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"REDUCTIO AD ABSURDUM" AND ŁUKASIEWICZ'S MODALITIES

1. Introduction

The present article contains part of results from my lecture delivered at II Flemish-Polish workshop on Ontological Foundation of Paraconsistency. In the lecture presented at the First Flemish-Polish Workshop (see [24]) as well as in a series of other works ([19, 20, 21, 22, 23]) I studied the class **Jhn** of non-trivial extensions of minimal or Johansson's logic. Minimal logic (we will denote it **Lj**) suggested for the fist time by Kolmogorov [15] and received the name "Johansson's logic" after the work [13] by Ingebrigt Johansson is a paraconsistent analog of Heyting's intuitionistic logic **Li**. These logics have common positive fragment but different axioms for negation. In **Li** the negation is defined by the laws $(p \supset q) \supset ((p \supset \neg q) \supset \neg p)$ (*reductio ad absurdum*) and $\neg p \supset (p \supset q)$ (*ex contradictione quodlibet*), while **Lj** have only first of these two axioms.. In this way minimal logic is paraconsistent according to the generally accepted definition of paraconsistent logics as logics admitting inconsistent but non-trivial theories. But it lies on the border line of paraconsistency. Usually (see, e.g. [27]) the above definition of paraconsistency is augmented with the remark that it is not absolutely satisfiable, and namely **Lj** serves as a counterexample, because we have in **Lj** for arbitrary formulae φ and ψ,

$$\{\varphi, \neg\varphi\} \vdash \neg\psi.$$

This condition means that although inconsistent **Lj**-theories may be non-trivial, they are trivial with respect to negation. Any negated formula is provable in any inconsistent **Lj**-theory.

Due to this "paraconsistent paradox" of minimal logic it looks natural to finish an investigation of the class of **Lj**-extensions with an attempt to overcome it. We try to do it by emerging the class of **Lj**-extensions in a more general class of paraconsistent logics and pointing out some special property distinguishing extensions of minimal logic in the latter class. We suggest also that negation in logics from the above mentioned class should preserve the most essential property of intuitionistic negation, namely, that the negation must be defined as reduction to absurdity. In intuitionistic logic, the negation is characterized by three important features: 1) we assert $\neg\varphi$ if supposing φ leads to absurdity; 2) absurdity may be explicated as propositional constant $\bot$; 3) absurdity implies everything, $\bot \supset p$, or $\neg p \supset (p \supset q)$. Note that item 3 implies, in fact, item 2. If any contradiction implies everything, all contradictions are equivalent, and so we may use a propositional constant do denote arbitrary contradictory or absurd statement. In minimal logic **Lj** we omit item 3, which allows to distinguish nonequivalent contradictions. In an extension **L** of **Lj**, the contradiction $\varphi \wedge \neg\varphi$ is equivalent to $\bot$ if and only if φ is provable in $\mathbf{L_{neg}} = \mathbf{L} + \{\bot\}$, so called negative counterpart of **L** (see [23]), moreover, it was stated in [23] that $\mathbf{L_{neg}}$ may be treated as a logic of contradictions of **L**. On the other hand, the item 2 is preserved in **Lj** which results, in particular, in the above mentioned "paraconsistent paradox". In this situation it looks natural to make the next step and to resign not only from item 3, but also from its consequence, item 2. This gives rise to a question, how in this case the absurdity should be explicated in a logical system?

I suggest an answer, which will be based on the interrelations between the logic of classical refutability (hereafter, **Le**) and modal logic *Ł* of Łukasiewicz. The logic **Le** of classical refutability can be obtained from **Lj** by extending its positive fragment up to the classical positive logic. It was suggested by P. Bernays [3] and received his name from H. B. Curry [7]. Namely **Le** was the starting point in the author's investigations of the class of **Lj**-extensions. In [19], **Le** was characterized via the class of Peirce-Johansson algebras (see Section 2) and it was proved on the base of this characterization that **Le** is the greatest paraconsistent extension of minimal logic having only two proper non-trivial extensions: classical logic **Lk**, and maximal negative logic $\mathbf{Lmn}(= \mathbf{Le} + \{\neg p\})$. The above characterization implies that the simplest matrix for **Le** has at least four elements, and so **Le** can be considered as a four-valued logic as well as Łukasiewicz's modal logic. This fact and the work [14] by A. Karpenko, where he considered isomorphs (see Section 2) of classical logic in three-valued Bochvar's logic $\mathbf{B_3}$ and suggested a hypothesis that namely different isomorphs of classical logic definable in a given finite-valued logic determine its paraconsistent structure, inspired the work [20]. In this work, it was proved that one can define in **Le** only one isomorph of classical logic and two isomorphs of maximal negative logic. These isomorphs, on one hand, correspond to translations of **Lk** and **Lmn** into **Le** playing a key role in studying the structure of the class **Jhn** (see [21]). On the other hand, the isomorphs were induced by mappings which can be identified in a natural way with modalities of L, which easily implies the fact that **Le** is definitially equivalent to the positive fragment of L and that **Le** extended by the classical negation is definitially equivalent to L. At this point I must acknowledge to Prof. Jozef M. Font, who informed me that such interrelations between **Le** and L were stated previously by J. Porte [25, 26]. For our goals the most essential is the fact that the necessity operator of L may be identified with the contradiction operator $C(\varphi) \rightleftharpoons \varphi \wedge \neg\varphi$ of the logic of classical refutability. This leads to the idea of considering a contradiction operator C as a primary logical connective and defining negation as reduction to this operator, $\neg\varphi \rightleftharpoons \varphi \supset C(\varphi)$. In this way, we emerge the class **Jhn** into the class of so called C-logics and show that logics from **Jhn** will be distinguished by some paradoxical property of C considered as a modal operator. So the more natural modal properties of contradiction operator result in a more satisfiable consequence relation from the point of view of paraconsistent logic. Further, it turns out that the same negation may be defined via different operators. There is a sense to distinguish, for example, between contradiction operator C that should satisfy the axiom $C(\varphi) \supset \varphi$ and more general absurdity operator, for which similar axiom is not obeying.

In the forthcoming article we will show that in some well known paracosistent logics, namely, in the logic **CLuN** by D. Batens [1, 2], in Sette's maximal paraconsistent logic P^1 [28, 30], and in Da Costa's C_1 negation can be defined via a suitable contradiction or absurdity operator. Interesting that the negation of **CLuN** may be defined via absurdity operator about which nothing is postulated. In this way **CLuN** may be considered as a counterpart of **Le**, where the negation is defined via the constant $\bot$ about which nothing is postulated.

2. Preliminaries

Our considerations will be restricted to the propositional case. We will consider several propositional languages, all of which are extensions of positive language $\mathcal{L}^+ \rightleftharpoons \{\supset, \vee, \wedge\}$. The equivalence $\equiv$ is definable as $\varphi \equiv \psi \rightleftharpoons (\varphi \supset \psi) \wedge (\psi \supset \varphi)$. We denote by $\mathcal{L}^\neg$ the language obtained by adding to $\mathcal{L}^+$ the symbol $\neg$ for negation, $\mathcal{L}^{\neg} = \{\supset, \vee, \wedge, \neg\}$ and by $\mathcal{L}^{\perp}$ the extension of $\mathcal{L}^{+}$ via the constant $\perp$ for absurdity, $\mathcal{L}^{\perp} = \{\supset, \vee, \wedge, \perp\}$. Further languages will be introduced in the last section. For any language $\mathcal{L}$ under consideration by a logic in the language $\mathcal{L}$ we mean a set of $\mathcal{L}$-formulae closed under the rules of substitution and *modus ponens*. With every logic $\mathbf{L}$ one can associate in a standard way a consequence relation $\vdash_{\mathbf{L}}$. The relation $X \vdash_{\mathbf{L}} \varphi$ means that one can pass in a finite number of steps to a formula φ using elements of X, theorems of $\mathbf{L}$ and the rule of *modus ponens*. For a logic $\mathbf{L}$ and a set X of formulae in the language of $\mathbf{L}$, $\mathbf{L} + \{X\}$ denotes the minimal extension of $\mathbf{L}$ containing X.

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be propositional languages, $\mathbf{L_1}$ and $\mathbf{L_2}$ logics in languages $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively, and let $\theta : \mathcal{L}_1 \to \mathcal{L}_2$ and $\rho : \mathcal{L}_2 \to \mathcal{L}_1$ be translations from the language $\mathcal{L}_1$ into the language $\mathbf{L}_2$ and visa versa, i.e., simply mappings from the set of $\mathcal{L}_1$-formulae to the set $\mathcal{L}_2$-formulae and visa versa. We will say that $\mathbf{L_1}$ is *faithfully embedded into* $\mathbf{L_2}$ *via* θ if for any set X of $\mathcal{L}_1$-formulae and $\mathcal{L}_1$-formula φ we have the equivalence

$$X \vdash_{\mathbf{L_1}} \varphi \;\Leftrightarrow\; \theta(X) \vdash_{\mathbf{L}} \theta(\varphi),$$

where $\theta(X) = \{\theta(\psi) \mid \psi \in X\}$. Further, the logic $\mathbf{L_1}$ is said to be *definitially equivalent to* $\mathbf{L_2}$ *via translations* θ *and* ρ if $\mathbf{L_1}$ is faithfully embedded into $\mathbf{L_2}$ via θ, $\mathbf{L_2}$ is faithfully embedded into $\mathbf{L_1}$ via ρ, and moreover, for any $\mathcal{L}_1$-formula φ and $\mathcal{L}_2$-formula ψ we have the equivalences

$$\mathbf{L_1} \vdash \varphi \equiv \rho\theta(\varphi) \text{ and } \mathbf{L_2} \vdash \psi \equiv \theta\rho(\psi).$$

A translation α from the language $\mathcal{L}_1$ to the language $\mathcal{L}_2$ *preserves propositional variables* if $\alpha(p) = p$ for any propositional variable p and it *preserves an n-ary connective* $*$ if $* \in \mathcal{L}^1 \cap \mathcal{L}^2$ and

$$\alpha(*(\varphi_1, \ldots, \varphi_n)) = *(\alpha\varphi_1, \ldots \alpha\varphi_n)$$

for all formulae $\varphi_1, \ldots, \varphi_n$.

Now we recall the definitions of basic logics. The *positive logic* $\mathbf{Lp}$ is a logic in the language $\mathcal{L}^+$ axiomatized by the following formulae.

1. $p \supset (q \supset p)$
2. $(p \supset (q \supset r)) \supset ((p \supset q) \supset (p \supset r))$
3. $(p \wedge q) \supset p$
4. $(p \wedge q) \supset q$

5. $(p \supset q) \supset ((p \supset r) \supset (p \supset (q \wedge r)))$

6. $p \supset (p \vee q)$

7. $q \supset (p \vee q)$

8. $(p \supset r) \supset ((q \supset r) \supset ((p \vee q) \supset r))$

The *classical positive logic* $\mathbf{Lk}^+$ is also a logic in the language $\mathcal{L}^+$ and can be axiomatized modulo $\mathbf{Lp}$ by either of the following two axioms:

P. $((p \supset q) \supset p) \supset p$ (Peirce law)

E. $p \vee (p \supset q)$ (extended law of excluded middle)

The version $\mathbf{Lj}^\perp$ of minimal logic in the language $\mathcal{L}^\perp$ can be defined as a logic axiomatized by the axioms 1–8 above. The equivalent version of minimal logic $\mathbf{Lj}^\neg$ in the language $\mathcal{L}^\neg$ with negation can be axiomatized by the axioms 1–8 and the following axiom:

A. $(p \supset q) \supset ((p \supset \neg q) \supset \neg p)$ (*reductio ad absurdum*)

To make precise the statement about the equivalence of two defined above versions of the minimal logic we define the translations θ from the language $\mathcal{L}^\neg$ to $\mathcal{L}^\perp$ and ρ from the language $\mathcal{L}^\perp$ to $\mathcal{L}^\neg$ as follows. Both translations θ and ρ preserve propositional variables and connectives of $\mathbf{L}^+$ and, additionally, we have

$$\theta(\neg\varphi) = \theta(\varphi) \supset \perp \text{ and } \rho(\perp) = \neg(p_0 \supset p_0),$$

where φ is an arbitrary $\mathcal{L}^\neg$-formula and p_0 is some fixed propositional variable.

Proposition 2.1. *Logics* $\mathbf{Lj}^\neg$ *and* $\mathbf{Lj}^\perp$ *are definitially equivalent via translations* θ *and* ρ.

Due to this fact we may freely pass in the following from the language $\mathcal{L}^\perp$ to the language $\mathcal{L}^\neg$ and visa versa. Therefore, we will omit the superscripts in the denotation of minimal logic and will not explicitly indicate with which version of minimal logic or of its extension we deal in a moment.

The *intuitionistic logic* $\mathbf{Li}$ can be axiomatized modulo minimal logic in the language $\mathcal{L}^\perp$ as follows $\mathbf{Li} = \mathbf{Lj} + \{\perp \supset p\}$; and in the language $\mathcal{L}^\neg$ as follows $\mathbf{Li} = \mathbf{Lj} + \{\neg p \supset (p \supset q)\}$.

The *classical logic* **Lk** and the *logic of classical refutability* **Le** can be axiomatized modulo intuitionistic logic **Li** and minimal logic **Lj**, respectively, via either the Peirce law **P** or the generalized law of excluded middle **E**.

$$\mathbf{Lk} = \mathbf{Li} + \{p \vee (p \supset q)\}, \mathbf{Le} = \mathbf{Lj} + \{p \vee (p \supset q)\}$$

Semantics for logics under considerations may be provided by matrices with unique distinguished value, therefore, we deal not with matrices but with algebras unit element of which corresponds to the only distinguished value. As usually, for an $\mathcal{L}$-formulae φ and algebra **A** of the language $\mathcal{L} \cup \{1\}$, φ is an *identity of* **A** if $V(\varphi) = 1$ for any **A**-valuation V. A *logic of* **A**, **LA**, is defined as a set of all its identities and a *logic of* a class $\mathcal{K}$ of algebras, $\mathbf{L}\mathcal{K}$, is an intersection of logics of algebras from $\mathcal{K}$. An algebra **A** is called a *(characteristic) model of* a logic **L** if $\mathbf{L} \subseteq \mathbf{LA}$ ($\mathbf{L} = \mathbf{LA}$). A logic **L** *is characterized by* a class $\mathcal{K}$ of algebras if $\mathbf{L} = \mathbf{L}\mathcal{K}$.

As is known the minimal logic **Lj** is characterized by the class of j-algebras (or pseudo-complemented algebras), i.e., implicative lattices in which the constant $\perp$ is interpreted as an arbitrary element of basic set. In [19], it was stated that the logic **Le** of classical refutability is characterized by the class of Peirce-Johansson (or pj-)algebras. A pj-algebra is an implicative lattice satisfying the Peirce law and with $\perp$ interpreted as an arbitrary element of its basic set. This characterization easily implies that any characteristic model of **Le** must contain at least four elements: unit element 1, $0 (= \perp = \neg 1)$, some element -1 under 0, because **Le** is paraconsistent, and finally, element $a = 0 \supset -1$ incomparable with 0. It turns out that the four-element lattice $\mathbf{4}'$ with universe $\{1, 0, a, -1\}$ and $\perp$ interpreted as 0 is really a characteristic model for **Le**.

In [20], the author studied isomorphs definable in **Le**, which was inspired by the work [14] by A. Karpenko.

The term "isomorph" was used in the first monograph on multi-valued logics [29] by N. Rescher, and now it looks a bit archaic. Let $\mathbf{L}_1$ and $\mathbf{L}_2$ be logics and let $\mathbf{L}_2$ be given via its logical matrix. Due to N. Rescher [29], an *isomorph* of the logic $\mathbf{L}_1$ in the logic $\mathbf{L}_2$ is a definition of a matrix for $\mathbf{L}_1$ in the matrix for $\mathbf{L}_2$ with the help of term operations only.

Consider the following term operations on $\mathbf{4}'$:

$$\varepsilon(x) \rightleftharpoons \neg\neg x (= x \vee \perp),\ \delta(x) \rightleftharpoons \neg\neg x \supset x (= \perp \supset x), \tau(x) \rightleftharpoons x \wedge \perp.$$

The first of these mappings allows to define an isomorph (we denote it $\mathbf{Lk}_\varepsilon$) of classical logic in **Le**. Defining operations $\vee_\varepsilon$, $\wedge_\varepsilon$, $\supset_\varepsilon$, and $\neg_\varepsilon$ as follows:

$$x *_\varepsilon y \rightleftharpoons \varepsilon(x) * \varepsilon(y),\ * \in \{\vee, \wedge, \supset, \neg\},\ \neg_\varepsilon x \rightleftharpoons \neg\varepsilon(x),$$

we obtain the following truth tables:

$\wedge_\varepsilon$	1	a	0	-1
1	1	1	0	0
a	1	1	0	0
0	0	0	0	0
-1	0	0	0	0

$\vee_\varepsilon$	1	a	0	-1
1	1	1	1	1
a	1	1	1	1
0	1	1	0	0
-1	1	1	0	0

$\supset_\varepsilon$	1	a	0	-1
1	1	1	0	0
a	1	1	0	0
0	1	1	1	1
-1	1	1	1	1

$\neg_\varepsilon$	
1	0
a	0
0	1
-1	1

As we can see, rows and columns corresponding to elements 1 and a are identical. The same holds for elements 0 and -1. So, identifying these pairs of elements we obtain two-valued matrices of operations of classical logic.

The other two mappings allow to define isomorphs $\mathbf{Lmn}_\delta$ and $\mathbf{Lmn}_\tau$ of maximal negative logic in **Le**. Recall that **Lmn** is characterized via the two element negative algebra $\mathbf{2}'$ differing from the two element model of **Lk** only by the negation operation, which is identically true in this case. The corresponding operations looks as follows.

$$x *_\delta y \rightleftharpoons \delta(x) * \delta(y),\ * \in \{\vee, \wedge, \supset\},\ \neg_\delta x \rightleftharpoons \neg\delta(x);$$

$$x *_\tau y \rightleftharpoons \tau(x) * \tau(y),\ * \in \{\vee, \wedge\},$$

$$x \supset_\tau y \rightleftharpoons \tau(x \supset y) = \tau(\tau(x) \supset \tau(y)),\ \neg_\tau \rightleftharpoons \tau(\neg x).$$

There is a close connection between interdefinability of logical matrices and mutual translations of logics. They were studied in details in the works of P. Wojtylak [31, 32]. The isomorphs defined above leads to the following translations of classical and maximal negative logics into **Le**.

Proposition 2.2. *For any formula φ, the following equivalences hold:*

1. $\mathbf{Lk} \vdash \varphi \ \Leftrightarrow\ \mathbf{Le} \vdash \neg\neg\varphi$,

2. $\mathbf{Lmn} \vdash \varphi \ \Leftrightarrow\ \mathbf{Le} \vdash \neg\neg\varphi \supset \varphi$

3. $\mathbf{Lmn} \vdash \varphi \ \Leftrightarrow\ \mathbf{Le} \vdash \bot \supset (\varphi \wedge \bot) \ \Leftrightarrow\ \mathbf{Le} \vdash \bot \supset \varphi$

As was mentioned these translations play a key role in studying the class of **Lj**-extensions, on the other hand, term operations ε, δ, and τ inducing translations may be identified in a natural way with Łukasiewicz modalities, which will be considered in the next section.

3. Le and modal logic of Łukasiewicz

Łukasiewicz's modal logic $Ł$ was introduced in [16] (see also [17]). J. Łukasiewicz intended to construct a system of modal logic, which is an extension of classical propositional logic with two interdefinable modal operators of "necessity" L and of "possibility" M, $Lp \equiv \neg M \neg p$ and $Mp \equiv \neg L \neg p$. These operators, in turn, should satisfy the conditions

$$\vdash Lp \supset p, \ \vdash p \supset Mp,$$

and

$$\nvdash p \supset Lp, \nvdash Mp \supset p.$$

He used the matrix approach to define his system. Since the ordinary two-element matrix **2** for classical logic does not allow to define modal operators satisfying the above mentioned conditions, Łukasiewicz took the four-element matrix $\mathbf{2} \times \mathbf{2}$.

Note that if we identify the elements of $\mathbf{2} \times \mathbf{2}$ and the elements of **Le**-model $\mathbf{4}'$ in the following way $(1,1) \to 1$, $(1,0) \to a$, $(0,1) \to 0$, and $(0,0) \to -1$, the binary operations of classical logic and of **Le** will coincide on $\mathbf{4}'$, only negations will act differently on the set of truth values $\{1, a, 0, -1\}$.

Modalities L and M and dual modalities W and V are defined on $\mathbf{2} \times \mathbf{2}$ via the following truth-tables:

x	Lx	Mx	Wx	Vx
1	0	1	a	1
a	-1	a	a	1
0	0	1	-1	0
-1	-1	a	-1	0

These pairs of modalities are interdefinable, e.g., as follows

$$Wp \equiv p \wedge M(\sim (p \supset p)), \ Lp \equiv p \wedge V(\sim (p \supset p)),$$

where $\sim$ is the classical negation, and if we will use modalities W and V instead of L and M, respectively, we obtain the logic with the same set of tautologies. Taking into account the above identification of truth-values of logics $Ł$ and **Le** we can easily see that the action of modalities L, M, and V on the set of truth-values will coincide, respectively, with the mappings τ, θ, and ε definable in the logic of classical refutability, which were used in Section 2 to construct isomorphs of classical (ε) and maximal negative logic (τ, θ) in **Le**. On the other hand, the modality W can not be defined in **Le**, because algebra $\mathbf{4}'$ has a subalgebra with the universe $\{1, 0\}$, i.e., the set $\{1, 0\}$ is closed under all operators definable in **Le**. Thus, we can see that there is a very close interrelation between L and **Le**. Porte [25, 26] was the first, who paid attention on the above mentioned interrelation, but he had Łukasiewicz's modal logic as a starting point of his considerations. More exactly, in [25], it was proved that the modalities of L can be defined in a system obtained from the classical logic by the addition a constant, Ω in Porte's notation, about which nothing is postulated, and conversely, that the modalities of L can be used to define the constant Ω. In [26], Porte noted that his constant Ω is close to the negation of the logic of classical refutability and stated that **Le** is definitially equivalent to the positive part of L.

Extend the language $\mathcal{L}^{\perp}$ of the logic **Le** via a new negation symbol $\sim$, $\mathcal{L}^{\perp,\sim} \rightleftharpoons \mathcal{L}^{\perp} \cup \{\sim\}$, and define the logic $\mathbf{Le}^{\sim}$ as a logic in the language $\mathcal{L}^{\perp,\sim}$ with axiom schemes of the logic **Le** and, additionally, axiom schemes of classical negation for $\sim$:

$1^{\sim}$. $(p \supset q) \supset ((p \supset\sim q) \supset\sim p)$

$2^{\sim}$. $\sim p \supset (p \supset q)$

$3^{\sim}$. $p \vee \sim p$

Recall that the logic L is considered in the language $\mathcal{L}^{+} + \{L, \sim\}$, and L^{+} denotes its positive fragment in the language $\mathcal{L}^{+} + \{L\}$. Note that the second modality M can be defined via L in L^{+} as follows

$$Mp \equiv L(p \supset p) \supset p.$$

Define a translation $\theta : \mathcal{L}^{\perp,\sim} \to \mathcal{L}^{+} + \{L, \sim\}$ from the language of $\mathbf{Le}^{\sim}$ to the language of Łukasiewicz's modal logic and an inverse translation ρ from the language of L to the language of $\mathbf{Le}^{\sim}$ in such a way that they preserve propositional variables and all classical connectives and, moreover,

$$\theta(\perp) = L(p \supset p),\ \rho(L\varphi) = \rho\varphi \wedge \neg\rho\varphi,$$

where p is a propositional variable and φ is an arbitrary formula in the language $\mathcal{L}^{+} + \{L, \sim\}$.

Defining algebras $\mathbf{4}^{\sim}$ and $\mathbf{4}^{L}$ as expansions of the algebra $\mathbf{4}'$ via the classical negation and, respectively, via the modal operator L, we state the following facts.

Proposition 3.3. *The algebra* $\mathbf{4}^\sim$ *is a characteristic model for* $\mathbf{Le}^\sim$*.*

Proof. Obviously, $\mathbf{Le}^\sim$ has only two subdirectly irreducible models, $\mathbf{2}^\sim$ and $\mathbf{2}'^\sim$, which are expansions of the Boolean algebra $\mathbf{2}$ and the negative algebra $\mathbf{2}'$ via additional Boolean negation $\sim$. Therefore, $\mathbf{Le}^\sim = \mathbf{L2}^\sim \cap \mathbf{L2}'^\sim$. Taking into account $\mathbf{4}^\sim \cong \mathbf{2}^\sim \times \mathbf{2}'^\sim$ we immediately arrive at $\mathbf{L4}^\sim = \mathbf{Le}^\sim$.

The proposition is proved.

Proposition 3.4. *The logic* $\mathbf{Le}^\sim$ *is a conservative extension of* $\mathbf{Le}$*.*

Proof. If φ is a formula in the language $\mathcal{L}^\perp$ and $\mathbf{Le}^\sim \vdash \varphi$, then $\mathbf{4}^\sim \models \varphi$, and also $\mathbf{4}' \models \varphi$, because the connective $\sim$ do not occur in φ. Since $\mathbf{L4}' = \mathbf{Le}$, we have $\varphi \in \mathbf{Le}$, which completes the proof.

Now we give in a precise form the slightly modified version of Porte's results [25, 26].

Proposition 3.5. *The logics* $\mathbf{Le}^\sim$ *and Ł are definitially equivalent via* θ *and* ρ*, the logic* $\mathbf{Le}$ *is definitially equivalent to the positive fragment of Ł via the same translations.*

The **proof** is by direct verification, because all logics involved in this statement are four-valued.

The only difference of the above translations from Porte's translations is the item $\rho(L\varphi) = \rho\varphi \wedge \neg\rho\varphi$. The original version was $\rho(L\varphi) = \rho\varphi \wedge \perp$, but, obviously, $\mathbf{Lj} \vdash (\varphi \wedge \neg\varphi) \equiv (\varphi \wedge \perp)$.

Proposition 3.5 means, first of all, that there is a very close connection between Łukasiewicz's modalities and the paraconsistent negation of the logic of classical refutability. From this fact Porte [26, pp. 87–88.] infer rather categorical conclusion that "the modalities of the *Ł*-system are very far from what everybody calls "possibility" and "necessity" and/or that the weak negation of CR (the logic of classical refutability) is very far from what everybody calls "negation"." As was noted above Łukasiewicz constructed his system so as to satisfy the minimal list of requirements for modal operators, which gives rise to a long history of critics of Łukasiewicz's modal logic *Ł*, but special discussing this logic lays outside the present research. The resent work by Font and Hajek [11] may be recommended to the reader to get an acquaintance of the topic. What concern the critics of the negation of classical refutability based on its similarity to modal operators, in the last years such similarity was not considered as something "negative" and was intensively studied. For example, K. Dosen in a series of works [8, 9, 10] treated the negation namely as a modal operator. The interrelation between $\mathbf{Le}$ and *Ł* stated in Proposition 3.5 will be used in the next section to suggest the way of generalization the notion of negation, which allow us to overcome the paradox of minimal logic mentioned in the introduction.

4. Paradox of minimal logic and generalized absurdity

Recall that by the paradox of minimal logic we mean the following property of **Lj**: from any contradiction one can infer in **Lj** an arbitrary negated formula, i.e., for any φ and ψ, we have

$$\varphi,\ \neg\varphi \vdash_{\mathbf{Lj}} \neg\psi.$$

Due to this property the negation have no sense in inconsistent **Lj**-theories, because all negated formulae are provable in them. This property is conditioned, on the one hand, by the axiom for implication

$$p \supset (q \supset p)$$

(which is called some times "the positive implication paradox") and, on the other hand, by the unrestricted law *reductio ad absurdum*,

$$(p \supset q) \supset ((p \supset \neg q) \supset \neg p),$$

saying that if a formula implies a contradiction, one can negate this formula without any restriction on the nature of this contradiction. Indeed, let T be some inconsistent **Lj**-theory and $\varphi, \neg\varphi \in T$. Then for an arbitrary formula ψ we can infer in T the implications $\psi \supset \varphi$ and $\psi \supset \neg\varphi$ using the positive paradox, and then we infer $\neg\psi$ applying *reductio ad absurdum.*

Of course, one can try to overcome the above mentioned paradox via rejecting the positive implication paradox and passing in this way in the field of relevant logics. But in our research we choose another way, we leave intuitionistic implication unchanged and consider possible ways of restricting *reductio ad absurdum.* It is worth to note that this idea were exploited many times in investigations in the field of paraconsistency. For example, such well-known paraconsistent logics as Sette's P^1 [28, 30] and Da Costa's C_1 [4] have *reductio ad absurdum* restricted to complex formulae, in case of P^1, and to formulae "behaving consistently", in case of C_1.

Our main idea is based on the correspondence stated in the previous section between the necessity operator L of Łukasiewicz's modal logic and the contradiction operator C, $C(\varphi) \rightleftharpoons \varphi\wedge\neg\varphi$, in the logic of classical refutability. The negation in **Le** as well as in an arbitrary extension of minimal logic can be defined via the constant "contradiction", $\neg\varphi \rightleftharpoons \varphi \supset\bot$, but also it can be defined via the contradiction operator $C(\varphi)$, $\neg\varphi \rightleftharpoons \varphi \supset C(\varphi)$. Clear that

$$\mathbf{Lj} \vdash \neg\varphi \equiv \varphi \supset C(\varphi).$$

This leads to the idea of defining a negation via the contradiction operator C considered as a primary logical connective, $\neg\varphi \rightleftharpoons \varphi \supset C(\varphi)$. It will be shown below that unrestricted *reductio ad absurdum* for negation defined as above exactly corresponds to some paradoxical properties of $C(\varphi)$ considered as a modal operator.

The results of the previous section allow to identify **Le** with a subsystem of Ł and we will have after such identification

$$Ł \vdash C(\varphi) \equiv L(\varphi) \text{ and } Ł \vdash \neg\varphi \equiv \varphi \supset L(\varphi).$$

We have also the following properties of modal operators L and M:

$$Ł \vdash L(\varphi \wedge \psi) \equiv L(\varphi) \wedge \psi \text{ and } Ł \vdash M(\varphi \vee \psi) \equiv M(\varphi) \vee \psi.$$

Each of these properties can be inferred from the other modulo classical logic and relation defining M through L and both of these properties have a paradoxical nature. Indeed, if we accept the conjunction of two conditions as necessary, it means from the intuitive point of view more then stating that one of these conditions is necessary and the second simply takes place. In a similar way, assuming that it is possible that one of the two conditions takes place should be weaker than the alternative of one of the conditions and the possibility of the other. Interesting that numerous authors criticized Łukasiewicz's modalities did not pay attention on these paradoxes.

For any formula φ, we have

$$\varphi \equiv \varphi \wedge (p \supset p) \text{ and } \varphi \equiv \varphi \vee \sim (p \supset p),$$

from which we infer using the above paradoxical properties,

$$Ł \vdash L(\varphi) \equiv \varphi \wedge L(p \supset p) \text{ and } Ł \vdash M(\varphi) \equiv \varphi \vee M(\sim (p \supset p)).$$

The first of these equivalences corresponds to the relation $C(\varphi) \equiv \varphi \wedge \bot \equiv \varphi \wedge \neg(p \supset p)$ for the contradiction operator in **Le**. Thus, the paradoxical properties envisaged above allow also to define the negation trough the constant "contradiction".

Now we consider the language $\mathcal{L}^C = \langle \vee, \wedge, \supset, C\rangle$, i.e., $\mathcal{L}^+$ extended by a contradiction operator C, and define a *C-logic* as a logic in this language containing axioms of **Lp** and the formula $C(p) \supset p$. We say that C is *extensional* in **L** if **L** is closed under the rule

$$\frac{\varphi \equiv \psi}{C(\varphi) \equiv C(\psi)}.$$

The following facts take place.

Lemma 4.6. *Let* **L** *be a* C*-logic and*

$$C(p \wedge q) \equiv C(p) \wedge q,\ C(p \wedge q) \equiv p \wedge C(q) \in \mathbf{L},$$

then C *is extensional in* **L**.

Proof. Let **L** satisfies the conditions of the lemma. Assume $\varphi \equiv \psi$. Using the first equivalence from the condition and the axiom $C(p) \supset p$ we have

$$C(\varphi \wedge \psi) \equiv C(\varphi) \wedge \psi \equiv C(\varphi) \wedge \varphi \equiv C(\varphi).$$

In a similar way, we infer from the equivalence $C(p \wedge q) \equiv p \wedge C(q)$ that $C(\varphi \wedge \psi) \equiv C(\psi)$, and, finally, $C(\varphi) \equiv C(\psi)$, which completes the proof.

Proposition 4.7. *Let* **L** *be a* C*-logic and we define* $\neg\varphi \rightleftharpoons \varphi \supset C(\varphi)$*, then*

$$C(p) \equiv p \wedge \neg p \in \mathbf{L}, \tag{1}$$

moreover,

$$(p \supset q) \supset ((p \supset \neg q) \supset \neg p) \in \mathbf{L}$$

iff

$$C(p \wedge q) \equiv C(p) \wedge q,\ C(p \wedge q) \equiv p \wedge C(q) \in \mathbf{L}.$$

Proof. From the well-known property $p \wedge (p \supset q) \equiv p \wedge q$ of the intuitionistic implication and the axiom $C(p) \supset p$ we immediately obtain the equivalence $C(p) \equiv p \wedge \neg p$, which justifies the name "contradiction operator" for C.

Assume that $(p \supset q) \supset ((p \supset \neg q) \supset \neg p) \in \mathbf{L}$ and show that in this case the negation may be defined via the constant, $\neg\varphi \equiv \varphi \supset \bot$, where $\bot \rightleftharpoons C(p_0 \supset p_0)$ for some fixed variable p_0. Substituting the definition of negation in *reductio ad absurdum* and exporting the second premise we obtain

$$((p \supset q) \wedge (p \supset (q \supset C(q)))) \supset (p \supset C(p)) \equiv$$

$$\equiv (p \supset (q \wedge (q \supset C(q))) \supset (p \supset C(p)).$$

The above equivalence follows from the tautology $p \supset (q \wedge r) \equiv (p \supset q) \wedge (p \supset q)$ which holds in **Lp**, and so in every C-logic. Due to (1) the latter is equivalent to

$$(p \supset C(q)) \supset (p \supset C(p)). \tag{2}$$

Substituting the tautology $p_0 \supset p_0$ for q we obtain

$$\mathbf{L} \vdash (p \supset C(p_0 \supset p_0)) \supset \neg p.$$

To state the inverse implication we substitute $p_0 \supset p_0$ for p and p for q in 2

$$((p_0 \supset p_0) \supset C(p)) \supset ((p_0 \supset p_0)) \supset C(p_0 \supset p_0)),$$

from which we have $C(p) \supset C(p_0 \supset p_0)$. Taking into account that the intuitionistic implication is increasing in the second argument we arrive at

$$\neg p \supset (p \supset C(p_0 \supset p_0))$$

and, finally, at

$$\neg p \equiv p \supset \bot.$$

Further, from this fact and the equivalence $C(p) \equiv p \wedge \neg p$ we infer as in **Lj** that $\mathbf{L} \vdash C(p) \equiv p \wedge \bot$. This equivalence implies in a trivial fashion the extensionality of C in $\mathbf{L}$, moreover,

$$C(p \wedge q) \equiv (p \wedge q) \wedge \bot \equiv (p \wedge \bot) \wedge q \equiv C(p) \wedge q.$$

The second desired equivalence easily follows from the just proved one by extensionality.

Now we assume that the equivalences $C(p \wedge q) \equiv C(p) \wedge q$ and $C(p \wedge q) \equiv p \wedge C(q)$ hold in $\mathbf{L}$. By Lemma 4.6 the operator C will be extensional in $\mathbf{L}$. Defining $\bot \rightleftharpoons C(p_0 \supset p_0)$ and using extensionality we obtain

$$C(p) \equiv C((p_0 \supset p_0) \wedge p) \equiv C(p_0 \supset p_0) \wedge p \equiv p \wedge \bot$$

and, further,

$$\neg p \equiv p \supset C(p) \equiv p \supset (p \wedge \bot) \equiv p \supset \bot.$$

The equivalence $\neg p \equiv p \supset \bot$ easily implies the unrestricted version of *reductio ad absurdum*.

The proposition is proved.

As was noted in the beginning of this section the paradox of minimal logic is conditioned by the "positive implication paradox" and the unrestricted version of *reductio ad absurdum*. In the definition of C-logics we leave the intuitionistic implication unchanged, therefore, a C-logic meets the paradox of minimal logic if and only if the negation defined via its operator C satisfy the unrestricted version of *reductio ad absurdum*. In this way, we infer from the last proposition

Corollary 4.8. *A C-logic $\mathbf{L}$ satisfy for all formulae φ and ψ the condition*

$$\varphi,\ \neg\varphi \vdash_{\mathbf{L}} \neg\psi$$

iff

$$C(p \wedge q) \equiv C(p) \wedge q,\ C(p \wedge q) \equiv p \wedge C(q) \in \mathbf{L}.$$

As we can see from this corollary deleting paradoxical property of C allow to avoid the paradox of minimal logic. In other words, if the operator C of contradiction has more natural properties from the point of view of modal logic, then the negation defined via C generates more satisfiable inference relation from the point of view of paraconsistent logic.

It was noted above that in the minimal logic the negation may be defined in two equivalent ways: via the constant $\bot$ and via the non-constant operator $C(\varphi)$ of contradiction. This simple observation leads naturally to the distinction between the contradiction operator and what we will call the *absurdity operator*, which we denote $A(\varphi)$. In case of $\mathbf{Lj}$, the absurdity operator is constant ($A(\varphi) \equiv \bot$). There are different intuitions behind these operators. By $C(\varphi)$ we mean a contradiction expressed in terms of φ, i.e., a simultaneous stating φ and its negation $\neg\varphi$, from which follows that $C(\varphi)$ should imply φ. By $A(\varphi)$ we mean such statement that reducing φ to it, i.e., proving the implication $\varphi \supset A(\varphi)$, is enough to negate φ. This understanding of $A(\varphi)$ do not assume that $A(\varphi)$ implies φ. The formulae $A(\varphi)$ and φ may be incomparable, as it takes place in case of $\mathbf{Lj}$, where the formulae φ and $\bot$ are incomparable in general case. In fact, the contradiction operator C can be considered as a special case of an absurdity operator satisfying an additional assumption that $C(\varphi) \supset \varphi$ for any φ. Moreover, if a negation of some system can be defined in terms of an absurdity operator, such negation itself can be taken as an absurdity operator, which will produce the same negation. To present these considerations in a precise form we define an A-language as a positive language $\mathcal{L}^+$ with additional unary operator A, $\mathcal{L}^A \rightleftharpoons \mathcal{L}^+ \cap \{A\}$, and an A-*logic* as a logic in the language $\mathcal{L}^A$ containing axioms of positive logic $\mathbf{Lp}$.

Proposition 4.9. *Let $\mathbf{L}$ be an A-logic. Define the contradiction operator C as $C(\varphi) \rightleftharpoons \varphi \wedge A(\varphi)$ and the negation $\neg$ as $\neg\varphi \rightleftharpoons \varphi \supset A(\varphi)$. The following* *formulae are provable in* **L***:*

$$\neg p \equiv p \supset \neg p \text{ and } \neg p \equiv p \supset C(p);$$

$$C(p) \equiv p \wedge \neg p \text{ and } C(p) \supset p;$$

$$C(p) \supset A(p) \text{ and } A(p) \supset \neg p.$$

All the formulae listed in the proposition can be easily inferred from the given definitions and axioms of **Lp**. For example, the equivalence $\neg p \equiv p \supset \neg p$ is an abbreviation for

$$p \supset A(p) \equiv p \supset (p \supset A(p))$$

and the latter is simply a partial case of the contraction law. This proposition shows that C defined in terms of the absurdity operator A can be really considered as a contradiction operator corresponding to the negation defined via A and that there is the whole "interval" of operators, from C to $\neg$ defining the same negation.

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