Logic and Logical Philosophy
Volume 2 (1994), 103–115

Yaroslav Shramko

RELEVANT PROPERTIES

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Received September 22, 1994

1. Introduction*

I would like to start my paper with the following statement of Barry Smith: "Relevance logic has become ontologically fertile." ([3, p. 45]) This estimation has been made just in connection with the problems of logical analysis of properties and relations. Apparently, J. M. Dunn was the first who tried to apply the relevance logic to analysis of these problems. In [2] he proposed to distinguish between relevant and irrelevant predications. As an example he considers the following pair of statements:

(1) Socrates is such that he is wise.
(2) Alcibiades is such that Socrates is wise.[1]

Dunn maintains that in the first case we deal with RELEVANT predication whereas the second case is a glaring example of IRRELEVANT predication. Dunn demonstrates that, using an apparatus of relevant logic, it is possible to realize a formal distinction between statements like (1) and (2). His analysis rests on some important notions of the lambda-calculus, too. Below I will use his results, but now let me dwell at greater length on the informal background of the problem.

2. Towards a Classification of Properties

There are many ways to classify properties. For example, one can make a division of properties into essential and accidental ones as well as into intrinsic and external ones. Dunn, however, remarks that "the distinction between necessary and nonnecessary properties is not the only way to sort out those properties which have an intimate life with an object from those which do not" ([2, p. 445]).

Let us consider a classical example and ask, whether

(3) Caesar is a prime number,

[*] Section headings introduced by the editors.

[1] Originally Dunn's paper has been published in Journal of Philosophical Logic (see [2]). In that version the statement (2) was: *Reagan is such that Socrates is wise*. The paper has been reproduced (with small alterations) in the second volume of *Entailment*, p. 445–472 (see [1]). In my paper I will refer just to this latter version.

or not? Apparently it is hardly appropriate to discuss the «problem» whether the property of *being a prime number* is necessary or nonnecessary to such an «object» as Caesar. This property simply HAS NOTHING TO DO with Caesar, in other words, one can say that it is IRRELEVANT to it. Nowadays the notion of RELEVANCE is more widely adopted in various fields. One of these fields is the theory of logical entailment. Apparently the demand that premises should be relevant to the conclusion is well-founded indeed.

Also the following demand looks very natural: IF WE ASCRIBE A PROPERTY TO SOME OBJECT, THIS PROPERTY SHOULD BE RELEVANT TO THIS OBJECT.

Intuitively many embarrassing and even paradoxical cases well-known to researches in the area of semantics and logical analysis of natural languages can be interpreted in terms of *relevance* and *irrelevance*. For example, considering the property

(4) *has stopped beating his father*

with respect to a person who has never beaten his father, one can maintain that this property is irrelevant to such a person. Another well-known example:

(5) The present King of France is (or is not) bald.

But it seems to me that it is not only irrelevance between objects and properties that is a factor in the above examples. All these examples illustrate the situation of so-called TRUTH-VALUE GAPS, where we find it difficult to ascribe some property to an object. However, a quite different situation is possible: an object REALLY HAS some property, but this property is still irrelevant to this object.

Consider the following situation: Someone, let us say, John (he is 50 years of age) lives all his life in London, and has never gone abroad. Suppose, he makes the following statement:

(6) I have not been in Berlin for 60 years.

This statement is obviously true – John REALLY has not been in Berlin during the last 60 years. But intuitively we are prone to consider the property of *(not) having been in Berlin for last 60 years* as irrelevant to a person who has never been in Berlin at all and is 50 years old. Moreover, when we consider the statement

(7) John has not been in Berlin for a long time.

we can decide that here we have the same irrelevance as in the case (6). Nevertheless (7) is true as well.

Thus, we can draw the following preliminary conclusion: relevance does not depend on truth, and relevance has to be defined and described without using the notion of *truth*. In other words:

Proposition Fa may be true, however the property F can be irrelevant to the object a, but also proposition Fa may be false, whereas the property F is relevant to the object a.

The next observation concerns the structure of the notion of *relevance*. Relevance itself is not a property, it is a RELATION, moreover, it is a TERNARY relation. Conformably to the subject of my paper, relevance can be interpreted as a relation between property, object and CONTEXT. A given property F is not simply relevant (or irrelevant) to a given object a, but is relevant (or irrelevant) to the object a IN A GIVEN CONTEXT α. From my point of view, the role of context is very important for the analysis of the notion of relevance. The same property can be relevant to an object a in one context, but irrelevant to this object in another context. Consider the following argument:

(8) All that is green is gratifying to the eye.
This picture is green.
Hence, this picture is gratifying to the eye.

Modern psychological investigations confirm the fact that the green colour is gratifying to the eye. But a picture can be green, and yet at the same time not gratifying to the eye (e.g. due to its repulsive content). The point is that IN THE CONTEXT OF ARGUMENT (8) the property *green* is irrelevant to such an object as a picture. But one can easily imagine a context when just COLOUR is relevant to some picture.

Thus, the problem of relevance conformably to properties is not quite an ontological problem. The notion of relevance itself has a strongly pronounced epistemological and pragmatic shade. As Dunn has it:

> It may well be that the relation of relevant implication is not part of the objective ontological universe, but rather is in some fundamental sense subjective and mind-dependent. Relevance may indeed only be a rough-and-ready way of dividing up the items in the universe according to human concerns [...]. ([1, p. 446])

Nevertheless, constructing a natural and effective formal apparatus that would allow us to distinguish between relevant and irrelevant properties in one context or another, could give us a possibility to elucidate some ontological problems connected with the analysis of properties and relations.

3. Making Use of λ-calculus

Let me return to Dunn's analysis of predication. Dunn considers the lambda--conversion principle in connection with the investigation of properties. Using lambda abstraction, any formula Fx can be made into a predicate:

$$\lambda x\, Fx.$$

(Dunn reads this as *the property of being (an x such that x is) F*.)

According to the lambda-conversion principle we have:

$$(\lambda x\, Fx)a \;\Leftrightarrow\; Fa.$$

Let a be Socrates and F be the property *wise*. Then $\lambda x\, Fx$ means *the property of being an x such that x is wise*, and $(\lambda x\, Fx)a$ means *Socrates is such that he is wise*.

But it is supposed in general that lambda-abstraction can be applied even to formulas which do not contain free occurrences of x. Then we have

$$\lambda x\, A,$$

where A is a sentence. (Dunn reads this as *the property ascribed to x is saying that A*.]

Lambda-conversion for closed formulas is simply

$$(\lambda x\, A)a \;\Leftrightarrow\; A.$$

Now let a be Alcibiades and A be the sentence *Socrates is wise*. Then $(\lambda x\, A)a$ means *Alcibiades is such that Socrates is wise*.

Thus, the classical lambda calculus gives no tools for distinguishing between (1) and (2). But sentence (2) seems to be at least STRANGE. Dunn shows that here we have a case of irrelevant predication, since the validity of (2) depends on the so-called *Positive Paradox of Relevance*:

$$A \vdash B \rightarrow A,$$

whereas the validity of (1) does not. Indeed, there is a strict analogy between (1) and (2) and the following statements:

(1′) If anyone is Socrates, then he is wise.
(2′) If anyone is Alcibiades, then Socrates is wise.

We obtain (1′) by means of the following argument:

(1″) Socrates is wise. Therefore, if $x =$ Socrates then x is wise.

which is an instance of Indiscernability:

$$Fa \vdash x = a \rightarrow Fx,$$

But the corresponding argument for (2′) is a clear instance of the Positive Paradox:

(2″) Socrates is wise.
Therefore, if $x =$ Alcibiades then Socrates is wise.

Dunn remarks that an attempt to distinguish between (1) and (2) by means of *restriction* saying that formation of a lambda-expression $\lambda x\, A$ is allowed only when A actually has at least one free occurrence of the variable x fails, because of equivalencies such as:

(9) $A \Leftrightarrow A \wedge (Fx \vee \neg Fx)$,
(10) $A \Leftrightarrow A \wedge (A \vee Fx)$.

I will not reproduce here all of Dunn's argumentation, and simply turn to the definition of relevant predication which he has proposed:

(11) $(\varrho x\, Ax)a \Leftrightarrow \forall_x(x = a \rightarrow Ax)$.

According to Dunn the expression $(\varrho x\, Ax)a$ is read as follows: *a relevantly has the property of being (an x) such that A*. He points out that his definition of relevant predication "is in line with the common medieval treatment of affirmative 'categorical propositions' with singular terms as universal affirmatives" ([1, p. 454]).

The point of Dunn's approach consists in using just relevant implication. Due to certain features of relevant implication any attempt to ascribe relevantly e.g. the property of *being such that Socrates is wise* to Alcibiades:

$$\forall_x(x = a \rightarrow p)$$

fails, because we cannot infer p from $x = a$ within relevant logic (even if p is true!): "An x's being identical to Alcibiades has nothing to do with Socrates' being wise." ([1, p. 454])

4. Dunn's Analysis Revisited

For all its merits. Dunn's analysis seem to be too rough. Of course, definition (11) works well, when we wish to distinguish between statements (1) and (2). But this definition dose not allow us to draw subtler distinctions, in particular those which would take into account the role of context. Dunn's approach ontologizes the relevance of properties. In accordance with his approach a property recognized once as relevant, remains so for ever. E.g., he considers the famous statement of Juliet (or Shakespeare):

(12) A rose by any other name would smell as sweet,

and qualifies *sweet smell* as a relevant property of a rose. However, let us consider a rose in a context of some "flower competition", where exclusively the outward appearance of flowers is taken into account. In such a context the smell (whatever it is) is an irrelevant property of a rose, because the presence of a sweet smell has no influence on the determination of the winner.

Thus, it would be desirable to construct a formal apparatus that would allow us to realize such subtler distinction.

First of all I dwell on the problem of the choice of a suitable system of relevant logic. Dunn uses the system **R** (of relevant logic). He considers the system **E** (which combines both relevance and necessity) as too strong for his purposes. But he does not exclude a possibility of using the system **E** for an analysis of e.g. essential predication. As for me, I would rather prefer just system **E**, since I consider it (following Anderson and Belnap) as an adequate explication of the notion of relevant logical entailment. However, this problem is most likely a domestic affair of relevant logic itself. Therefore I will use the sign $\rightarrow$ as a symbol of some relevant implication, implying that there is a certain scope for further determination of what kind of relevant implication it is.

My point of departure is an observation that the semantical interpretation of formulas within the classical approach is essentially found on a precondition of uniformity (homogeneity) of a domain of interpretation of formulas. It is supposed that a domain is made simply of objects, and all these objects are of one *modus vivendi.* Thus, every predicate is interpreted on the whole domain, i.e. it is considered as interpreted on the whole domain of interpretation. Such a view is fully justified when we deal with propositions of e.g. mathematical theories. But it is apparently inappropriate for the semantical consideration of natural language. Therefore it would be quite natural to suppose that every predicate has ITS OWN domain of interpretation.

I start with the consideration of unary atomic predicates. Let D be a general domain. Then for every atomic predicate F we can pick out its own subdomain:

$$D[F] \subseteq D.$$

$D[F]$ just consists of these objects from D in relation to which the predicate F is defined.

Then we can consider the role of context. Every predicate is used in certain context. And this context restricts a domain of possible interpretation of a predicate even more. Thus, for the given predicate F and a given context α we have the domain of F in context α:

$$D[F, \alpha] \subseteq D[F].$$

I will use $D[F, \alpha]$ to denote a DOMAIN OF DEPENDENCE of predicate F in context α. That is, $a \in D[F, \alpha]$ means that the PREDICATE F DEPENDS ON THE OBJECT a IN CONTEXT α.

Now, what about ascribing properties to objects? In the early seventies a notion of *strict functors*, that should "really depends on their arguments" has been discussed. Dunn remarks that the discussion of these matters "was not always perfectly clear" ([1, p. 464]), but he still uses in his analysis some important ideas of this discussion. One of these ideas is just an idea of dependence of a formula on its variables. Dunn considers the notion of a *strict formula*, and expresses the so-called *Strict Proposal*: ONLY STRICT FORMULA CAN DETERMINE PROPERTIES.

I will interpret this proposal as follows: PROPERTY F CAN BE (RELEVANTLY) ASCRIBED TO AN OBJECT a (IN CONTEXT α), IF F DEPENDS ON a (IN CONTEXT α):

(13) $$a \in D[F, \alpha] \ \rightarrow \ Fa.$$

Furthermore, taking into account the results of Dunn's analysis in such a way as to prevent irrelevance in Dunn's sense, we obtain the following:

(14) $$\forall_x (x \in D[F, \alpha] \ \rightarrow \ (x = a \ \rightarrow \ Fx)).$$

Now let a be a rose, and F be a property of sweet smell. Remember the above-mentioned context α of a flower competition where ONLY the external appearance of flowers is taken into account, and smell of participants has no influence on the distribution of prizes. Then $D[F,\alpha] = \emptyset$. Surely, from the circumstance that x belongs to the empty set it is impossible in relevant logic to derive Fx, even if Fx is true. That is, the property of sweet smell cannot be relevantly ascribed to a rose in this context.

5. Having Properties

The next problem concerns the following question: when does an object ACTUALLY relevantly have the property F in context α? Obviously, this takes place when the property F can be relevantly ascribed to an object a in context α, and F really depends on this object in this context. Using Dunn's notation, let me write $(\varrho x\, Fx)a\backslash\alpha$ for the fact that a relevantly has the property F in context α. Then we are led to the following definition:

(15) $\quad (\varrho x\, Fx)a\backslash\alpha \;\Leftrightarrow\; a \in D[F,\alpha] \;\wedge\; \forall_x\, (x \in D[F,\alpha] \rightarrow (x = a \rightarrow Fx)).$

The following statement can be easily proved:

(16) $\quad (\varrho x\, Fx)a\backslash\alpha \;\rightarrow\; Fa.$

In words: if an object a relevantly has the property of being an x such that F in the context α, then a simply has the property F.

Now consider an object such as the number 4, and consider the property *is a prime number*. Let Fx mean *x is a prime number*. The domain of the predicate F in the context of arithmetic (α) — $D[F,\alpha]$ — consists of numbers. Of course, $4 \in D[F,\alpha]$. And, surely, 4 is not a prime number, that is 4 does not have the property F. But intuitively one can maintain that 4 RELEVANTLY does not have the property F. This corresponds to the following scheme:

(17) $\quad 4 \in D[F,\alpha] \;\wedge\; \forall_x\, (x \in D[F,\alpha] \;\rightarrow\; (x = 4 \;\rightarrow\; \neg Fx)).$

What I would like to emphasize here is the difference between the following expression:

> An object a relevantly has (has not) the property F in the context α.

and

> The property F is (is not) relevant to an object a in the context α.

Indeed, the property of *being a prime number* is relevant to ALL numbers, both prime and not prime. This motivates the following definition:

Let $Rel[F, a, \alpha]$ means *property F is relevant to an object a in the context α*. Then

(18) $$Rel[F, a, \alpha] \Leftrightarrow \forall_x (x \in D[F, \alpha] \rightarrow (x = a \rightarrow Fx \vee \neg Fx)).$$

Now it becomes clear why we should use just relevant implication in the above definitions. If we had used the material implication instead of a relevant one, these definitions would have been trivial in the lemma that especially holds for (18). $Fx \vee \neg Fx$ is the law of excluded middle, and if we change $\rightarrow$ to material implication, then we obtain a tautology instead of the right-hand side of this definitions. However, this is not true for $\rightarrow$, because in relevant logic one cannot derive $Fx \vee \neg Fx$ from every proposition. Within relevant logic $Fx \vee \neg Fx$ is derived only from those propositions, from which it is REALLY derived.

Let a be *the present King of France*, and F be the property of *being bald*. Consider the proposition

(19) The present King of France is bald.

It is clear that in the context (α) of this proposition $D[F, \alpha]$ consists of people who live now, i.e., $a \notin D[F, \alpha]$. Consider an arbitrary x, such that $x \in D[F, \alpha]$. Since $x \neq a$, we cannot relevantly derive neither Fx nor $\neg Fx$ only from a supposition that $x = a$. Thus, the property *bald* is irrelevant to *the present King of France* in the given context.

The same holds for the property *has stopped beating his father*. $D[F, \alpha]$ for this property consists of people who have beaten their father before, hence this property is irrelevant for a person who has never beaten his father.

6. Applications

An approach, developed here, can be used for the analysis of certain well known logical and semantical paradoxes. Consider e.g. Russell's Paradox. Let F be the property of *being a normal set*, and let a be the set of all normal sets whereas α is the context of intuitive set-theory. Then $D[F, \alpha]$ consists of sets. Suppose F is relevant to a in the context α. Then we have the following sketch of an inference:

1.	$\forall_x (x \in D[F,\alpha] \rightarrow (x = a \rightarrow Fx \vee \neg Fx))$	supposition
2.	$a \in D[F,\alpha] \rightarrow (a = a \rightarrow Fa \vee \neg Fa)$	1, substitution
3.	$a = a \rightarrow (a \in D[F,\alpha] \rightarrow Fa \vee \neg Fa)$	2, commutation
4.	$a = a$	reflexivity of identity
5.	$a \in D[F,\alpha] \rightarrow Fa \vee \neg Fa$	3, 4, MP
6.	$\neg(Fa \vee \neg Fa) \rightarrow a \notin D[F,\alpha]$	5, contraposition
7.	$Fa \wedge \neg Fa \rightarrow a \notin D[F,\alpha]$	6, De Morgan's law
8.	$Fa \wedge \neg Fa$	Russell's Paradox
9.	$a \notin D[F,\alpha]$	7, 8, MP
10.	$\forall_x (x = a \wedge a \notin D[F,\alpha] \rightarrow x \notin D[F,\alpha])$	substitution axiom
11.	$\forall_x (x = a \wedge x \in D[F,\alpha] \rightarrow a \in D[F,\alpha])$	10, contraposition
12.	$\exists_x (x = a \wedge x \in D[F,\alpha]) \rightarrow a \in D[F,\alpha]$	11
13.	$\neg \exists_x (x = a \wedge x \in D[F,\alpha])$	9, 12, MT

Thus a supposition that the property *normal* is relevant to the set of all normal sets has led us to a result such an object as «the set of all normal sets» does not exist. The paradox of the Liar can be analyzed in the same way. If we suppose that the property of *being false* is relevant to the expression *The present proposition is false*, we obtain a result that this expression does not express a proposition.

The next problem that should be resolved is the problem of extending the above definitions to compound formulas.

First of all we have to consider the definition of dependence of formulas on an object. As to negative formulas we have to conclude that a formula $\neg P$ depends just on the same objects that formula P depends on:

(20) $$D[\neg P,\alpha] = D[P,\alpha].$$

Hence, formula $P \vee Q$ depends on just the same objects as those on which formula $P \wedge Q$ depends (proceeding from De Morgan's laws). As to conjunctive formulas, we can suppose two possible definitions:

(21) $$D[P \wedge Q,\alpha] = D[P,\alpha] \cap D[Q,\alpha],$$

(21′) $D[P \wedge Q,\alpha] = D[P,\alpha]$, when $D[P,\alpha] = D[Q,\alpha]$
(but otherwise on no object).

Dunn points to both of these possibilities ([1, p. 465]). He prefers the latter one, and so do I.

Considering implication, one can propose the following natural definition:

(22) $D[P \to Q, \alpha] = D[Q, \alpha]$, when $D[P, \alpha] \subseteq D[Q, \alpha]$
(but otherwise on no set of objects).

In conclusion I would like to stress that the further investigation of relevant properties (I have presented here only a preliminary sketch of such an investigation) can be very useful for the analysis of many logical and methodological problems. Let me mention here some of these possible applications:

1. A theory of scientific models. As it is well-known, the notion of model plays an important role in scientific practice. Wartofsky [4] maintains a restriction to relatively RELEVANT PROPERTIES. When we construct a scientific model, we have to pick out (from ALL the properties that are responsible for (as Wartofsky says) the MODEL RELATION existing between two entities. These properties are just the RELEVANT PROPERTIES in the context of a given model.

2. Paradox of confirmation. Von Wright [5] has introduced the term *range of relevance* of generalization for resolving the paradox of confirmation. He has supposed that only objects in this range can be considered as REAL confirmations or refutations. But every generalization can be interpreted as ascribing a property to objects. Thus, we can formally present von Wright's range of relevance as a set of objects on which some predicate depends.

3. Set theory. As it is well-known, a nonrestricted axiom of comprehension:

$$\exists_y \forall_x (x \in y \Leftrightarrow Fx)$$

is the cause of the Russell's paradox. Therefore one could consider the following restriction on this axiom: *where F is relevant to x.*

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Yaroslav Shramko
Pedagogical Institute
Krivoi Rog
324086, The Ukraine
e-mail: kff@kpi.dnepropetrovsk.ua