inference steps, so-called *extension steps*. The first extension step solves the initial goal $\neg p(\varepsilon!\alpha)$ with the substitution $\{\alpha \mapsto b * \beta\}$ but opens a new goal $q((\varepsilon!b)!c)$. This goal is solved by extension step 2, which in turn opens the goal $p(\varepsilon!\gamma)$. The latter is solved by the last inference. For more details see [17] and [20].

Definition 4.5. (Pools, hooks) A *hook* for a matrix M is a pair (p, Γ) where p is a partial path in an amplification M' of M and Γ is a sub-clause of a clause $\Gamma' \in M'$ such that $p \cap \Gamma = \emptyset$. The hook (p, Γ) will be denoted by $(p \perp \Gamma)$. The partial path p is called the *current path*. The elements of Γ are called *goals*. The *set of paths represented by the hook* $(p \perp \Gamma)$ is the set $\{p' \mid \exists L(p \cup \{L\} \subset p', p' \cap \Gamma = \{L\}), p'$ is a path through $M'\}$. It will be denoted by $Paths_{(p \perp \Gamma)}$. A hook $(p \perp \emptyset)$ will be called a *solved hook*, and a hook of the form $(\emptyset \perp \Gamma)$ is called an *initial hook*.

An inference step chooses a hook, removes it from the pool, and eventually produces some new hooks. The rules of a calculus describe how to construct new hooks from a chosen hook.

Definition 4.6. ($\mathscr{T}$-connection inference) Let $\mathcal{U}$ be a complete set of $\mathscr{T}$-connections and M a matrix. A *$\mathscr{T}$-connection inference* is an inference rule of the form

$$\frac{(p \perp \Gamma_0, L_0) \qquad \Gamma_1 \cup \{L_1\}, \ldots, \Gamma_n \cup \{L_n\}}{(p \perp \Gamma_0), (p, L_0 \perp \Gamma_1), \ldots, (p, L_0, \ldots, L_{n-1} \perp \Gamma_n)} \quad \sigma$$

where (1) $(p \perp \Gamma_0, L_0)$ is a hook, called the *chosen hook*, (2) if $0 < n$ then the clauses $\Gamma_1 \cup \{L_1\}, \ldots, \Gamma_n \cup \{L_n\}$ are copies of clauses from M, called the *extension clauses*, (3) σ is a substitution, (4) the hooks $(p \perp \Gamma_0)$, $(p, L_0 \perp \Gamma_1)$, …, $(p, L_0, \ldots, L_{n-1} \perp \Gamma_n)$ are called *new hooks* and (5) there exists a sub-path q of p such that $u \in \mathcal{U}$ and $\sigma(u)$ is $\mathscr{T}$-complementary for the partial path $u = q \cup \{L_0, \ldots, L_n\}$. A $\mathscr{T}$-connection inference is called an *extension step* if $n \neq 0$ and a *reduction step* else.

Example 4.6. Let us return to the sample derivation in Figure 5. In that example an equational theory $\mathscr{T}$ has been assumed which contains the equation $(\varepsilon!\alpha)!\beta = \varepsilon!(\alpha * \beta)$. Let $\mathcal{U}$ be the set of all unordered pairs of literals $\{p(t_1, \ldots, t_n), \neg p(t'_1, \ldots, t'_n)\}$ such that for each i with $1 \leq i \leq n$ the terms t_i and t'_i are $\mathscr{T}$-unifiable. A theory extension is an inference rule of the form

$$\frac{(p \perp L_0, \Gamma_0) \qquad L_1, \Gamma_1}{(p \perp \Gamma_0),\ (p, L_0 \perp \Gamma_1)} \quad \sigma$$

where (1) L_1, Γ_1 is a copy of a clause from M, called the *extension clause* and (2) for $u = \{L_0, L_1\}$ holds $u \in \mathcal{U}$ and $\sigma(u)$ is $\mathcal{T}$-complementary. A theory reduction rule has the form

$$\frac{(p \perp L_0, \Gamma_0)}{(p \perp \Gamma_0)}\ \sigma$$

where for some literal $L_1 \in p$ and $u = \{L_0, L_1\}$ holds $u \in \mathcal{U}$ and $\sigma(u)$ is theory complementary.

Definition 4.7. (Rule application) A rule

$$\frac{h \qquad \Gamma_1, \ldots, \Gamma_n}{H}\ \sigma$$

may be applied to a pool P if $h \in P$. The new pool is obtained from P by removing h, then adjoining those hooks from H which are not solved and finally applying the substitution σ to the resulting pool. The clause copies used in an inference within a derivation must have always a set of new variables, i.e. those not occurring already in the pool. Moreover if $u \in \mathcal{U}$ is the $\mathcal{T}$-connection chosen in the considered rule application then the variables from $\mathrm{Var}(\sigma) \setminus \mathrm{Var}(u)$ must not occur in P.

An *initial pool* in a derivation consists of a single initial hook. Now a *derivation* may be defined as a sequence of rule applications which starts from an initial pool. A derivation is called *ground* if the unifier in every $\mathcal{T}$-connection step is empty. A derivation is *successful* if its last element is the empty pool. The calculus is sound, because in every state of a derivation the pool represents all paths, such that there still have to be found theory connections spanning them.

Proposition 4.4. (Soundness) *The theory connection calculus is sound.*

The completeness proof consists of the steps Herbrand theorem, ground completeness and lifting lemma. The Herbrand theorem (4.1) and the lifting lemma rely on the completeness of a given set of theory connections $\mathcal{U}$ and the solvability of the theory unification problem in $\mathcal{U}$. The proof of the ground completeness relies on the properties of minimal spanning matings. The following result may be found already in [20].

Theorem 4.2. (General Completeness theorem) *Suppose that for a theory $\mathcal{T}$ and a query language $\mathcal{Q}$ there is given a decidable set $\mathcal{U}$ of $\mathcal{T}$-connections which is $\mathcal{T}$-complete w.r.t. $\mathcal{Q}$ and the $\mathcal{T}$-unification problem in $\mathcal{U}$ is solvable.*

Then for every $\mathscr{T}$-unsatisfiable query from $\mathcal{Q}$ exists a clause $\Gamma \in \mathcal{Q}$ and a successful derivation starting from the initial pool $\{(\perp \Gamma)\}$ such that in each inference according to Definition 4.6 for the chosen connection u holds $u \in \mathcal{U}$ and the chosen T-unifier σ is an element of the complete set of $\mathscr{T}$-unifiers S_u for u.

For an outline of the completeness proof we conclude from the Herbrand theorem (4.1) that for each $\mathscr{T}$-unsatisfiable matrix $M \in \mathcal{Q}$ exist an amplification M', a ground substitution σ and a $\mathcal{U}$-mating U spanning M' such that σ is a minimal $\mathscr{T}$-unifier of u for each $u \in U$. We assume U to be minimal with respect to inclusion and construct a derivation satisfying the following invariant: There exist a minimal subset $U' \subseteq U$ and a substitution σ' such that for each unsolved goal L in a hook $(p \perp L, \Gamma)$ exist $u \in U'$, σ'' and σ''' such that $L \in \sigma'(u)$, σ'' is a minimal $\mathscr{T}$-unifier, $\sigma'' \in S_{\sigma'(u)}$, and $\sigma =_{\mathsf{var}(M')} \sigma'\sigma''\sigma'''$. This general theorem will be specialized to the case of hybrid theories. In Section 5 we introduce sufficient criteria for obtaining a complete set of theory connections for a hybrid theory if those are given for its constituents. The criteria can be applied to the target logic of the algebraic translation of multi-modal logic and of extended multi-modal logic of [9] (cf. Section 3.1).

5. Combining theories

In Section 3 we have discussed examples justifying the treatment of the background reasoner as a hybrid system itself. Let us now forge precise notions from the observations made for the target logic of the algebraic translation of multi-modal reasoning. Our goal is to construct a $\mathscr{T} \cup \Re$-reasoner from a $\mathscr{T}$-reasoner and a $\Re$-reasoner. A formula will be considered as consisting of a $\mathscr{T}$-layer and an $\Re$-layer. The intended $\mathscr{T} \cup \Re$-reasoner should try to find a $\mathcal{U}_{\mathscr{T}}$-connection if the current goal is in the $\mathscr{T}$-layer and a $\mathcal{U}_{\Re}$-connection if the current goal is in the $\Re$-layer. We formulate sufficient conditions such $\mathcal{U}_{\mathscr{T}} \cup \mathcal{U}_{\Re}$ is a complete set of T$\cup\Re$-connections for $\mathcal{Q}$ if so are $\mathcal{U}_{\mathscr{T}}$ for $\mathcal{Q}_{\mathscr{T}}$ and $\mathcal{U}_{\Re}$ for $\mathcal{Q}_{\Re}$. Moreover the theory unification problems in both $\mathcal{U}_{\mathscr{T}}$ and $\mathcal{U}_{\Re}$ should not interfere. The last condition will us allow to use just the unification algorithms for the connections belonging to one of both layers without change.

Definition 5.1. Let a theory be given by its sub-theories $\mathscr{T}$ and $\Re$ which are formulated within the signatures Σ and Δ respectively. Then we say that $\mathscr{T}$ and $\Re$ *form a hybrid theory* in the union $\Sigma \cup \Delta$ of both signatures.

Definition 5.2. Let the theories $\mathscr{T}$ and $\Re$ form a hybrid theory in the union $\Sigma \cup \Delta$ of their signatures and let $\mathcal{Q}$ be a query language formulated in a signature which contains $\Sigma \cup \Delta$.

Every clause C in a matrix $M \in \mathcal{Q}$ contains then two sub-clauses $C_{\mathscr{T}}$ and $C_{\Re}$ consisting of literals L expressed in signature Σ (respectively L' expressed in signature Δ). The set of nonempty sub-clauses $C_{\mathscr{T}}$ of M will be called the $\mathscr{T}$-layer of M. Analogously will be defined the $\Re$-layer of M. By $\mathcal{Q}_{\mathscr{T}}$ (analogously $\mathcal{Q}_{\Re}$) will be denoted the set of all matrices being the $\mathscr{T}$-layer (respectively the $\Re$-layer) of a query from $\mathcal{Q}$. $\mathcal{Q}_{\mathscr{T}}$ (analogously $\mathcal{Q}_{\Re}$) will be called the *$\mathscr{T}$-layer* (respectively the $\Re$-layer) of $\mathcal{Q}$. If for a matrix $M \in \mathcal{Q}$ every of its clauses is the union of its $\mathscr{T}$- and $\Re$-layers then M will be called *covered by its $\mathscr{T}$- and $\Re$-layers*. If every matrix $M \in \mathcal{Q}$ is covered by its $\mathscr{T}$- and $\Re$-layers then query language $\mathcal{Q}$ is said to be *covered by its $\mathscr{T}$- and $\Re$-layers*.

Example 5.1. In the wise men puzzle in Figure 2 signatures Σ of the $\mathscr{T}$-layer and Δ of the $\Re$-layer share the function symbols $!$, ε, f_1, f_2, a and b. Σ contains w as the single predicate symbol, Δ contains k and the equality symbol $=$. The target language of the algebraic translation of multi-modal logic is covered by its $\mathscr{T}$- and $\Re$-layers. Since the sets of predicate symbols of the $\mathscr{T}$-layer and the $\Re$-layer are disjoint, for each literal L the sets of $\mathscr{T}$- and of $\Re$-connections L might belong to are disjoint.

Example 5.2. In the safe puzzle from Figure 4 the signatures Σ of the $\mathscr{T}$-layer and Δ of the $\Re$-layer share the equality symbol $=$ and the function symbols ϕ, ψ, h, i, n, s, d, p, joe, 1 and ε. Thus, the sets of predicate symbols for the $\mathscr{T}$-layer and the $\Re$-layer are not disjoint. Σ contains moreover w, c and o and Δ contains k, u, a and as as predicate symbols. The target language of the algebraic translation of extended multi-modal logic is covered by its $\mathscr{T}$- and $\Re$-layers. Again we may show that for each literal L the sets of $\mathscr{T}$-connections and of $\Re$-connections L can belong to are disjoint. Definition 5.3 introduces a notion for this property.

Definition 5.3. Let $\mathscr{T}$ and $\Re$ form a hybrid theory in the union $\Sigma \cup \Delta$ of signatures. Let $\mathcal{Q}$ be a query language formulated in a signature containing both signatures Σ and Δ. Moreover, let $\mathcal{U}_{\mathscr{T}}$ and $\mathcal{U}_{\Re}$ be sets of $\mathscr{T}$-connections and of $\Re$-connections. We say that $\mathcal{U}_{\mathscr{T}}$ and $\mathcal{U}_{\Re}$ are *separated w.r.t.* Q if and only if there does not exist connections $u \in \mathcal{U}_{\mathscr{T}}$ and $u' \in \mathcal{U}_{\Re}$ with $\emptyset \neq u \cap u'$.

The following propositions 5.1 and 5.2 give sufficient criteria for the theory completeness of the union of sets of theory connections that are theory complete with respect to the constituent sub-theories of a hybrid theory. The case of the target logic of the multi-modal logic will be covered by Proposition 5.1. The criterion Proposition 5.2 covers the case of the target logic of the algebraic translation of extended multi-modal logic.

Definition 5.4. Let M be a set of instances of clauses and U a mating in M. For every literal L in M we define the *set R_L of clauses reachable from L via U* as the least set being closed with respect to the following condition: If there exists a connection $u \in U$ such that one of the literals of u is L or a literal in a clause being element of R_L then also any clause containing a literal of u different from L belongs to R_L.

Proposition 5.1. *Let theories $\mathscr{T}$ and $\Re$ be expressed in the signatures Σ and Δ respectively form a hybrid theory such that $\mathscr{T} \cup \Re$ is consistent. The query language $\mathcal{Q}$ is formulated in the union $\Sigma \cup \Delta$ of signatures. Moreover suppose that:*

(1) *The sets of $\mathscr{T}$-connections $\mathcal{U}_{\mathscr{T}}$ and of $\Re$-connections $\mathcal{U}_{\Re}$ are complete w.r.t. $\mathcal{Q}_{\mathscr{T}}$ and $\mathcal{Q}_{\Re}$ respectively.*
(2) *In $\mathcal{Q}$ equality literals occur only negative.*
(3) *In both theories positive equality literals may occur only within conditional equations.*
(4) *The sets of predicate symbols occurring in $\mathscr{T} \cup \mathcal{Q}_{\mathscr{T}}$ and $\Re \cup \mathcal{Q}_{\Re}$ are disjoint.*
(5) *If equality occurs in $\mathscr{T} \cup \Re$ then let $\mathscr{T}_1$ be that of the sub-theories $\mathscr{T}$ and $\Re$ that does not contain equality and $\mathcal{U}_1$ be the set of theory connections for that sub-theory. Moreover let $\mathscr{E}$ be the set of equational axioms in $\mathscr{T} \cup \Re$. For every $u \in \mathcal{U}_1$ and substitution σ holds $\mathscr{E} \cup \mathscr{T}_1 \vDash \sigma(\bigvee \bar{u})$ if and only if $\mathscr{T}_1 \vDash \sigma(\bigvee \bar{u})$.*

Then the sets of $\mathscr{T}$-connections $\mathcal{U}_{\mathscr{T}}$ and $\Re$-connections $\mathcal{U}_{\Re}$ are separated with respect to $\mathcal{Q}$ and $\mathcal{U}_{\mathscr{T}} \cup \mathcal{U}_{\Re}$ is $\mathscr{T}, \Re$-complete with respect to $\mathcal{Q}$.

Proof. Let us suppose that theories $\mathscr{T}$ and $\Re$, signatures Σ and Δ, query language $\mathcal{Q}$ and the sets of $\mathscr{T}$-connections $\mathcal{U}_{\mathscr{T}}$ and of $\Re$-connections $\mathcal{U}_{\Re}$ satisfy the assumptions of the proposition. In order to show that $\mathcal{U}_{\mathscr{T}}$ and $\mathcal{U}_{\Re}$ are separated with respect to $\mathcal{Q}$ it is sufficient to observe that the sets of predicate symbols occurring in $\mathscr{T} \cup \mathcal{Q}_{\mathscr{T}}$ and $\Re \cup \mathcal{Q}_{\Re}$ are disjoint. In order to show that $\mathcal{U}_{\mathscr{T}} \cup \mathcal{U}_{\Re}$ is $\mathscr{T}, \Re$-complete with respect to $\mathcal{Q}$ we show first of all that $\mathcal{U}_{\mathscr{T}} \cup \mathcal{U}_{\Re}$ has property 2.1 formulated in Definition 4.3. Let

p be a $\mathscr{T},\Re$-complementary ground path. We have to show there exists a sub-path such that $u \in \mathcal{U}_{\mathscr{T}} \cup \mathcal{U}_{\Re}$. We consider p as a set of unit clauses. By the compactness theorem for first-order logic there exists a finite set M of instances of clauses of $\mathscr{T}$ and of $\Re$ and a minimal mating U spanning $M \cup p$. Let u be the multi-set of all literals of p which are element of a connection in U. Then u is not empty because of the consistency of $\mathscr{T} \cup \Re$. Because the sets of predicate symbols occurring in $\mathscr{T} \cup \mathcal{Q}_{\mathscr{T}}$ and $\Re \cup \mathcal{Q}_{\Re}$ are disjoint either for every connection $u' \in U$ holds $u \in \mathcal{Q}_{\Sigma}$ or for every connection $u' \in U$ holds $u \in \mathcal{Q}_{\Delta}$. Therefore, u is either element of $\mathcal{Q}_{\mathscr{T}}$ or of $\mathcal{Q}_{\Re}$. If $u \in \mathcal{Q}_{\mathscr{T}}$ (the case $u \in \mathcal{Q}_{\Re}$ may be treated analogously) then there exists $u'' \in \mathcal{U}_{\mathscr{T}}$ such that $u'' \subseteq u$, and therefore $u'' \subseteq p$, because $\mathcal{U}_{\mathscr{T}}$ is $\mathscr{T}$-complete with respect to $\mathcal{Q}_{\mathscr{T}}$. Both $\mathcal{U}_{\mathscr{T}}$ and $\mathcal{U}_{\Re}$ satisfy condition 2.2 of Definition 4.3. Therefore also $\mathcal{U}_{\mathscr{T}} \cup \mathcal{U}_{\Re}$ has this property. □

Example 5.3. Let us observe that in a matrix belonging to the target language of the algebraic translation of multi-modal logic theory connections either are in the non-sort part, i.e. those discussed in Example 4.1, or in the sort part, i.e. those discussed in Example 4.2. This is obvious because both parts of the hybrid theory are expressed by use of disjoint sub-sets of predicate symbols and equality does not occur in the query language. Therefore, in order to obtain a complete set of theory connections for the hybrid theory consisting of $\mathscr{T}$ and $\Re$ it is sufficient to take just the union of the complete sets of theory connections $\mathcal{U}_{\mathscr{T}}$ and $\mathcal{U}_{\Re}$.

Proposition 5.2. *Let theories $\mathscr{T}$ and $\Re$ be expressed in the signatures Σ and Δ respectively form a hybrid theory such that $\mathscr{T} \cup \Re$ is consistent. The query language $\mathcal{Q}$ is formulated in the union $\Sigma \cup \Delta$ of signatures. Moreover suppose that:*

(1) *The sets of $\mathscr{T}$-connections $\mathcal{U}_{\mathscr{T}}$ and of $\Re$-connections $\mathcal{U}_{\Re}$ are complete w.r.t. $\mathcal{Q}_{\mathscr{T}}$ and $\mathcal{Q}_{\Re}$ respectively.*

(2) *In $\mathcal{Q}$ equality literals may occur only negative.*

(3) *In both theories $\mathscr{T}$ and $\Re$ positive equality literals may occur only within conditional equations.*

(4) *The sets of non-equational predicate symbols occurring in $\mathscr{T} \cup \mathcal{Q}_{\mathscr{T}}$ and $\Re \cup \mathcal{Q}_{\Re}$ are disjoint.*

(5) *If $\mathscr{T}_{=+}$ (and $\Re_{=+}$) are the sets of non-negative equational clauses in $\mathscr{T}$ (and $\Re$ respectively) then hold $\mathscr{T} \models \Re_{=+}$ and $\Re \models \mathscr{T}_{=+}$.*

Then the set $\mathcal{U}_{\mathscr{T}} \cup \mathcal{U}_{\Re}$ is $\mathscr{T},\Re$-complete with respect to $\mathcal{Q}$.

Proof. Analogously to the proof of Proposition 5.1 it is sufficient to show that for every $\mathscr{T},\Re$-complementary ground path $p \in \mathcal{Q}$ exists $u \in \mathcal{U}_{\mathscr{T}} \cup \mathcal{U}_{\Re}$ such that $u \subseteq p$. Let p be such path. We consider p as a set of unit clauses. By the compactness theorem for first-order logic there exists a finite set M of ground instances of clauses of $\mathscr{T}$ and $\Re$ and a minimal mating U spanning $M \cup p$. Let u' be the multi-set of all literals of p which are element of a connection in U. Then u' is not empty because of the consistency of $\mathscr{T} \cup \Re$. In order to complete the proof it will be sufficient to show that u' is $\mathscr{T}$-unsatisfiable or $\Re$-unsatisfiable. We prepare this proof by the following three claims.

Claim 1: Let L be a negative equational literal in $M \cup p$. Then every clause reachable from L via U is a conditional equation and $R_L \models \neg L$.

Proof: Immediately from assumption (3) follows that R_L consists of conditional equations only. The minimality of U ensures that U also is spanning $R_L \cup \bar{L}$. Therefore $R_L \models \neg L$.

Claim 2: Under the assumptions of claim 1 holds: If $L \in \mathcal{Q}_{\mathscr{T}}$ — the opposite case may be treated by symmetry — then every conditional equation e reachable from L and being element of $\Re$ may be substituted by a set of clauses $R'_e \subseteq \mathscr{T}$ such that $R'_e \models e$.

Proof: Follows immediately from assumption (5).

Claim 3: Let $L \in p$ be an non-equational literal. If $L \in \mathcal{Q}_{\mathscr{T}}$ — again the opposite case may be treated by symmetry — then every clause $\Gamma \in (\mathscr{T} \cup \Re) \cap R_L$ containing non-equational literals satisfies $\Gamma \in \mathscr{T}$.

Proof: From the assumptions (2), (3) and (4) and follows that the predicate symbol of every non-equational literal in any clause reachable from L belongs to Σ. The assumptions (3) and (4) are important for this conclusion because they ensure that any clause reachable from an equational literal in a non-equational clause reachable from L is a conditional equation. We complete the proof by the following case analysis.

Case 1: The sub-path u' of p contains an equality literal L. By assumption (2) L is negative and by claim 1 all clauses in R_L are conditional equations and $\mathcal{R}_L \models \neg L$. Therefore $u' = \{L\}$ because U is a minimal mating spanning $M \cup p$. Suppose that R_L contains clauses being instances of clauses from $\mathscr{T}$ (the case symmetric case may be proved analogously). Then according to assumption (4) every element e of R_L being not an instance of a clause in $\mathscr{T}$ may be substituted by a set of conditional equations implying e. Therefore

$\mathcal{T} \models \bar{u}'$. Because $\mathcal{U}_\mathcal{T}$ is $\mathcal{T}$-complete with respect to $\mathcal{Q}_\mathcal{T}$ there exists a $\mathcal{T}$-connection $u'' \subseteq u'$ and therefore $u'' \subseteq p$.

Case 2: The subpath u' of p does not contain any equality literal L. We suppose that u' contains a $\mathcal{T}$-layer literal — the opposite case may be treated by symmetry reasoning. Then by claim 3 every non-equational clause in $R_L \cap (\mathcal{T} \cup \Re)$ is element of $\mathcal{T}$. By claim 2 every equational clause e being element of $R_L \setminus \mathcal{T}$ may be substituted by a finite subset $R'_e \subseteq \mathcal{T}$ such that $R'_e \models e$. Therefore $\mathcal{T} \models \bigvee \bar{u}'$. □

Now we discuss briefly the unification problem in sets of hybrid theory connections. We restrict our attention to the case that for given theories $\mathcal{T}$ and $\Re$ a complete set of theory connections is given by the union of sets of theory connections that are complete with respect to the respective theories. What we have in mind is that unification of a theory connection u is either $\mathcal{T}$-unification if u is a $\mathcal{T}$-connection or $\Re$-unification otherwise. This leads to the notion of non-interfering unification problems.

Definition 5.5. Let $\mathcal{U}_\Re$ and $\mathcal{U}_\mathcal{T}$ be sets of theory connections for the components of a hybrid theory $\mathcal{T}, \Re$. We say that *the unification problems in $\mathcal{U}_\Re$ and $\mathcal{U}_\mathcal{T}$ do not interfere* if and only if

(1) for every $u \in \mathcal{U}_\mathcal{T}$ and for every substitution σ holds: σ is a $\mathcal{T}$-unifier of u if and only if σ is $\mathcal{T}, \Re$-unifier of u and

(2) for every $u \in \mathcal{U}_\Re$ and for every substitution σ holds: σ is a $\Re$-unifier of u if and only if σ is $\mathcal{T}, \Re$-unifier of u.

Let $\mathcal{U}_\mathcal{T} \cup \mathcal{U}_\Re$ be the set of theory connections discussed in Section 3.1 for the target logic of the algebraic translation of multi-modal logic. Then the unification problems in $\mathcal{U}_\mathcal{T}$ and $\mathcal{U}_\Re$ do not interfere. Let $\mathcal{U}_\mathcal{T}$ and $\mathcal{U}_\Re$ be the sets of theory connections discussed in Section 3.2 for the sub-theories $\mathcal{T}$ and $\Re$ of the target logic of the algebraic translation of extended multi-modal logic. Then the unification problems in $\mathcal{U}_\mathcal{T}$ and $\mathcal{U}_\Re$ do not interfere.

Proposition 5.3. *Let theories $\mathcal{T}$ and $\Re$, which are expressed in the signatures Σ and Δ respectively, form a hybrid theory, such that $\mathcal{T} \cup \Re$ is consistent. The query language $\mathcal{Q}$ is formulated in the union $\Sigma \cup \Delta$ of signatures. Moreover suppose that the assumptions (1)–(5) of Proposition 5.1 (resp. 5.2) are satisfied. Then the unification problems in $\mathcal{U}_\mathcal{T}$ and $\mathcal{U}_\Re$ do not interfere.*

Proof. In the non-trivial direction of the equivalence to be proved we have to show that every $\mathscr{T} \cup \Re$-unifier of a $\mathscr{T}$-connection $u \in \mathcal{U}_{\mathscr{T}}$ is a $\mathscr{T}$-unifier of u and that every $\mathscr{T} \cup \Re$-unifier of a $\Re$-connection $u \in \mathcal{U}_{\Re}$ is a $\Re$-unifier of u. The latter claim is satisfied because $\mathscr{T}$ and $\Re$ have no common predicate symbols and $\Re$ does not contain the equality sign. The former claim follows from assumption (5). □

Now a completeness theorem for hybrid theories may be proved.

Theorem 5.1. (Completeness theorem for hybrid theories) *Let $\mathcal{Q}$ be a query language expressed in a signature containing Σ and Δ. Moreover, let $\mathcal{Q}_{\Re}$ and $\mathcal{Q}_{\mathscr{T}}$ be the $\Re$-layer and $\mathscr{T}$-layer of $\mathcal{Q}$ respectively. Let $\mathcal{U}_{\Re}$ and $\mathcal{U}_{\mathscr{T}}$ be complete sets of $\Re$-connections and $\mathscr{T}$-connections which satisfy the assumptions either of Proposition 5.1 or 5.2. Then for every $\mathscr{T}, \Re$-unsatisfiable query $M \in \mathcal{Q}$ exists a clause $\Gamma \in M$ and a successful derivation starting from the initial pool $\{(\perp \Gamma)\}$ such that in each inference according to Definition 4.6 for the chosen connection u holds either $u \in U_{\Re}$ or $u \in \mathcal{U}_{\mathscr{T}}$ and for the chosen theory unifier $\sigma \in S_u$, with S_u being the set of $\mathscr{T}$-unifiers or, respectively, $\Re$-unifiers.*

Proof. Due to Proposition 5.1 and 5.2 the set of $\mathscr{T}, \Re$-connections $\mathcal{U}_{\Re} \cup \mathcal{U}_{\mathscr{T}}$ is $\mathscr{T}, \Re$-complete w.r.t. query language $\mathcal{Q}$. Due to Proposition 5.3 the unification problem in $\mathcal{U}_{\Re} \cup \mathcal{U}_{\mathscr{T}}$ is solvable and applying the $\mathscr{T}$-unification procedure to $\mathcal{U}_{\mathscr{T}}$-connections and the $\Re$-unification procedure to $\mathcal{U}_{\Re}$-connections provides a solution to the $\mathcal{U}_{\Re} \cup \mathcal{U}_{\mathscr{T}}$-unification problem. Thus the assumptions of Theorem 4.2 are satisfied and the calculus for the hybrid theory is complete. □

Let $\mathcal{U}_{\mathscr{T}}$ and $\mathcal{U}_{\Re}$ be either the set of theory connections discussed in Section 3.1 for the target logic of the algebraic translation of multi-modal logic or those discussed in Section 3.2 for the sub-theories $\mathscr{T}$ and $\Re$ of the target logic of the algebraic translation of extended multi-modal logic. Then for both cases we obtain a complete calculi instantiating the theory pool calculus (cf. Section 4.3) as a corollary of Theorem 5.1.

6. Concluding remarks

A prover for multi-modal logic has been implemented by a joint effort of research groups in Leipzig and Caen. We used the calculi description interface

CaPrI of the PTTP-prover ProCom [16]. The algebraic translation of Francoise Debart and Patrice Enjalbert from multi-modal logic to a language of constrained clauses has been implemented by Zoltán Rigó [21]. The translation generates a constraint theory that provides information about the interaction between modalities, the properties of the occurring modalities and the dependencies introduced by Skolemization. For reasoning in the non-constraint part of a matrix being element of the target language an A1-unification algorithm due to Francoise Debart and Patrice Enjalbert [8] is used. The algorithm has been tuned for this application. The used implementation is due to Gilbert Boyreau [5]. ProCom and his interface has been implemented by Gerd Neugebauer. He also integrated constraint reasoning into ProCom.

We examined the algebraic translation of multi-modal logic into a fragment of first-order logic. To the target of this translation we applied a general framework which allows to build-in theories into provers which are based on the connection method. For this purpose we introduced the notion of a hybrid theory. We obtained a completeness result for a connection method based calculus dealing with hybrid theories. A brief overview about an implementation has been given. Ongoing research considers the combination of theories given syntactically with those given semantically.

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