we obtain $\bigvee_{z\in M} z \sqsubseteq x$, by (A4). $\square$

It is easily observable that from (A1), (A2) and (A4) follows

(16) $$a \sqsubseteq b \iff \bigwedge_{x\in M} (x \sqsubseteq a \;\Rightarrow\; x \between b).$$

Moreover, by Fact 1, from (16) we obtain

(def$_\between$ $\sqsubseteq$) $$a \sqsubseteq b \iff \bigwedge_{x\in M} (x \between a \;\Rightarrow\; x \between b).$$

So from (A3) and (def$_\between$ $\sqsubseteq$) it follows that

(ext$_\between$) $$\bigwedge_{x\in M} (x \between a \Leftrightarrow x \between b) \implies a = b\,.$$

We will show that from (def$_\between$ $\sqsubseteq$) and (def $\between$) follows the following thesis, whose «strenght» will be unveiled in lemmas 1–3.

(17) $$a \between b \iff \bigvee_{y\in M} \bigwedge_{x\in M} \big(x \between y \;\Rightarrow\; (x \between a \wedge x \between b)\big).$$

Indeed, let $a \between b$. Then there is y such that $y \sqsubseteq a$ and $y \sqsubseteq b$. From this and from (def$_\between$ $\sqsubseteq$) we have $\bigwedge_x(x \between y \Rightarrow x \between a)$ and $\bigwedge_x(x \between y \Rightarrow x \between b)$. Thus $\bigwedge_x(x \between y \Rightarrow (x \between a \wedge x \between b))$. Conversely, let $\bigvee_y \bigwedge_x(x \between y \;\Rightarrow\; (x \between a \wedge x \between b))$, thus after making some logical transformations: $\bigvee_y(\bigwedge_x(x \between y \Rightarrow x \between a) \wedge \bigwedge_x(x \between y \Rightarrow x \between b))$. Hence by (def$_\between$ $\sqsubseteq$), we have $\bigvee_y(y \sqsubseteq a \wedge y \sqsubseteq b)$, that is $a \between b$, by (def $\between$).

LEMMA 1 (cf. [7]). *Let M be an arbitrary non-empty set and let $\sqsubseteq$ and $\between$ be binary relations in M. Then for the relations $\sqsubseteq$ and $\between$ the following sets of conditions are equivalent*

(i) (A2), (A4) *and* (def $\between$)*;*

(ii) (def$_\between$ $\sqsubseteq$) *and* (def $\between$)*;*

(iii) (def$_\between$ $\sqsubseteq$) *and* (17).

PROOF. We have already shown "(i) $\Rightarrow$ (ii)" and "(i) $\Rightarrow$ (iii)".

"(ii) $\Rightarrow$ (i)" It is clear that the conditions (A1) and (A2) are logical consequences of the condition (def$_{\between}$ $\sqsubseteq$). Therefore we have to prove the condition (A4). To do this we will show that $\bigwedge_x (x \sqsubseteq a \Rightarrow x \between b)$ entails $\bigwedge_x (x \between a \Rightarrow x \between b)$, which entails $a \sqsubseteq b$, by (def$_{\between}$ $\sqsubseteq$). Therefore, by contraposition, we obtain (A4).

Let us assume that $\bigwedge_x (x \sqsubseteq a \Rightarrow x \between b)$ and $c \between a$. Then by (def $\between$), there is some y such that $y \sqsubseteq c$ and $y \sqsubseteq a$. From this and from the first assumption $y \between b$. Therefore—by (6) which we derive from (A2)—we have $c \between b$.

"(iii) $\Rightarrow$ (ii)" From (17) and (def$_{\between}$ $\sqsubseteq$) we will infer (def $\between$). Let $a \between b$. Then, by (17), $\bigvee_y \bigwedge_x (x \between c \Rightarrow (x \between a \wedge x \between b))$. After logical transformations we obtain: $\bigvee_y (\bigwedge_x (x \between y \Rightarrow x \between a) \wedge \bigwedge_x (x \between y \Rightarrow x \between b))$. Hence, by (def$_{\between}$ $\sqsubseteq$), we have $\bigvee_y (y \sqsubseteq a \wedge y \sqsubseteq b)$. Conversely, let $\bigvee_y (y \sqsubseteq a \wedge y \sqsubseteq b)$. By (def$_{\between}$ $\sqsubseteq$) we obtain $\bigvee_y (\bigwedge_x (x \between y \Rightarrow x \between a) \wedge \bigwedge_x (x \between y \Rightarrow x \between b))$. From this, after logical transformations, we have $\bigvee_y \bigwedge_x (x \between c \Rightarrow (x \between a \wedge x \between b))$. Therefore, by (17), we have $a \between b$. $\square$

LEMMA 2 (cf. [7], [8, Theorem IV.3.1]). *Let M be an arbitrary non-empty set, and $\sqsubseteq$ and $\between$ be binary relations in M which satisfy condition* (def $\between$). *Moreover, let $\mathfrak{F}$ and $\mathfrak{F}_\star$ be relations in $M \times 2^M$ which satisfy conditions* (def $\mathfrak{F}$) *and* (def $\mathfrak{F}_\star$), *respectively. Then:*

(i) *If $\sqsubseteq$ and $\between$ satisfy* (A2) *and* (A4), *then $\mathfrak{F}_\star \subseteq \mathfrak{F}$.*

(ii) *Let $\sqsubseteq$ and $\between$ satisfy* (A1), (A2) *and for all $a, b \in M$: if $b \,\mathfrak{F}_\star\, \{a, b\}$ then $b \,\mathfrak{F}\, \{a, b\}$, i.e. $a \sqsubseteq b$. Then* (A4) *and $\mathfrak{F}_\star \subseteq \mathfrak{F}$ hold.*

(iii) *If $\sqsubseteq$ and $\between$ satisfy* (A1), (A2) *and $\mathfrak{F}_\star \subseteq \mathfrak{F}$, then* (A4) *holds.*

PROOF. Clearly the relation $\between$ satisfying (def $\between$) is symmetrical.

"(i)" If $\sqsubseteq$ and $\between$ satisfy (A2), (A4) and (def $\between$), then, by Fact 3 and Lemma 1, $\sqsubseteq$ and $\between$ also satisfy the condition (A1) and (def$_{\between}$ $\sqsubseteq$). Let $a \,\mathfrak{F}_\star\, X$, i.e., $\bigwedge_{y \in M} (y \between a \Leftrightarrow \bigvee_{x \in X} x \between y)$. The condition $\bigwedge_{y \in M} (\bigvee_{x \in X} x \between y \Rightarrow y \between a)$ entails $\bigwedge_{x \in X} \bigwedge_{y \in M} (y \between x \Rightarrow y \between a)$. Hence $\bigwedge_{x \in X} x \sqsubseteq a$, by (def$_{\between}$ $\sqsubseteq$). Moreover, the condition $\bigwedge_{y \in M} (y \between a \Rightarrow \bigvee_{x \in X} x \between y)$ entails $\bigwedge_{y \in M} (y \sqsubseteq a \Rightarrow \bigvee_{x \in X} x \between y)$, by (A1) and (def $\between$). So, $a \,\mathfrak{F}\, X$.

"(ii)" We prove (A4) by contraposition. Suppose that $\bigwedge_{x \in M} (x \sqsubseteq a \Rightarrow x \between b)$. Hence, by Fact 1(i), we obtain $\bigwedge_{x \in M} (x \between a \Rightarrow x \between b)$. Therefore, $\bigwedge_{x \in M} (x \between b \Leftrightarrow x \between a \vee x \between b)$; and so $b \,\mathfrak{F}_\star\, \{a, b\}$, by (def $\mathfrak{F}_\star$). So $b \,\mathfrak{F}\, \{a, b\}$. Hence $a \sqsubseteq b$, by (def $\mathfrak{F}$). Furthermore, having (A4) we use (i).

"(iii)" Directly by (ii). $\square$

COROLLARY 1. *If a given quasi-partial order $\sqsubseteq$ is separative then $\mathfrak{F} = \mathfrak{F}_\star$.*

PROOF. In virtue of Fact 2, from (A2) and (def $\between$) we obtain $\mathfrak{F} \subseteq \mathfrak{F}_\star$. Moreover, by Lemma 2(i), we obtain $\mathfrak{F}_\star \subseteq \mathfrak{F}$. □

LEMMA 3 (cf. [7]). *Let M be an arbitrary non-empty set and let $\sqsubseteq$ and $\between$ be binary relations in M. Then the following sets of conditions are equivalent*

(i) (A2), (A3) (A4) *and* (def $\between$)*;*

(ii) (def$_\between$ $\sqsubseteq$), (ext$_\between$) *and* (17).

PROOF. We have already shown "(i) ⇒ (ii)".

"(ii) ⇒ (i)" By Lemma 1 it is enough to notice that, in a clear way, the condition (A3) follows from (def$_\between$ $\sqsubseteq$) and (ext$_\between$). □

Let us define one more auxiliary relation $\sqsubset$ of *being a proper part*. For any $a, b \in M$

(def $\sqsubset$) $$a \sqsubset b \iff a \sqsubseteq b \,\wedge\, a \neq b\,.$$

By (A3) definition (def $\sqsubset$) is equivalent to the condition

(def′ $\sqsubset$) $$a \sqsubset b \iff a \sqsubseteq b \,\wedge\, b \not\sqsubseteq a\,.$$

The reflexivity of identity relation gives us the irreflexivity of $\sqsubset$, i.e. for any $a \in M$

(irr$_\sqsubset$) $$a \not\sqsubset a\,.$$

It follows from (A3) that the relation $\sqsubset$ is asymmetric, i.e. for any $a, b \in M$

(as$_\sqsubset$) $$a \sqsubset b \implies b \not\sqsubset a\,.$$

From (A2) and (A3) it follows that relation $\sqsubset$ is transitive, i.e. for any $a, b, c \in M$

(t$_\sqsubset$) $$a \sqsubset b \,\wedge\, b \sqsubset c \implies a \sqsubset c\,.$$

Finally, from (A1) and (A3) it follows that for $a, b, \in M$

(def$_\sqsubset$ $\sqsubseteq$) $$a \sqsubseteq b \iff a \sqsubset b \,\vee\, a = b\,.$$

The following fact is well known.

LEMMA 4. *Let M be an arbitrary non-empty set and let $\sqsubseteq$ and $\sqsubset$ be binary relations in M. Then for relations $\sqsubseteq$ and $\sqsubset$ the following sets of conditions are equivalent:*

(i) (A1)–(A3) *and* (def $\sqsubset$)*;*

(ii) (A1)–(A3) *and* (def′ $\sqsubset$)*;*

(iii) ($\mathrm{irr}_\sqsubset$), ($\mathrm{t}_\sqsubset$) *and* ($\mathrm{def}_\sqsubset \sqsubseteq$);
(iv) ($\mathrm{as}_\sqsubset$), ($\mathrm{t}_\sqsubset$) *and* ($\mathrm{def}_\sqsubset \sqsubseteq$).

PROOF. We have already shown that "(i) ⇔ (ii)" and "(i) ⇒ (iii) & (iv)".

"(iii) ⇔ (iv)" From ($\mathrm{irr}_\sqsubset$) and ($\mathrm{t}_\sqsubset$) we have ($\mathrm{as}_\sqsubset$), moreover from ($\mathrm{as}_\sqsubset$) we obtain ($\mathrm{irr}_\sqsubset$).

"(iv) ⇒ (i)" (A1), (A2) and (A3) are obtained from ($\mathrm{def}_\sqsubset \sqsubseteq$), ($\mathrm{t}_\sqsubset$) and ($\mathrm{as}_\sqsubset$). Moreover, from ($\mathrm{as}_\sqsubset$) and ($\mathrm{def}_\sqsubset \sqsubseteq$) we have (def $\sqsubset$). □

Using (A1), (A3) and (A4) we obtain the following law of separation[5]

$$\text{(WSP)} \qquad a \sqsubset b \implies \bigvee_{x \in M} (x \sqsubset b \wedge \neg\, x \between a).$$

Indeed, let $a \sqsubset b$, i.e., $a \sqsubseteq b$ and $b \neq a$. Then by (A3), we have (a) $b \not\sqsubseteq a$, while by (A1) and (def $\between$), we have (b) $b \between a$. From (a), by (A4), there is an x such that (c) $x \sqsubseteq b$ and (d) $\neg\, x \between a$. From (b) and (d) we obtain $x \neq b$. From this and from (c) we have $x \sqsubset b$.

5. Fusion relation in separative partial orders

We will prove that from (A2) and (A4) follows the auxiliary thesis

$$(18) \qquad \Big(\bigwedge_{z \in M} (z \sqsubseteq a \Rightarrow \bigvee_{x \in X} x \between z) \wedge X \subseteq Y \wedge \bigwedge_{y \in Y} y \sqsubseteq b \Big) \implies a \sqsubseteq b\,.$$

Indeed, from the first two assumptions we have: $\bigwedge_{z\in M}(z \sqsubseteq a \Rightarrow \bigvee_{y \in Y} y \between z)$. So, by the last assumption, we have that $\bigwedge_z (z \sqsubseteq a \Rightarrow \bigvee_y (y \sqsubseteq b \wedge y \between z))$. Now from (6) we obtain: $\bigwedge_z (z \sqsubseteq a \Rightarrow z \between b)$. Thus, by (A4), we have $a \sqsubseteq b$.

Directly from (18) and (A3) it follows that relation $\mathcal{F}$ is a function of the second argument, i.e., for any $a, b \in M$ and $X \subseteq M$

$$(\text{fun } \mathcal{F}) \qquad a \,\mathcal{F}\, X \wedge b \,\mathcal{F}\, X \implies a = b\,.$$

Indeed, let $a \,\mathcal{F}\, X$ and $b \,\mathcal{F}\, X$. Since we have $\bigwedge_y (y \sqsubseteq a \Rightarrow \bigvee_{x\in X} x \between y)$ and $\bigwedge_{x \in X} x \sqsubseteq b$, so $a \sqsubseteq b$, by (18). Similarly, since $\bigwedge_y (y \sqsubseteq b \Rightarrow \bigvee_{x \in X} x \between y)$ and $\bigwedge_{x\in X} x \sqsubseteq a$, thus $b \sqsubseteq a$. Therefore $a = b$, by (A3).

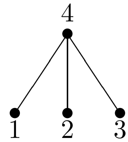
Moreover from (18) it follows that

$$(19) \qquad \mathcal{F} \subseteq \mathcal{S}\,.$$

[5] This formula was called by Simons in [9] "Weak Supplementation Principle", and formula (A4) "Strong Supplementation Principle".

Indeed, let $a \mathcal{F} X$. Then $\bigwedge_{x \in X} x \sqsubseteq a$. Let us take an arbitrary b such that $\bigwedge_{x \in X} x \sqsubseteq b$. From (def $\mathcal{F}$) and (18) we obtain $a \sqsubseteq b$. Therefore $a \mathcal{S} X$.

The inclusion that is inverse to (19) is not deducible from (A1)–(A4). For example in a partial order depicted below in which $M = \{1, 2, 3, 4\}$ and $\sqsubseteq := \{\langle x, x\rangle : x \in M\} \cup \{\langle 1, 4\rangle, \langle 2, 4\rangle, \langle 3, 4\rangle\}$, set $\{1, 2\}$ has the upper bound equal 4 while it has no mereological fusion.



Yet in case if there exist both upper bound and mereological fusion then they are equal to each other, i.e., for any $a, b \in M$

$$a \mathcal{F} X \wedge b \mathcal{S} X \implies a = b\,. \tag{20}$$

Indeed by (19), if $a \mathcal{F} X$, then $a \mathcal{S} X$. Therefore $a = b$, by (11).

Let us notice that from (A1), (A3) and (A4) it follows that in non-degenerate structures (i.e., such that their domains contain at least two elements) there is no smallest element

$$\bigvee_{x,y \in M} x \neq y \implies \neg \bigvee_{x \in M} \bigwedge_{y \in M} x \sqsubseteq y\,. \tag{21}$$

Indeed, let us assume that the domain has at least two elements and assume for contradiction that there is such b that (†) $\bigwedge_{y \in M} b \sqsubseteq y$. From the assumptions and from (A3) we infer that there is such a that $a \not\sqsubseteq b$. Hence, by (A4), for some c it is the case that $c \sqsubseteq a$ and $\neg\, c \between b$. By (†) we have $b \sqsubseteq c$. Hence, we get contradiction: $c \between b$, by (A1) and (def $\between$).

From (21) we obtain the following conclusion

$$\mathrm{Card}(M) > 1 \implies \neg \bigvee_{x} x \mathcal{S} \emptyset\,. \tag{22}$$

Indeed, assume for contradiction, that there is x such that $x \mathcal{S} \emptyset$. Then by definition of the relation $\mathcal{S}$ we have $\bigwedge_{y \in M} x \sqsubseteq y$. And this contradicts (21).

6. The axiom of fusion existence

A pair $\mathfrak{M} = \langle M, \sqsubseteq \rangle$ is called *mereological structure* if and only if it satisfies axioms (A2)–(A4) (by Fact 3 we obtain (A1)) and the following condition

$$\bigwedge_{\emptyset \neq X \subseteq M} \bigvee_{z \in M} z \mathcal{F} X\,, \tag{A5}$$

i.e., for every non-empty distributive subset of whose elements are among the elements of the domain there exists its mereological fusion.

The formulae (A5) and (fun $\mathfrak{F}$) entail the following sentence

$$\text{(A5\&fun}\mathfrak{F}\text{)} \qquad \bigwedge_{\emptyset \neq X \subseteq M} \bigvee_{x \in M} \Big(x \,\mathfrak{F}\, X \wedge \bigwedge_{y \in M} (y \,\mathfrak{F}\, X \Rightarrow x = y) \Big),$$

i.e., every non-empty subset of the domain has exactly one fusion.

Remark 2. It is evident that (A5) and (fun $\mathfrak{F}$) follow from (A5&fun$\mathfrak{F}$), (A1), and (def $\mathfrak{F}$), by (13). □

Remark 3. Let us mention here that taking the relation $\sqsubseteq$ as a primitive one Leśniewski accepted the following four axioms: (A2), (A3), (fun $\mathfrak{F}$), (A5), and (def $\mathfrak{F}$) (cf. axioms (a)–(d) and Definition (f) in [6, p. 82; English version p. 321]).[6] Leśniewski proved that from these axioms follows the formula (A1) (cf. [6], Theorem m, p. 85; English version p. 324). Later Tarski noticed that the axiom (A3) is redundant (cf. [10], second footnote). Further, in Theorem 1, we will prove that the set of the three axioms (A2), (fun $\mathfrak{F}$) and (A5) is equivalent to (A2)–(A5). □

In mereological structures for any $a \in M$ and $X \subseteq M$ it is the case that

$$\text{(23)} \qquad \begin{aligned} a \,\mathfrak{F}\, X \iff X \neq \emptyset \wedge \bigwedge_{x \in X} x \sqsubseteq a \wedge \\ \bigwedge_{y \in M} \Big(\bigwedge_{x \in X} x \sqsubseteq y \wedge y \sqsubseteq a \Rightarrow y = a \Big). \end{aligned}$$

Indeed, if $a \,\mathfrak{F}\, X$, then $X \neq \emptyset$ and $a \,\mathcal{S}\, X$, respectively by (13) and (19). It follows from the second fact that if $\bigwedge_{x \in X} x \sqsubseteq y$ then $a \sqsubseteq y$. Therefore if $y \sqsubseteq a$, then $y = a$ by (A3). Conversely, let us assume the right side of the equivalence being proved. Since $X \neq \emptyset$, then by (A5) there is b such that $b \,\mathfrak{F}\, X$, i.e. $\bigwedge_{x \in X} x \sqsubseteq b$ and $\bigwedge_y (y \sqsubseteq b \Rightarrow \bigvee_{x \in X} x \between y)$. Now in (18) we put $a := b$, $b := a$ and $Y := X$. Since the antecedent in (18) is satisfied, then $b \sqsubseteq a$. By the assumption we obtain that $a = b$, that is $a \,\mathfrak{F}\, X$.[7]

Let us notice that in mereological structures *being fusion* of elements of a given non-empty set coincides with *being supremum* of elements of this set, i.e. for any $a \in M$ and $X \subseteq M$

$$\text{(24)} \qquad a \,\mathfrak{F}\, X \iff X \neq \emptyset \wedge a \,\mathcal{S}\, X\,.$$

[6] We of course pass over the difference between set theoretical language and language of Leśniewski's system.

[7] In [6, p. 87, thesis (a); English version p. 327] Leśniewski points out that K. Kuratowski was first to prove (23), in a peculiar language of the original system (cf. footnote 1).

Indeed, if $a \mathcal{F} X$, then $X \neq \emptyset$ and $a \mathcal{S} X$, by (13) and (19). Conversely, assume that $X \neq \emptyset$ and $a \mathcal{S} X$. Then, by (A5), there is b such that $b \mathcal{F} X$. Therefore $a = b$, by (20). So $a \mathcal{F} X$.[8]

Using definite description, thesis (24) can be formulated as follows

$$\bigwedge_{X\neq\emptyset} (\imath x)\, x \mathcal{F} X = (\imath x)\, x \mathcal{S} X\,. \tag{25}$$

Indeed, for every $X \neq \emptyset$, by (A5&fun$\mathcal{F}$), there exists exactly one x such that $x \mathcal{F} X$. Now, by (11) and (24), x is the only element such that $x \mathcal{S} X$.[9]

Moreover, in non-degenerate mereological structures $\mathcal{F} = \mathcal{S}$

$$\mathrm{Card}(M) > 1 \implies \mathcal{F} = \mathcal{S}\,. \tag{26}$$

Indeed, by (19), $\mathcal{F} \subseteq \mathcal{S}$. Conversely, assume that $a \mathcal{S} X$. Then $X \neq \emptyset$, by (22). Hence $a \mathcal{F} X$, by (24). Therefore it is also the case that $\mathcal{S} \subseteq \mathcal{F}$.

In a mereological structure $\langle M, \sqsubseteq\rangle$ we can distinguish greatest element 1 with respect to the relation $\sqsubseteq$

$$1 := (\imath x)\, x \mathcal{F} M\,. \tag{def 1}$$

Therefore, by (21), we arrive at the following conclusion

$$\mathrm{Card}(M) \neq 2\,. \tag{27}$$

7. Other axiomatizations

7.1. Axiomatizations with primitive $\sqsubseteq$

As we already mentioned it in Remark 3, Leśniewski showed that the formula (A1) is a consequence of the axioms (A2), (A3), (fun $\mathcal{F}$) and (A5). Moreover, Tarski noticed that (A3) is redundant in this set. The conditions (A2) and (A5&fun$\mathcal{F}$) together form the axiomatization of mereological structures adopted by Tarski in [10].

[8] Fact (24), written in specific language of original Leśniewski's system, was proved for the first time by Tarski in two formulations: when relation $\mathcal{S}$ is defined by means of (def $\mathcal{S}$) or condition (9) (cf. [6], p. 87, theses (b) and (c) respectively; English version p. 327).

[9] Of course thesis (25) does not say that in axiom (A5) instead of the relation $\mathcal{F}$ one can take relation $\mathcal{S}$, i.e., in place of (A5) accept as the axiom ($*$): $\bigwedge_{\emptyset\neq X\subseteq M} \bigvee_{x\in M} x \mathcal{S} X$. The set of conditions (A1)–(A4) and ($*$) is essentially weaker than set (A1)–(A5). For example, the structure on p. 221 satisfies the first set but not the condition (A5).

FACT 4 (cf. [7, Theorem 1]). *For any non-empty set M, any relation $\sqsubseteq$ in $M \times M$ and two relations $\between$ and $\mathfrak{F}$ defined respectively by* (def $\between$) *and* (def $\mathfrak{F}$)*:*

(i) *If $\sqsubseteq$ and $\mathfrak{F}$ satisfy conditions* (A2) *and* (A5&fun$\mathfrak{F}$)*, then $\sqsubseteq$ satisfies conditions* (A1) *and* (A3)*.*

(ii) *If $\sqsubseteq$ and $\mathfrak{F}$ satisfy conditions* (A2), (fun $\mathfrak{F}$) *and* (A5)*, then $\sqsubseteq$ satisfies conditions* (A1) *and* (A3)*.*

PROOF. (i) Notice that (A5) follows from (A5&fun$\mathfrak{F}$).

(A1): Let us take an arbitrary $a \in M$. Since $\{a\} \neq \emptyset$, then by (A5) there exists b such that $b \,\mathfrak{F}\, \{a\}$. From (def $\mathfrak{F}$) we obtain that 1° $a \sqsubseteq b$ and 2° $\bigwedge_x(x \sqsubseteq b \Rightarrow x \between a)$. Hence $a \between a$. Applying (def $\between$) we get $\bigvee_x x \sqsubseteq a$, that is $A := \{x \in M : x \sqsubseteq a\} \neq \emptyset$. Therefore by (A5) there exists c such that $c \,\mathfrak{F}\, A$. Now from (def $\mathfrak{F}$) we have $\bigwedge_x(x \sqsubseteq a \Rightarrow x \sqsubseteq c)$ and $\bigwedge_x(x \sqsubseteq c \Rightarrow \bigvee_y(y \sqsubseteq a \wedge y \between x))$. Hence 3° $\bigwedge_x(x \sqsubseteq a \Rightarrow \bigvee_y(y \sqsubseteq a \wedge y \between x))$. It is enough, using (def $\mathfrak{F}$), to conclude that $a \,\mathfrak{F}\, A$.

Since $A \neq \emptyset$, so by (A5&fun$\mathfrak{F}$) we have that for any z in M, if $z \,\mathfrak{F}\, A$ then $a = z$. We will show that also $b \,\mathfrak{F}\, A$. From this we will get $a = b$ which, by 1°, will give us $a \sqsubseteq a$.

Indeed, firstly, by 1° and (A2), it is the case that $\bigwedge_x(x \sqsubseteq a \Rightarrow x \sqsubseteq b)$. Secondly, by 2° and (def $\between$), we have $\bigwedge_x(x \sqsubseteq b \Rightarrow \bigvee_z(z \sqsubseteq a \wedge z \sqsubseteq x))$. From this and from 3° we obtain $\bigwedge_x(x \sqsubseteq b \Rightarrow \bigvee_{y,z}(y \sqsubseteq a \wedge y \between z \wedge z \sqsubseteq x))$. Therefore $\bigwedge_x(x \sqsubseteq b \Rightarrow \bigvee_{y,z,u}(y \sqsubseteq a \wedge u \sqsubseteq y \wedge u \sqsubseteq z \wedge z \sqsubseteq x))$. Hence, by (A2) and (def $\between$), we have $\bigwedge_x(x \sqsubseteq b \Rightarrow \bigvee_y(y \sqsubseteq a \wedge y \between x))$. From both above facts and from (def $\mathfrak{F}$) it follows that $b \,\mathfrak{F}\, A$.

(A3): Let $a \sqsubseteq b$ and $b \sqsubseteq a$. Then firstly, we have 1° $\bigwedge_x(x \sqsubseteq a \Rightarrow x \sqsubseteq b)$, by (A2). Secondly, from (A1) follows (15), i.e. $a \,\mathfrak{F}\, \{x \in M : x \sqsubseteq a\}$, that is $\bigwedge_{x \in M}(x \sqsubseteq a \Rightarrow \bigvee_{y \in M}(y \sqsubseteq a \wedge y \between x))$. From this and from (A2) and $b \sqsubseteq a$ we get $\bigwedge_x(x \sqsubseteq b \Rightarrow \bigvee_y(y \sqsubseteq a \wedge y \between x))$. From this and from 1° we have $b \,\mathfrak{F}\, \{x \in M : x \sqsubseteq a\}$. Moreover, by (A1), we have $\{x \in M : x \sqsubseteq a\} \neq \emptyset$. Therefore $a = b$, by (A5&fun$\mathfrak{F}$).

(ii) It follows from (i), because (A5) and (fun $\mathfrak{F}$) entail (A5&fun$\mathfrak{F}$). □

We will prove that the system (A2)–(A5) is equivalent to the system adopted by Tarski in [10].

THEOREM 1 ([7]). *For every non-empty set M, every relation $\sqsubseteq$ in $M \times M$ and two relations $\between$ and $\mathfrak{F}$ defined respectively by* (def $\between$) *and* (def $\mathfrak{F}$) *the three following sets of conditions are equivalent:*

(i) (A2)–(A5),

(ii) (A2), (fun $\mathcal{F}$) *and* (A5),

(iii) (A2) *and* (A5&fun$\mathcal{F}$).[10]

PROOF. "(i) $\Rightarrow$ (ii)" On p. 220 we proved that (fun $\mathcal{F}$) follows from (A2), (A3), and (A4).

"(ii) $\Rightarrow$ (iii)" It is evident that (A5&fun$\mathcal{F}$) follows from (A5) and (fun $\mathcal{F}$).

"(iii) $\Rightarrow$ (i)" (A5) follows from (A5&fun$\mathcal{F}$). Moreover, by Fact 4, we obtain (A1) and (A3).

(A4): Assume towards contradiction that for some a and b from M it is the case that 1° $a \not\sqsubseteq b$ and 2° $\bigwedge_z(z \sqsubseteq a \Rightarrow z \between b)$.

By (A5) there exists c such that $c\,\mathcal{F}\,\{x : x \sqsubseteq b\} \cup \{a\}$, i.e. we have 3° $\bigwedge_x(x \sqsubseteq b \Rightarrow x \sqsubseteq c)$, 4° $a \sqsubseteq c$ and 5° $\bigwedge_x(x \sqsubseteq c \Rightarrow \bigvee_y((y \sqsubseteq b \vee y = a) \wedge y \between x)$. From 1° and 4° it follows that $b \neq c$. Further we will show that $c\,\mathcal{F}\,\{x : x \sqsubseteq b\}$. And this gives a contradiction since by (A1) we have $\{x \in M : x \sqsubseteq b\} \neq \emptyset$ and $b\,\mathcal{F}\,\{x \in M : x \sqsubseteq b\}$, that is $b = c$, by (A5&fun$\mathcal{F}$).

Let us take an arbitrary x such that $x \sqsubseteq c$. By 5° there exists y such that (a) $y \between x$ and either (b′) $y \sqsubseteq b$ or (b″) $y = a$. In case (b″), from (a) we get that $a \between x$. Hence there exists z such that (c) $z \sqsubseteq a$ and (d) $z \sqsubseteq x$. From (c) and 2° we have $z \between b$. Hence there is u such that (e) $u \sqsubseteq z$ and (f) $u \sqsubseteq b$. From (e), (d) and (A2) we have $u \sqsubseteq x$ which together with (A1) gives $u \between x$. From this and from (f), (a) and (b′) it follows that $\bigvee_v(v \sqsubseteq b \wedge v \between x)$. Therefore, taking 3° into account, we have $c\,\mathcal{F}\,\{x : x \sqsubseteq b\}$. □

Remark 4. In the set (A1)–(A3), (fun $\mathcal{F}$) and (A5) two axioms for partial orders, (A1) and (A3), follow from (A2), (fun $\mathcal{F}$) and (A5), that is, in other words, from (A2) and (A5&fun$\mathcal{F}$). Moreover, (A1) follows from (A2)–(A4). In the set (A2)–(A5) the axiom (A3) cannot be omitted. □

By Corollary 1, in separative partial orders the axiom (A5) is equivalent to the following formula

$$\text{(A5}_\star\text{)} \qquad \bigwedge_{\emptyset \neq X \subseteq M} \bigvee_{z \in M} z\,\mathcal{F}_\star\,X\,,$$

since in these structures $\mathcal{F} = \mathcal{F}_\star$.

[10] If the relation $\mathcal{F}$ were replaced by $\mathcal{F}_\star$, then the theorem would not be true (cf. [8]). Moreover, referring to the footnote 9, let us notice that the counterpart of this theorem for the relation $\mathcal{S}$ does not hold as well. Indeed, replacing in formula (A5&fun$\mathcal{F}$) the relation $\mathcal{F}$ by the relation $\mathcal{S}$ we obtain formula (∗∗): $\bigwedge_{\emptyset\neq X\subseteq M}\bigvee_{x\in M}(x\,\mathcal{S}\,X \wedge \bigwedge_{y\in M}(y\,\mathcal{S}\,X \Rightarrow x = y))$. It clearly follows from (11) that in partial orders conditions (∗) and (∗∗) are equivalent.

The above fact, facts 3 and 4, and lemmas 1–4 show different possible axiomatizations of the theory of mereological structures.

Of course, the set of formulas (A2), (A3) (A4) and (A5$_\star$) is an axiomatization of mereological structures. Indeed, by Fact 2 and Lemma 2, we have $\mathfrak{F} = \mathfrak{F}_\star$. Hence, (A5$_\star$) entails (A5).

In [8, Theorem IV.5.1] it was proved that in axiomatization of mereological structures instead of "Strong Supplementation Principle" (A4) one can accept "Weak Supplementation Principle" (WSP). Then the axiom (A3) can be omitted. So, the set of formulas (A2), (WSP) and (A5) is an axiomatization of mereological structures.

Notice that the system (A1), (A2), (A3), (WSP) and (A5$_\star$) is essentially weaker than the system (A2), (WSP) and (A5) (resp. (A2)–(A5)). Simply the set (A2)–(A4) is essentially stronger than the set (A1)–(A3) and (WSP), since the last one does not guarantee that the Lemma 2 holds. In partial orders it can only be proved that $\mathfrak{F} \subseteq \mathfrak{F}_\star$. The equality of these relations in partial orders is actually equivalent to the formula (A4) (cf. [8], Theorem IV.3.1, Corollary IV.3.1 and Fact IV.3.4).

Taking as a primitive one the relation of *being a part* $\sqsubseteq$ and defining relation *having a common part* $\between$ by means of (def $\between$), one can use axioms (A3), (def$_\between$ $\sqsubseteq$) and (A5$_\star$).[11] Indeed, by Lemma 1, from (def$_\between$ $\sqsubseteq$) and (def $\between$) we have (A2) and (A4). Moreover, by Fact 2 and Lemma 2, we have $\mathfrak{F} = \mathfrak{F}_\star$. Hence, (A5$_\star$) entails (A5).

7.2. Axiomatizations with primitive $\between$

Axiomatizing mereological structures it is possible to accept the relation $\between$ as a primitive one, and take as axioms the formulae (def $\between$), (A3), (def$_\between$ $\sqsubseteq$) and (A5$_\star$), where (def$_\between$ $\sqsubseteq$) can be treated as a definition of the relation $\sqsubseteq$.[12]

[11] The set of axioms just mentioned was used by Eberle in [2], while he treated mereology as an elementary theory. Therefore instead of non-elementary axiom (A5$_\star$) Eberle accepted infinite number of elementary axioms that represented one axiom schema. In his axioms in place of '$x \in X$' one finds an arbitrary formula $\varphi(x)$ of the first order language with identity '=', one specific predicate '<' and with at least one free variable 'x'. Taking such an approach—from a structure-theoretical point of view—we only assume the existence of fusion for so called elementarily definable sets of the form $\{x \in M : \varphi(x)\}$, while these can be sets that are defined with parameters when a formula $\varphi(x)$ contains other free variables. We write about the difference between both approaches to mereology in [8].

[12] This solution is similar to the one accepted in [4] by Leonard and Goodman. The only difference is that they took as a primitive relation the complement of the relation $\between$ (one of their axioms asserts this). In [4] the relation $\sqsubseteq$ is defined by means of some formula

Different axiomatization, where $\between$ is a primitive relation, was used in Goodman's book [3]. One can assume that it consists of formulae ($\mathrm{ext}_{\between}$), (17) and ($\mathrm{A5}_{\star}$), while the relation $\sqsubseteq$ is defined by means of the formula ($\mathrm{def}_{\between} \sqsubseteq$).[13] Indeed, by Lemma 3, from ($\mathrm{def}_{\between} \sqsubseteq$), ($\mathrm{ext}_{\between}$) and (17) we obtain (A2), (A3), (A4) and (def $\between$). Moreover, by Fact 2 and Lemma 2, we have $\mathfrak{F} = \mathfrak{F}_{\star}$. Hence, ($\mathrm{A5}_{\star}$) entails (A5).

7.3. Axiomatizations with primitive $\sqsubset$

In the very first set of axioms for mereology, that was proposed by its creator Leśniewski, a primitive one is the relation of *being a proper part* $\sqsubset$. This relation satisfies the axioms of a strict partial order ($\mathrm{as}_{\sqsubset}$) and ($\mathrm{t}_{\sqsubset}$) and, moreover, (fun $\mathfrak{F}$) and (A5) (cf. [5], pp. 263–265; English version pp. 230–232). Leśniewski proved ([6], pp. 82–86; English version p. 321–326), that it is definitionally equivalent to the set composed of (A2), (A3), (fun $\mathfrak{F}$) and (A5) which axiomatizes the relation $\sqsubseteq$. To the first set as a definition of relation $\sqsubseteq$ the formula ($\mathrm{def}_{\sqsubset} \sqsubseteq$) was added, while to the second, as a definition of $\sqsubset$ was added (def $\sqsubset$).

Indeed, Lemma 4 says that in an axiomatization of mereological structures one can take relation $\sqsubset$ as a primitive one. In such a case instead of the axioms of partial orders (A1)–(A3) we take the axioms of strict orders, i.e., either the pair ($\mathrm{irr}_{\sqsubset}$) and ($\mathrm{t}_{\sqsubset}$) or the pair ($\mathrm{as}_{\sqsubset}$) and ($\mathrm{t}_{\sqsubset}$), and we define relation $\sqsubseteq$ by ($\mathrm{def}_{\sqsubset} \sqsubseteq$). To the axioms of strict partial orders we add a pair of axioms (A4) and (A5). In [8, Theorem IV.1.1] it was proved that replacing axiom (A4) by the formula (fun $\mathfrak{F}$) we also get the axiomatization of mereological structures.

Moreover, in [8, Theorem IV.1.1] it was proved that in axiomatization of mereological structures instead of "Strong Supplementation Principle" (A4) one can accept "Weak Supplementation Principle" (WSP). Yet the axioms ($\mathrm{irr}_{\sqsubset}$) and ($\mathrm{as}_{\sqsubset}$) follow from ($\mathrm{t}_{\sqsubset}$) and (WSP).

FACT 5 (cf. [8, Lemma II.4.1]). *Let M be a non-empty set, let $\sqsubset$ be a binary relation in M, and let $\sqsubseteq$ and $\between$ be binary relations in M defined respectively by* ($\mathrm{def}_{\sqsubset} \sqsubseteq$) *and* (def $\between$).

(i) *If the relation $\sqsubset$ satisfies* (WSP), *then it is irreflexive, i.e., it satisfies the condition* ($\mathrm{irr}_{\sqsubset}$).

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that arises from ($\mathrm{def}_{\between} \sqsubseteq$) by contraposing it and $\between$ is defined by (def $\between$).

[13] Similarly to Eberle (cf. footnote 11), Goodman in [3] deals with some elementary theory.

(ii) *If the relation $\sqsubset$ satisfies* (WSP) *and* ($\mathrm{t}_{\sqsubset}$), *then it is asymmetric, i.e., it satisfies the condition* ($\mathrm{as}_{\sqsubset}$).

Proof. (i) Assume towards contradiction that for some a we have $a \sqsubset a$. Then, by (WSP), for some x we have $x \sqsubset a$ and $\neg\, x \between a$. Hence, by ($\mathrm{def}_{\sqsubset}\ \sqsubseteq$) and (def $\between$), we have $x \between a$, since $x \sqsubseteq x$ and $x \sqsubseteq a$.

(ii) The conditions ($\mathrm{irr}_{\sqsubset}$) and ($\mathrm{t}_{\sqsubset}$) entail ($\mathrm{as}_{\sqsubset}$). $\square$

So the set of formulas ($\mathrm{t}_{\sqsubset}$), (WSP) and (A5) is an axiomatization of mereology. Yet after such a change one cannot—contrary to what is suggested by Simons in [9, p. 37]—replace the axiom (A5) by the formula ($\mathrm{A5}_{\star}$), because the set of the axioms ($\mathrm{t}_{\sqsubset}$), (WSP) and ($\mathrm{A5}_{\star}$) is essentially weaker than the set ($\mathrm{t}_{\sqsubset}$), (WSP) and (A5). Simply the set ($\mathrm{as}_{\sqsubset}$), ($\mathrm{t}_{\sqsubset}$) and (A4) is essentially stronger than the set ($\mathrm{as}_{\sqsubset}$), ($\mathrm{t}_{\sqsubset}$) and (WSP), since the letter does not guarantee that $\mathcal{F} = \mathcal{F}_{\star}$. In strict partial orders it can only be proved that $\mathcal{F} \subseteq \mathcal{F}_{\star}$. The equality of these relations in partial orders is actually equivalent to the formula (A4) (cf. [8], Theorem IV.3.1, Corollary IV.3.1 and Fact IV.3.4).

Remark 5 (cf. [7, footnote 12]). Simons in [9] called *Classical Extensional Mereology* the theory whose primitive relation is $\sqsubset$ and whose axioms are ($\mathrm{as}_{\sqsubset}$), ($\mathrm{t}_{\sqsubset}$), (WSP) and

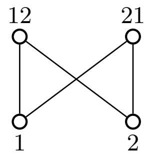
$$(\mathrm{A5}') \qquad \bigwedge_{\emptyset \neq X \subseteq M} \ \bigvee_{z \in M} \ \bigwedge_{y \in M} \big(y \between z \Leftrightarrow \bigvee_{x \in X} x \between y\big)\,,$$

while relations $\sqsubseteq$ and $\between$ are defined by means ($\mathrm{def}_{\sqsubset}\ \sqsubseteq$) and (def $\between$). Of course, the axiom (A5′) can be replaced by formulas ($\mathrm{A5}_{\star}$) and (def $\mathcal{F}_{\star}$).

In [9, p. 64] on the list presenting theses of the system one can find formulae (16) and ($\mathrm{def}_{\between}\ \sqsubseteq$) (respectively as SCT13 and SCT15). In [8, Ch. IV] we proved that they are consequences of neither axioms nor definitions accepted by Simons.

Simons in [9] does not prove the fact that formulae (16) and ($\mathrm{def}_{\between}\ \sqsubseteq$) are theses of his system. He only quotes the lists of theorems given in [1] by Breitkopf. Yet Simons need not have excluded the third of Breitkopf's axioms maintaining on p. 36 in footnote 22 that this axiom is not independent from the others. Simons justification (p. 36) is not apt while having such a weak assumption as (WSP) added to partial order axioms. Simons made avail himself of an analogy of mutual definability of relations $\mathcal{S}$ and $\mathcal{I}$ in partial orders (cf. conditions (7) and (8)) which simply does not hold in this case. We present detailed analysis of this fact in [8, Ch. IV, §§ 3 and 8].

The following structure is a model of Simons' "Classical Extensional Mereology". However, the formulas (A5) and (fun $\mathfrak{F}$) are not true in it. Thus, it is not a mereological structure. We put $S := \{1, 2, 12, 21\}$ and $\sqsubset := \{\langle 1, 12\rangle, \langle 1, 21\rangle, \langle 2, 12\rangle, \langle 2, 21\rangle\}$.



The relations $\sqsubseteq$ and $\between$ are defined by (def$_\sqsubset$ $\sqsubseteq$) and (def $\between$). Of course, the formulas (irr$_\sqsubset$), (t$_\sqsubset$) and (WSP) are true in $\langle S, \sqsubset\rangle$. Obviously, the axiom (A5$'$) is true, for any singelton (i.e., $x\ \mathfrak{F}_\star\ \{x\}$). Moreover, for every at least two-element set $X \subseteq S$, we have $\{y : y \between 12\} = \bigcup_{x\in X}\{y : y \between x\} = \{y : y \between 21\} = S$ (i.e., $12\ \mathfrak{F}_\star\ X$ and $21\ \mathfrak{F}_\star\ X$). Thus, the axiom (A5$'$) is true in $\langle S, \sqsubset\rangle$, for any non-empty subset of S.

The sentence (A5) is not true in $\langle S, \sqsubset\rangle$, since there are no fusions of the sets S, $\{12, 21\}$, $\{1, 12, 21\}$ and $\{2, 12, 21\}$. Moreover, the condition (fun $\mathfrak{F}$) is not true, because $12\ \mathfrak{F}\ \{1, 2\}$ and $21\ \mathfrak{F}\ \{1, 2\}$.[14] □

8. Algebraic operations

Let $\mathfrak{M} = \langle M, \sqsubseteq\rangle$ be an arbitrary mereological structure. By means of (A5&fun$\mathfrak{F}$) and (25) we can define a unary operation $\bigsqcup\colon 2^M \setminus \{\emptyset\} \to M$

$$\text{(def }\textstyle\bigsqcup\text{)} \qquad X \neq \emptyset \implies \bigsqcup X := (\imath x)\, x\ \mathfrak{F}\ X = (\imath x)\, x\ \mathfrak{S}\ X\,.$$

Referring to part 3, if $\emptyset \neq X = \{x : \varphi(x)\}$, then the element $\bigsqcup X$ can be called a *collective set* (resp. *fusion*) determined by a condition $\varphi(x)$ and we can designate it, e.g. by $[\![x : \varphi(x)]\!]$.

Let us notice that for $a \in M$ and $\emptyset \neq X \subseteq M$ we have

$$a \sqsubseteq \bigsqcup X \iff \bigwedge_{y\in M} (y \sqsubseteq a \Rightarrow \bigvee_{x\in X} x \between y).$$

Indeed, let $a \sqsubseteq \bigsqcup X$ and $y \sqsubseteq a$. Then, by (A2), we have $y \sqsubseteq \bigsqcup X$. Therefore from (def $\mathfrak{F}$) it follows that $\bigvee_{x\in X} x \between y$. Conversely let $\bigwedge_{y\in M}(y \sqsubseteq a \Rightarrow \bigvee_{x\in X} x \between y)$. By $\mathfrak{F} = \mathfrak{F}_\star$, if $\bigvee_{x\in X} x \between y$, then $y \between \bigsqcup X$. Therefore we obtain $\bigwedge_{y\in M}(y \sqsubseteq a \Rightarrow y \between \bigsqcup X)$. Hence, by (A4), we have $a \sqsubseteq \bigsqcup X$.

[14] From Theorem 2 it follows that there is no four-element mereological structure (cf. footnote 15). Hence we also obtain that $\langle S, \sqsubset\rangle$ is not a mereological structure.

We will also introduce binary operation $\sqcup\colon M \times M \to M$

$$a \sqcup b := \bigsqcup\{a, b\}\,. \tag{def $\sqcup$}$$

Notice that for $a, b \in M$ we have

$$a \sqcup b = \bigsqcup\{x : x \sqsubseteq a \vee x \sqsubseteq b\}\,.$$

Indeed, applying just (1), (def $\mathcal{S}$) and simple logical transformations we get: $c\,\mathcal{S}\,\{x : x \sqsubseteq a \vee x \sqsubseteq b\}$ iff $c\,\mathcal{S}\,\{a, b\}$. Thanks to (25) instead of the relation $\mathcal{S}$ one can take $\mathcal{F}$.

If $a \between b$, then $\{x : x \sqsubseteq a \wedge x \sqsubseteq b\} \neq \emptyset$. Therefore, by (7), (A5&fun$\mathcal{F}$) and (25), we obtain

$$a \between b \implies (\imath x)\, x\, \mathcal{I}\, \{a, b\} = (\imath x)\, x\, \mathcal{F}\, \{x : x \sqsubseteq a \wedge x \sqsubseteq b\}\,.$$

So we can define a partial binary operation

$$a \between b \implies a \sqcap b := (\imath x)\, x\, \mathcal{I}\, \{a, b\}\,. \tag{def $\sqcap$}$$

If $b \not\sqsubseteq a$, then, by (A4), $\{x : x \sqsubseteq b \wedge \neg\, x \between a\} \neq \emptyset$. Therefore for $a \neq 1$ we have $\{x : \neg\, x \between a\} \neq \emptyset$. So we can define a unary partial operation

$$a \neq 1 \implies -a := \bigsqcup\{x : \neg\, x \between a\}\,. \tag{def $-$}$$

9. Mereological structures and complete Boolean algebras

By a Boolean algebra we mean an algebraic structure $\mathfrak{A} = \langle A, +, \bullet, -, 0, 1\rangle$, in which A is a non-empty set, $+$ and $\bullet$ are binary operations on A, $-$ is a unary operation on A, while 0 and 1 are elements of A; moreover, the following axioms are satisfied

(i)	$a + b = b + a,$	$a \bullet b = b \bullet a,$
(ii)	$a + (b + c) = (a + b) + c,$	$a \bullet (b \bullet c) = (a \bullet b) \bullet c,$
(iii)	$a \bullet (b + c) = (a \bullet b) + (a \bullet c),$	$a \bullet (b + c) = (a \bullet b) + (a \bullet c),$
(iv)	$a + 0 = a,$	$a \bullet 1 = a,$
(v)	$a + -a = 1,$	$a \bullet -a = 0.$

If $0 \neq 1$ (iff $\mathrm{Card}(A) > 1$), then algebra $\mathfrak{A}$ is called *non-degenerate*.

Boolean algebra is a set partially ordered by the relation $\leq$ by means of conditions

(def $\leq$) $$a \leq b \iff a + b = b \iff a \bullet b = a\,.$$

Let us remind that for any $a, b \in A$

$$a \leq b \iff -b \leq -a,$$
$$-(-a) = a.$$

Moreover

$$\begin{aligned}
1 &= (\imath x)\, x\, \mathcal{S}_{\leq}\, A\,,\\
0 &= (\imath x)\, x\, \mathcal{S}_{\leq}\, \emptyset\,,\\
a + b &= (\imath x)\, x\, \mathcal{S}_{\leq}\, \{a, b\}\,,\\
a \bullet b &= (\imath x)\, x\, \mathcal{I}_{\leq}\, \{a, b\}\,,\\
-a &= (\imath x)\, x\, \mathcal{S}_{\leq}\, \{y \in A : y \bullet a = 0\}\,,\\
-a &= (\imath x)\, x\, \mathcal{S}_{\leq}\, \{y \in A : \neg \textstyle\bigvee_{z \in A}(z \neq 0 \wedge z \leq y \wedge z \leq a)\}\,,
\end{aligned}$$

where $\mathcal{S}_{\leq}$ and $\mathcal{I}_{\leq}$ are relations of supremum and infimum with respect to the relation $\leq$.

We say that a Boolean algebra is complete if for an arbitrary set X in A there is such an $a \in A$ that $a\, \mathcal{S}_{\leq}\, X$ (then there exists as well such $b \in A$ that $b\, \mathcal{I}_{\leq}\, X$, it is b such that $b\, \mathcal{S}_{\leq}\, \{y \in A : \bigwedge_{x \in X} y \leq x\}$).

We will prove the theorem saying that after «adding» zero element to some mereological structure we will «turn» it into a non-degenerate complete Boolean algebra:[15]

THEOREM 2. *Let $\langle M, \sqsubseteq \rangle$ be a mereological structure. Let 0 be an arbitrary element not in M. Assume for $a, b \in M$*

$$a + b := a \sqcup b,$$
$$a + 0 := a =: 0 + a \quad \textit{and} \quad 0 + 0 := 0,$$
$$a \between b \implies a \bullet b := a \sqcap b,$$

[15]Tarski was the first one to notice this fact together with the other one presented in Theorem 3. He described these in [11] in some footnote which in an English translation of [11] can be found at pages 333–334. Cf. further remarks 6 and 7.

By this fact one can strengthen the result (27): *The number of elements of the universe of some finite mereological structure equals $2^n - 1$ for some natural number $n \geqslant 1$.* Indeed, after adding 1 to the number of elements of the universe («adding zero») we have to get number 2^n, since only such can be the cardinality of finite universes of non-degenerate Boolean algebras.

$$\neg\, a \between b \implies a \bullet b := \mathrm{o},$$
$$a \bullet \mathrm{o} := \mathrm{o} =: \mathrm{o} \bullet a \text{ and } \mathrm{o} \bullet \mathrm{o} := \mathrm{o},$$
$$a \neq 1 \implies -a := -a,$$
$$-1 := \mathrm{o} \text{ and } -\mathrm{o} := 1,$$
$$\mathrm{1} := 1.$$

Then $\langle M \cup \{\mathrm{o}\}, +, \bullet, -, \mathrm{o}, \mathrm{1}\rangle$ *is a non-degenerate complete Boolean algebra in which the relation* $\leq$ *defined by* (def $\leq$) *satisfies the following condition*

(†) $$a \leq b \iff a \sqsubseteq b \vee a = \mathrm{o}\,.$$

Remark 6. In the footnote mentioned earlier Tarski noticed that a structure $\langle M \cup \{\mathrm{o}\}, \leq\rangle$, where now $\leq$ is the relation defined directly by (†), will be a complete Boolean lattice (relational characterization of complete Boolean algebras). Let us remind that Tarski characterized mereological structure $\langle M, \sqsubseteq\rangle$ by means of axioms (A2) and (A5&fun$\mathfrak{F}$) (cf. Theorem 1). The whole gist of Tarski's observation was hidden in theorems 1 and 2 proved in [11] which say that a structure $\langle B, \leq\rangle$ is a complete Boolean lattice iff relation $\leq$ satisfies some conditions $\mathfrak{B}_2$ and $\mathfrak{B}_4^*$. Condition $\mathfrak{B}_2$ says that the relation $\leq$ is transitive, so it follows from (A2). Condition $\mathfrak{B}_4^*$ is a counterpart of the condition (A5&fun$\mathfrak{F}$). Roughly speaking, those conditions differ only in «adding or deleting zero» of a Boolean lattice. Details of the proof can be found in [8, Ch. III]. □

Proof of Theorem 2. We check (i)–(v) and (†) in a standard way. We will show that Boolean algebra $\langle M \cup \{\mathrm{o}\}, +, \bullet, -, \mathrm{o}, \mathrm{1}\rangle$ is complete. Indeed $\mathrm{o}\ \mathcal{S}_{\leq}\ \emptyset$. Moreover, if $X \neq \emptyset$, then, by (A5) and (25), there is such an $a \in M$, that $a\ \mathcal{S}_{\sqsubseteq}\ X$. Therefore $a\ \mathcal{S}_{\leq}\ X$. □

We will prove now that from a non-degenerate complete Boolean algebra by «deleting zero» we will obtain a mereological structure:[16]

Theorem 3. *Let* $\langle A, +, \bullet, -, \mathrm{o}, \mathrm{1}\rangle$ *be a non-degenerate complete Boolean algebra. Assume*
$$\sqsubseteq := \leq|_{A\setminus\{\mathrm{o}\}},$$
where the relation $\leq$ *is defined by* (def $\leq$). *Then* $\langle A \setminus \{\mathrm{o}\}, \sqsubseteq\rangle$ *is a mereological structure.*

[16] Of course zero element can be dropped only in non-degenerate Boolean algebras (there must be some element left in the domain after «deleting»). If a given Boolean algebra is not two-element ($\mathrm{Card}(A) \geqslant 4$; cf. footnote 15), then obtained mereological structure is non-degenerate.

Remark 7. Referring to Remark 6, Tarski noticed that if a structure $\langle B, \leq\rangle$ satisfies conditions $\mathfrak{B}_2$ and $\mathfrak{B}_4^*$, then the structure $\langle B \setminus \{\mathrm{o}\}, \sqsubseteq\rangle$ fulfills (A2) and (A5&fun$\mathfrak{F}$). Details of the proof can be found in [8, Ch. III]. $\square$

PROOF. It is evident that $\sqsubseteq$ is a partial order.

For (A4): Let $a, b \in A \setminus \{\mathrm{o}\}$ be such that $a \not\sqsubseteq b$. Then $a \bullet (-b) \neq \mathrm{o}$, $a \bullet (-b) \sqsubseteq a$ and there is no such $x \in A \setminus \{\mathrm{o}\}$, that $x \sqsubseteq a \bullet (-b)$ and $x \sqsubseteq b$. Therefore partial order $\sqsubseteq$ is separative.

For (A5): Let $\emptyset \neq X \subseteq A \setminus \{\mathrm{o}\}$. Since $\langle A, +, \bullet, -, \mathrm{o}, 1\rangle$ is a complete Boolean algebra, then there is such $a \in A \setminus \{\mathrm{o}\}$ that $a\, \mathcal{S}_{\leq}\, X$. We will show that $a\, \mathfrak{F}\, X$. *Primo*: $\bigwedge_{x\in X} x \leq a$. Hence $\bigwedge_{x\in X} x \sqsubseteq a$. *Secundo*: we have to prove that $\bigwedge_{x\in A\setminus\{\mathrm{o}\}}(x \sqsubseteq a \Rightarrow \bigvee_{y\in Y} y \between x)$. Let us take an arbitrary $c \neq \mathrm{o}$ such that $c \sqsubseteq a$. If $c \in X$, then the proof is finished since $c \between c$. Assume now that $c \notin X$ and (for contradiction) that $\neg \bigvee_{y\in X} y \between x$. By definition of relations $\between$ and $\sqsubseteq$ we have $\bigwedge_{y\in X} \neg \bigvee_{z\neq \mathrm{o}}(z \leq y \wedge z \leq c)$. Since $-c = (\imath x)\, x\, \mathcal{S}_{\leq}\, \{y : \neg \bigvee_{z\neq \mathrm{o}}(z \leq y \wedge z \leq c)\}$ and $X \subseteq \{y : \neg \bigvee_{z\neq \mathrm{o}}(z \leq y \wedge z \leq c)\}$, so $a \leq -c$. Moreover, by assumption we have $c \sqsubseteq a$. Therefore $c \leq -a$ and $c \leq a$. Hence $c \leq a \bullet -a = \mathrm{o}$, i.e., $c = \mathrm{o}$, and we obtained a contradiction. $\square$

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Andrzej Pietruszczak
Nicolaus Copernicus University
Department of Logic
ul. Asnyka 2
PL 87-100 Toruń, Poland
pietrusz@uni.torun.pl