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ON ALGORITHMIC PROPERTIES OF PROPOSITIONAL INCONSISTENCY-ADAPTIVE LOGICS

Abstract. The present paper is devoted to computational aspects of propositional inconsistency-adaptive logics. In particular, we prove (relativized versions of) some principal results on computational complexity of derivability in such logics, namely in cases of $CLuN^{\mathbf{r}}$ and $CLuN^{\mathbf{m}}$, i.e., $CLuN$ supplied with the reliability strategy and the minimal abnormality strategy, respectively.

Keywords: Inconsistency-adaptive logics, non-monotonic logic, dynamic reasoning, reliability strategy, minimal abnormality strategy, computational complexity, expressiveness.

1. Introduction

Adaptive logic is a well-developed approach to non-monotonic logic which can be considered as unifying for formalization of default reasoning (see [4]). Naturally, being non-monotonic, such logics usually have rather complex consequence relations, so it is surprising that there are only a few works devoted to investigating algorithmic complexity of adaptive logics.

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Historically, the first adaptive logics were inconsistency-adaptive (cf. [1]) and thus, with the present manuscript, we start the systematic study of algorithmic properties of this kind of logics (more precisely, of their propositional variants). As a point of departure, we consider several known results on adaptive logics complexity, but give alternative, simpler (than in the available literature on the subject) and purely algorithmic proofs for them. Simultaneously, we prove several theorems in a relativized form which may serve as a basis for the subsequent generalizations.

For instance, it is known [3] that the set of consequences derivable from a finite premiss set in the adaptive logic $CLuN^{\mathbf{r}}$ (having the weak paraconsistent logic $CLuN$ as its lower limit logic and supplied with the reliability strategy) is decidable: this was obtained by providing the goal directed proof procedure for $CLuN^{\mathbf{r}}$. A similar proof procedure for the minimal abnormality strategy was suggested in [8] and yields the decidability of the set of consequences of a finite premiss set in the corresponding adaptive logic $CLuN^{\mathbf{m}}$. The goal directed proof procedures (for $CLuN^{\mathbf{r}}$ and $CLuN^{\mathbf{m}}$) are rather complicated, involve many different parameters and both have various applications besides the decidability itself. Actually, however, all we need for getting decidability in these cases is the fact that only finitely many minimal disjunctions of abnormalities are $CLuN$-derivable from a finite set of premisses: this observation will be reflected in our own proofs of Propositions 3.1, 3.5 and 3.6 (see Section 3).

In their paper [5], L. Horsten and P. Welsh investigated the complexity of the sets of $CLuN^{\mathbf{r}}$- and $CLuN^{\mathbf{m}}$-consequences for an infinite recursive set of premisses: they argued that each of these is Σ^0_3 and that the estimation is exact, namely there is a recursive set Γ the collections of $CLuN^{\mathbf{r}}$- and $CLuN^{\mathbf{m}}$-consequences of which are both Σ^0_3-complete. Though it is easy to check their lower bound proof (i.e., that the problem is Σ^0_3-hard), the proof for the upper bound is hard to follow. The latter is quite complicated and is based on a fairly non-standard representation of the dynamic proof procedure for adaptive logics. Moreover, the Σ^0_3-complexity for $CLuN^{\mathbf{m}}$ contradicts the Π^1_1-hardness of the same problem established by P. Verdee [7] (which will be discussed below). In Section 3 we give a direct and explicit proof of the fact that generalizes the Σ^0_3 upper bound for the reliability strategy and relies on the standard format of adaptive logics (as in [2, 4]). The idea is the following. Let us start with the definition of the final derivability relation: a formula A

is finally derivable from a set of premisses Γ iff there is a finite stage of proof s (from Γ) such that A is derived on some unmarked (according to the reliability strategy) line i of this stage and for any finite extension t of s, there exists a further finite extension r (of t) in which i appears to be unmarked. The definition contains a Σ^0_3 prefix followed by a condition recursive modulo the predicate "to be a finite stage of proof from Γ" (which doesn't presuppose markings done): the proof of Theorem 3.7 and its corollaries provide the detailed analysis and demonstrate the technique needed. Then it only remains to notice that such predicate appears to be recursive in case of recursive Γ, and recursively enumerable (r.e.) in case of r.e. Γ (more generally, its algorithmic complexity is m-equivalent to the complexity of Γ). The obtained result agrees with the estimation for the reliability strategy claimed by Horsten and Welsh and generalizes it as well. However, this argumentation cannot be carried over to the minimal abnormality strategy, because the definition of final derivability involves infinite stages of proof (and, in effect, essentially exploits them). Verdee [7] proved that the collection of $CLuN^{\mathbf{m}}$-consequences is Π^1_1-hard for a suitable recursive set Γ. It turns out that this estimation is exact: in Theorem 3.15 we prove that for every set of premisses Γ, the set of its $CLuN^{\mathbf{m}}$-consequences is Π^1_1 w.r.t. Γ. On the other hand, if there are only finitely many formulas unreliable w.r.t. Γ, then the set of $CLuN^{\mathbf{m}}$-consequences of Γ will be again arithmetical modulo Γ (see Proposition 3.17).

2. Preliminaries

We assume the reader is acquainted with the basics of computability theory. Let us recall only the definition of the arithmetical hierarchy. An n-ary relation R on the set of natural numbers ω belongs to the class Σ^0_1 iff it is a projection of $n+1$-ary recursive relation, i.e.,

$$R \;=\; \{\langle x_1,\dots,x_n\rangle \mid \exists y(\langle x_1,\dots,x_n,y\rangle \in Q)\}$$

for some recursive relation $Q \subseteq \omega^{n+1}$. An n-ary relation $R \subseteq \omega^n$ belongs to the class Π^0_1 iff its complement $\omega^n \setminus R$ is in Σ^0_1. Next Σ^0_{n+1} consists of projections of Σ^0_n-relations, and elements of Π^0_{n+1} are exactly the complements of Σ^0_{n+1}-relations. Taking into account that every projection of Σ^0_n-relation is again a Σ^0_n-relation, one can easily obtain that any relation

$$\{\bar{z} \mid \exists x_1 \exists x_2 \dots \forall y_1 \forall y_2 \dots R(x_1,x_2\dots,y_1,y_2\dots,\bar{z})\}$$

defined via a recursive matrix R with the prefix containing n-alternations of quantifiers and starting with existential quantifier belongs to Σ^0_{n+1}, whereas the relation defined via a recursive matrix R with prefix containing n-alternations of quantifiers and starting with universal quantifier belongs to Π^0_{n+1}. The families of classes Σ^0_{n+1} and Π^0_{n+1} form the arithmetical hierarchy. Note that Σ^0_1 coincides with the class of r.e. relations.

If we start not with the family of all recursive sets, but with the family of sets recursive with respect to an oracle X, we will get the relativized arithmetical hierarchy consisting of classes $\Sigma^{0,X}_{n+1}$ and $\Pi^{0,X}_{n+1}$, $n \in \omega$.

A set which belongs to one of the classes of the arithmetical (w.r.t. X) hierarchy is called *arithmetical* (*w.r.t.* X).

The following representation of arithmetical sets is well-known. A set S is in Σ^0_n (Π^0_n) iff there is an arithmetical $\Sigma_n(\Pi_n)$-formula $A(x_1, \ldots, x_n)$ such that

$$S \;=\; \{\langle a_1, \ldots, a_n\rangle \mid \mathfrak{N} \models A(a_1, \ldots, a_n)\},$$

where $\mathfrak{N} = \langle \omega, +, \cdot, s, 0\rangle$ is the standard model of arithmetic. Thus, arithmetical sets are defined via the arithmetical first order formulas.

A set $S \subseteq \omega^n$ is said to be a Π^1_1-*set* iff

$$S \;=\; \{\langle a_1, \ldots, a_n\rangle \mid \mathfrak{N} \models \forall P\, A(P, a_1, \ldots, a_n)\},$$

where $A(P, x_1, \ldots, x_n)$ is a second order arithmetical formula with only one predicate variable P (so "$\forall P$" ranges over all subsets of naturals), and S is a $\Pi^{1,X}_1$-set iff

$$S \;=\; \{\langle a_1, \ldots, a_n\rangle \mid \mathfrak{N}^{\mathcal{X}} \models \forall P\; A(P, \mathcal{X}, a_1, \ldots, a_n)\},$$

where $\mathfrak{N}^{\mathcal{X}} = \langle \omega, +, \cdot, s, 0, X\rangle$ is the standard model of arithmetic enriched with the unary predicate symbol $\mathcal{X}$ interpreted by X and the formula A may contain occurrences of both P and $\mathcal{X}$.

Now we introduce the necessary adaptive logic terminology (cf. [2]). Fix some language $\mathcal{L}$ with the set of formulas $For_{\mathcal{L}}$. Let γ be a Gödel numbering of $For_{\mathcal{L}}$, i.e., γ is an effective one-to-one mapping from $For_{\mathcal{L}}$ onto ω with the property: $\gamma(A) < \gamma(B)$ whenever A is a proper subformula of B.

Let **LLL** be a *lower limit logic*, namely a monotonic logic in the language $\mathcal{L}$ with its consequence relation $\vdash_{\mathbf{LLL}}$ (between sets of $\mathcal{L}$-formulas), appropriate class of models, and its satisfiability relation $\vDash_{\mathbf{LLL}}$ (between the models and the formulas). In fact, the relation $\vdash_{\mathbf{LLL}}$ will be a subrelation of an adaptive consequence we intend to define.

Fix a set of formulas $\Omega \subseteq For_{\mathcal{L}}$ the elements of which will be called *abnormalities*. Usually it is assumed that the set Ω is distinguished by a logical form of formulas, e.g., consists of all formulas of the form $A \wedge \neg A$. This assumption guaranties the decidability of the set of abnormalities. For an **LLL**-model $\mathcal{M}$, put $Ab(\mathcal{M}) := \{A \in \Omega \mid \mathcal{M} \vDash A\}$.

Let $\Delta, \Gamma \subseteq For_{\mathcal{L}}$. We employ the following notation[1]:

$$\ell(\varphi) := \text{ the length of } \varphi \in For_{\mathcal{L}};$$
$$SubF(\Gamma) := \text{the set of all subformulas of formulas in } \Gamma;$$
$$\Delta \subseteq_{fin} \Gamma \;\text{ means “}\Delta \text{ is a finite subset of } \Gamma\text{”};$$
$$Dab(\Delta) := \textstyle\bigvee_{\varphi \in \Delta} \varphi, \text{ where } \Delta \subseteq_{fin} \Omega.$$

Formulas of the form $Dab\,(\Delta)$ are called *Dab*-formulas. Then $Dab\,(\Delta)$ is a *minimal Dab-consequence* of Γ iff $\Gamma \vdash_{\mathbf{LLL}} Dab\,(\Delta)$ and there is no $\Delta' \subset \Delta$ for which $\Gamma \vdash_{\mathbf{LLL}} Dab\,(\Delta')$. Set

$$U(\Gamma) := \{A \in For_{\mathcal{L}} \mid A \in \Delta \text{ for some minimal } Dab\text{-consequence } Dab(\Delta) \text{ of the set } \Gamma\}\,.$$

We say that the elements of $U(\Gamma)$ are *unreliable* with respect to Γ.

Let $\mathcal{M}$ be an **LLL**-model of the set of $\mathcal{L}$-formulas Γ, namely $\mathcal{M} \vDash \Gamma$. Then $\mathcal{M}$ is *reliable* iff $Ab(\mathcal{M}) \subseteq U(\Gamma)$, and $\mathcal{M}$ is *minimally abnormal* iff there is no other model $\mathcal{M}'$ of Γ with $Ab(\mathcal{M}') \subset Ab(\mathcal{M})$.

Now we are to define (semantically) two adaptive consequence relations: for $\Gamma \cup \{A\} \subseteq For_{\mathcal{L}}$, $\Gamma \vDash_{AL^{\mathbf{r}}} A$ iff $\mathcal{M} \vDash A$ for all reliable models $\mathcal{M}$ of Γ, and $\Gamma \vDash_{AL^{\mathbf{m}}} A$ iff $\mathcal{M} \vDash A$ for all minimally abnormal models $\mathcal{M}$ of Γ.

The relation $\vDash_{AL^{\mathbf{r}}}$ provides the semantics for the adaptive logic $AL^{\mathbf{r}}$ based on the lower limit logic **LLL**, the set of abnormalities Ω, and *the reliability strategy*. Similarly, the adaptive logic $AL^{\mathbf{m}}$ (corresponding to $\vDash_{AL^{\mathbf{m}}}$) is based on the same lower limit logic and set of abnormalities, but exploits a different strategy of handling abnormalities which is called *the minimal abnormality strategy*.

Next we have to define the proof procedures for the adaptive logics $AL^{\mathbf{r}}$ and $AL^{\mathbf{m}}$. Both of them significantly make use of the notion of a

[1] Below we presuppose that the logical connective “$\vee$” is in the language.

stage of proof (from a given set of premisses). For any $\Gamma \subseteq For_{\mathcal{L}}$, a *stage of proof from* Γ is represented by a sequence (finite or infinite) of lines, where each line is a quintuple with the following components: (i) a line number, (ii) a formula, (iii) line numbers for the premisses of a rule, (iv) the name of the rule, (v) a condition which is a finite set of abnormalities. Moreover, every line of a stage of proof s must be constructed from the previous lines using one of the following rules:

PREM If $A \in \Gamma$, one may add a line comprising the following elements: (i) an appropriate line number, (ii) A, (iii) —, (iv) PREM, and (v) $\varnothing$.

RU If $A_1, \ldots, A_n \vdash_{\mathbf{LLL}} B$ and $A_1, \ldots, A_n$ occur in s as the second elements of lines with numbers $i_1, \ldots, i_n$ that have conditions Δ_1, …, Δ_n, respectively, then one may add a line consisting of: (i) an appropriate line number, (ii) B, (iii) $i_1, \ldots, i_n$, (iv) RU, and (v) $\Delta_1 \cup \ldots \cup \Delta_n$.

RC If $A_1, \ldots, A_n \vdash_{\mathbf{LLL}} B \vee Dab(\Theta)$ and $A_1, \ldots, A_n$ occur in s as the second elements of lines with numbers $i_1, \ldots, i_n$ that have conditions Δ_1, …, Δ_n respectively, then one may add a line consisting of: (i) an appropriate line number, (ii) B, (iii) $i_1, \ldots, i_n$, (iv) RC, and (v) $\Delta_1 \cup \ldots \cup \Delta_n \cup \Theta$.

If s is a stage of proof that contains a line with number i, the second element being A and the fifth element Δ, we say that A *is derived in s at line i under condition* Δ. By an *extension* of a stage of proof s we mean a stage of proof t with the property: the sequence of lines of s forms a subsequence of that of t, when all the (i)-st and (iii)-rd components of lines in s are suitably renumbered.

Notice, the notion of a stage of proof does not depend on the strategy of handling abnormalities. Rather, the strategies are involved in the proof theory in the form of marking definitions.

Let s be a stage of proof from a premiss set Γ. For the reliability strategy, we first need to define the set U_s of formulas that are unreliable at s.[2] Say that $Dab\,(\Delta)$ is a *minimal Dab-consequence at s* iff $Dab\,(\Delta)$ has been derived at some line of s under the empty condition (i.e., $Dab\,(\Delta)$ is the second component of this line whilst the fifth component is empty) and there is no $\Delta' \subset \Delta$ for which $Dab\,(\Delta')$ has been derived in s under the empty condition. Let $U_s := \{A \in For_{\mathcal{L}} \mid A \in$

[2] We use here the notation U_s instead of the traditional $U_s(\Gamma)$ to emphasize the fact that this set is determined solely by the stage of proof s and the whole set of premisses Γ is not indeed required. Analogously, we write Φ_s instead of $\Phi_s(\Gamma)$ below.

Δ for some minimal *Dab*-formula $Dab(\Delta)$ at stage $s\}$. At times, when it doesn't lead to confusion, we call lines by their numbers.

Definition 2.1. Let a finite stage of proof s contain a line with number i and condition Δ. We say that this line i is **r**-*marked* (or *marked according to the reliability strategy*) *at stage* s iff $\Delta \cap U_s = \varnothing$.

Definition 2.2. A formula A is *finally* $AL^{\mathbf{r}}$*-derived at a finite stage of proof* s iff A is derived at some line i of s, which is not **r**-marked at s and any finite extension of s in which this line is **r**-marked may be further finitely extended in such a way that the line becomes **r**-unmarked again.

Definition 2.3. A formula A is *finally* $AL^{\mathbf{r}}$*-derivable from* Γ (written as $\Gamma \vdash_{AL^{\mathbf{r}}} A$) iff there exists a stage of proof s (from Γ) such that A is finally **r**-derived at some line of s.

Now we turn to the minimal abnormality strategy where infinite stages of proof play an important role.

First we need to say a few words on the so-called choice sets. Assume Σ is a family of sets. A set Δ is said to be a *choice set for* Σ iff for any $\varphi \in \Sigma$, $\Delta \cap \varphi \neq \varnothing$. Then such a choice set Δ is *minimal* (for Σ) iff there is no other choice set Δ' for Σ with $\Delta' \subset \Delta$. It is well-known that every family of finite sets has a minimal choice set (see, e.g., [4, Fact 5.1.2]). The next statement is an obvious strengthening of this latter result.

Proposition 2.4. *Let* Σ *be a family of sets. A choice set* Δ *for* Σ *is minimal iff for every* $a \in \Delta$*, there exists* $\varphi \in \Sigma$ *such that* $\Delta \cap \varphi = \{a\}$*.*

Suppose that s is a stage of proof from Γ and $\{Dab(\Delta_i) \mid i \in I\}$ is the family of all minimal *Dab*-formulas at s. Denote by Φ_s the set of all minimal choice sets for the family $\{\Delta_i \mid i \in I\}$.

Definition 2.5. Let a stage of proof s contain a line with number i and condition Δ. We say that this line i is **m**-*marked* (or *marked according to minimal abnormality strategy*) *at stage* s iff one of the following requirements is satisfied:

(i) there is no $\varphi \in \Phi_s$ such that $\varphi \cap \Delta = \varnothing$;
(ii) for some $\varphi \in \Phi_s$, there is no line in s at which A is derived under condition Θ with $\varphi \cap \Theta = \varnothing$.

Definition 2.6. A formula A is *finally* $AL^{\mathbf{m}}$*-derived at a stage of proof* s iff A is derived at some line i of s, which is not **m**-marked at s and any extension of s in which this line is **m**-marked may be further extended in such a way that the line becomes **m**-unmarked again.

Definition 2.7. A formula A is *finally* $AL^{\mathbf{m}}$*-derivable from* Γ (written as $\Gamma \vdash_{AL^{\mathbf{m}}} A$) iff there exists a stage of proof s (from Γ) such that A is finally **m**-derived at some line of s.

For an arbitrary set of formulas Γ, we denote by $\Phi(\Gamma)$ the set of all minimal choice sets for the family $\{\Delta_i \mid i \in I\}$, where $\{Dab(\Delta_i) \mid i \in I\}$ is the set of all minimal *Dab*-consequences of Γ. It is easy to reformulate the criterion for the final **m**-derivability as follows.

Proposition 2.8. *A formula A is finally $AL^{\mathbf{m}}$-derivable from Γ iff there exists a stage of proof s from Γ with the property: $\Phi_s = \Phi(\Gamma)$ and for every $\varphi \in \Phi(\Gamma)$, there is a line i of s such that A is derived at this line under condition Δ_i with $\varphi \cap \Delta_i = \varnothing$.*

Assume that

$$Cn_{AL^{\mathbf{r}}}(\Gamma) := \{A \mid \Gamma \vdash_{AL^{\mathbf{r}}} A\} \quad \text{and} \quad Cn_{AL^{\mathbf{m}}}(\Gamma) := \{A \mid \Gamma \vdash_{AL^{\mathbf{m}}} A\}\,.$$

We also write $Cn^{\mathbf{r}}(\Gamma)$ and $Cn^{\mathbf{m}}(\Gamma)$, for short, if it is clear from the context what kind of lower limit logic and abnormalities are used.

For many concrete lower limit logics and sets of abnormalities one can prove that the final $AL^{\mathbf{r}}$ ($AL^{\mathbf{m}}$)-derivability relation is strongly complete w.r.t. the proper semantics, i.e., that $\vdash_{AL^{\mathbf{r}}} = \vDash_{AL^{\mathbf{r}}}$ ($\vdash_{AL^{\mathbf{m}}} = \vDash_{AL^{\mathbf{m}}}$).

Perhaps the most standard choice for a lower limit logic and a collection of abnormalities (in propositional setting) is the propositional weak paraconsistent logic *CLuN* together with inconsistencies

$$\Omega := \{A \wedge \neg A \mid A \in For_{CL}\},$$

where For_{CL} is the set of formulas in the classical propositional language $\{\vee, \wedge, \rightarrow, \neg\}$ built up from the propositional variables *Prop*. Thus, we arrive at (propositional) inconsistency adaptive logics $CLuN^{\mathbf{r}}$ and $CLuN^{\mathbf{m}}$.

The logic *CLuN* can be viewed as the least subset of For_{CL} containing the axioms of classical positive logic with the only additional axiom for the negation, namely $p \vee \neg p$, and closed under the rules of substitution and *modus ponens*. The consequence relation $\vdash_{CLuN}$ associated with *CLuN* is defined as follows: for $\Gamma \cup \{A\} \subseteq For_{CL}$, $\Gamma \vdash_{CLuN} A$ holds iff A can be obtained in a finite number of steps from the elements of $CLuN \cup \Gamma$

using *modus ponens*. And for $\Gamma, \Delta \subseteq \mathit{For}_{CL}$, the relation $\Gamma \vdash_{CLuN} \Delta$ means that $\Gamma \vdash_{CLuN} A_1 \vee \ldots \vee A_n$ for some $\{A_1, \ldots, A_n\} \subseteq \Delta$.

Models of *CLuN* are simply valuations $v\colon \mathit{For}_{CL} \to \{0,1\}$ having the properties: for all $A, B \in \mathit{For}_{CL}$,

1. $v(A \wedge B) = 1$ iff $v(A) = 1$ and $v(B) = 1$;
2. $v(A \vee B) = 1$ iff $v(A) = 1$ or $v(B) = 1$;
3. $v(A \to B) = 1$ iff $v(A) = 0$ or $v(B) = 1$;
4. if $v(A) = 0$, then $v(\neg A) = 1$.

We write $v(\Gamma) = 1(0)$ iff $v(A) = 1(0)$ for all $A \in \Gamma$. Hence $\Gamma \vDash_{CLuN} A$ means that $v(\Gamma) = 0$ or $v(A) = 1$ for each *CLuN*-valuation v. Accordingly, for two sets of formulas Γ and Δ, $\Gamma \vDash_{CLuN} \Delta$ means that for every *CLuN*-valuation v, either $v(\Gamma) = 0$ or $v(A) = 1$ for some $A \in \Delta$.

The logic *CLuN* is strongly complete w.r.t. the semantics just described, i.e., for any $\Gamma, \Delta \subseteq \mathit{For}_{CL}$, we have

$$\Gamma \vdash_{CLuN} \Delta \iff \Gamma \vDash_{CLuN} \Delta\,.$$

Since the values $v(\Gamma)$ and $v(\Delta)$ are completely determined by the restriction of v to the subformulas $\mathit{SubF}(\Gamma \cup \Delta)$, the relation $\vdash_{CLuN}$ restricted to finite sets (for both premisses and consequences) is decidable.

The analogs of strong completeness results for the final $\mathit{CLuN}^{\mathbf{r}}$- and $\mathit{CLuN}^{\mathbf{m}}$-derivabilities were proved by D. Batens.

Theorem 2.9 ([1]). *For any $\Gamma \cup \{A\} \subseteq \mathit{For}_{CL}$, the equivalences hold:*

$$\Gamma \vdash_{CLuN^{\mathbf{r}}} A \iff \Gamma \vDash_{CLuN^{\mathbf{r}}} A\,,$$
$$\Gamma \vdash_{CLuN^{\mathbf{m}}} A \iff \Gamma \vDash_{CLuN^{\mathbf{m}}} A\,.$$

The next criterion for the final $\mathit{CLuN}^{\mathbf{r}}$-derivability is also useful (it can be viewed as a sort of 'compactness' for the non-monotonic logic $\mathit{CLuN}^{\mathbf{r}}$).

Theorem 2.10 ([1]). *For any $\Gamma \cup \{A\} \subseteq \mathit{For}_{CL}$, $\Gamma \vdash_{CLuN^{\mathbf{r}}} A$ iff there exists $\Delta \subseteq_{\mathit{fin}} \Omega$ such that $\Gamma \vdash_{CLuN} A \vee \mathit{Dab}(\Delta)$ and $\Delta \cap U(\Gamma) = \varnothing$.*

A similar criterion for the final $\mathit{CLuN}^{\mathbf{m}}$-derivability was provided in [1]. However, since the proof of this statement in [1] (and also in [4]) essentially exploits the presence of classical negation in the language involved, but the latter is not available in *CLuN* (according to our presentation), we give an alternative proof of this statement here.

THEOREM 2.11 ([1]). *For any* $\Gamma \cup \{A\} \subseteq For_{CL}$, $\Gamma \vDash_{CLuN^{\mathbf{m}}} A$ *iff for each* $\varphi \in \Phi(\Gamma)$, *there exists* $\Delta \subseteq_{fin} \Omega$ *such that* $\Gamma \vdash_{CLuN} A \vee Dab(\Delta)$ *and* $\Delta \cap \varphi = \varnothing$.

PROOF. $\boxed{\Rightarrow}$ Suppose there exists $\varphi \in \Phi(\Gamma)$ such that for every $\Delta \subseteq_{fin} \Omega \setminus \varphi$, we have $\Gamma \nvdash_{CLuN} A \vee Dab(\Delta)$. What it means is $\Gamma \nvdash_{CLuN} \{A\} \cup \Omega \setminus \varphi$. Hence, due to strong completeness for $CLuN$, there is a $CLuN$-valuation v with the property:

$$v(\Gamma) = 1 \quad \text{and} \quad v(\{A\} \cup \Omega \setminus \varphi) = 0\,.$$

Particularly, v is a model of Γ with $Ab(v) \subseteq \varphi$. Now if $B \wedge \neg B \in \varphi$ but $v(B \wedge \neg B) = 0$ (i.e., $B \wedge \neg B \notin Ab(v)$), by Proposition 2.4 there is a minimal Dab-consequence $Dab(\Theta)$ of Γ with $\varphi \cap \Theta = \{B \wedge \neg B\}$. On the other hand, $\Theta \setminus \{B \wedge \neg B\} \subseteq_{fin} \Omega \setminus \varphi$, so $v\left(\Theta \setminus \{B \wedge \neg B\}\right) = 0$. Then $Dab\left(\Theta\right)$ is false in a $CLuN$-model of Γ and can't be a $CLuN$-consequence of Γ which is a contradiction. Consequently, $Ab(v) = \varphi$. Notice, if v' is such that

$$v'\left(\Gamma\right) = 1 \quad \text{and} \quad Ab\left(v'\right) \subseteq Ab\left(v\right) = \varphi\,,$$

then $v'\left(\Omega \setminus \varphi\right) = 0$ and an argument similar to the above leads to $v' = v$. Thus, v is minimally abnormal and $\Gamma \nvDash_{CLuN^{\mathbf{m}}} A$.

Moreover, one can prove that *a set of abnormalities* φ *is in* $\Phi(\Gamma)$ *iff* φ *coincides with* $Ab(v)$ *for some minimally abnormal model* v *of* Γ.

Indeed, by the above argument we have that if $\Gamma \nvDash_{CLuN} Dab\left(\Delta\right)$ for each $\Delta \subseteq_{fin} \Omega \setminus \varphi$, then $\varphi = Ab(v)$ for an appropriate minimally abnormal model of Γ (one should omit "A" to get this). Suppose there exists $\Delta \subseteq_{fin} \Omega \setminus \varphi$ such that $\Gamma \vdash_{CLuN} Dab\left(\Delta\right)$, hence $\Gamma \vdash_{CLuN} Dab\left(\Delta'\right)$ where $\Delta' \subseteq \Delta$ and $Dab\left(\Delta'\right)$ is a minimal Dab-consequence of Γ. But in this case $\varphi \notin \Phi(\Gamma)$.

Inversely, if v is a minimally abnormal model of Γ, then $Ab(v)$ is a choice set for $\{\Delta_i \mid i \in I\}$, where $\{Dab(\Delta_i) \mid i \in I\}$ is the collection of all minimal Dab-consequences of Γ. If $Ab(v)$ is a proper supset of some $\varphi \in \Phi(\Gamma)$, then v is not minimally abnormal, since (by the direct implication) $\varphi = Ab(v')$ for a suitable model v' of Γ.

$\boxed{\Leftarrow}$ Assume that for every $\varphi \in \Phi(\Gamma)$, there exists $\Delta \subseteq_{fin} \Omega$ with the property: $\Gamma \vdash_{CLuN} A \vee Dab(\Delta)$ and $\Delta \cap \varphi = \varnothing$. If there is a minimally abnormal model v of Γ such that $v(A) = 0$, then $Ab(v) \in \Phi(\Gamma)$ and so $\Gamma \vdash_{CLuN} A \vee Dab(\Delta)$ for some $\Delta \subseteq_{fin} \Omega$ with $\Delta \cap Ab(v) = \varnothing$. Since $v(A) = 0$, we obtain $v(Dab(\Delta)) = 1$ which conflicts $\Delta \cap Ab(v) = \varnothing$. $\dashv$

Remark that we have also established the following

Corollary 2.12. *Let* $\Gamma \subseteq For_{CL}$. *Then*

$$\Phi(\Gamma) = \{Ab(v) \mid v \text{ is a minimally abnormal model of } \Gamma\},$$
$$U(\Gamma) = \bigcup \{Ab(v) \mid v \text{ is a minimally abnormal model of } \Gamma\}.$$

In particular, if v is a minimally abnormal model of Γ, then $Ab(v) \subseteq U(\Gamma)$. So every minimally abnormal model (of Γ) is also reliable one.

3. Complexity Bounds

The next simple observation plays an important part in providing the results of this section. For $\Gamma, \Delta \subseteq For_{CL}$, we denote

$$\Delta_\Gamma := \Delta \cap \{A \wedge \neg A \mid \neg A \in SubF(\Gamma)\}.$$

For instance,

$$\Omega_\Gamma := \{A \wedge \neg A \mid \neg A \in SubF(\Gamma)\}.$$

Proposition 3.1. *Let* $\Gamma \subseteq For_{CL}$ *and* $\Delta \subseteq_{fin} \Omega$. *Then* $\Gamma \vdash_{CLuN} Dab(\Delta)$ *entails* $\Gamma \vdash_{CLuN} Dab(\Delta_\Gamma)$.

Proof. Let v be a *CLuN*-valuation such that $v(\Gamma) = 1$. Now we want to show $v(Dab(\Delta_\Gamma)) = 1$.

Construct $v' \colon For_{CL} \to \{0, 1\}$ inductively as follows:

1. if p is a propositional symbol which does not appear in Γ, then $v'(p)$ is arbitrary (but, obviously, fixed; e.g., zero);
2. if $A \in SubF(\Gamma)$, then $v'(A) := v(A)$;
3. if A has the sort $A_1 \wedge A_2$, $A_1 \vee A_2$ or $A_1 \to A_2$, then $v'(A)$ is defined as for *CLuN*-valuations being given the values of $v'(A_1)$ and $v'(A_2)$;
4. if $\neg A \notin SubF(\Gamma)$, then $v'(\neg A) := 1 - v'(A)$.

It is straightforward that v' is a *CLuN*-valuation as well, and, since it acts just like v on the elements of $SubF(\Gamma)$, $v(A) = v'(A)$ for any $A \in \Delta_\Gamma$. Clearly, $v'(\Gamma) = v(\Gamma) = 1$ and $v'(Dab(\Delta_\Gamma)) = v(Dab(\Delta_\Gamma))$. In particular, v' is a model of Γ. Thus, by assumption, $v'(Dab(\Delta)) = 1$. On the other hand, $v'(A) = 0$ for all $A \in \Omega \setminus \Delta_\Gamma$, because in v' the negation behaves classically outside of $SubF(\Gamma)$. Hence $v'(Dab(\Delta \setminus \Delta_\Gamma)) = 0$, and so $v'(Dab(\Delta_\Gamma)) = 1$. Finally, we obtain the desired equality $v(Dab(\Delta_\Gamma)) = v'(Dab(\Delta_\Gamma)) = 1$. ⊣

Consequently, for a finite Γ, there are only finitely many minimal disjunctions of abnormalities that are derivable from Γ. By analogy one can establish the following

COROLLARY 3.2. *Let* $\Gamma \subseteq For_{CL}$ *and* $\Delta \subseteq_{fin} \Omega$. *Then* $\Gamma \vdash_{CLuN} A \vee Dab\,(\Delta)$ *entails* $\Gamma \vdash_{CLuN} A \vee Dab\,(\Delta_{\Gamma\cup\{A\}})$.

From the last Corollary and Theorem 2.10 we obtain

COROLLARY 3.3. *For any* $\Gamma \cup \{A\} \subseteq For_{CL}$, $\Gamma \vdash_{CLuN^{\mathbf{r}}} A$ *iff there exists* $\Delta \subseteq \Omega_{\Gamma\cup\{A\}}$ *such that* $\Gamma \vdash_{CLuN} A \vee Dab(\Delta)$ *and* $\Delta \cap U(\Gamma) = \varnothing$.

In the proof of Proposition 3.1, for every *CLuN*-model v of Γ, we've constructed another model v' of Γ with the property $Ab(v') \subseteq Ab(v) \cap \Omega_\Gamma$. This construction leads us naturally to

COROLLARY 3.4. *Let* $\Gamma \subseteq For_{CL}$ *and* v *be an arbitrary reliable or minimally abnormal CLuN-model of* Γ. *Then* $Ab\,(v) \subseteq \Omega_\Gamma$.

PROPOSITION 3.5. *The relation*

$$\{(\Gamma, A) \mid \Gamma \cup \{A\} \subseteq_{fin} For_{CL} \text{ and } \Gamma \vdash_{CLuN^{\mathbf{r}}} A\}$$

is decidable.

PROOF. By Proposition 3.1, if $Dab\,(\Delta')$ is a minimal *Dab*-consequence of Γ (in *CLuN*), then Δ' is a subset of the finite set Ω_Γ. Thus, in order to get all minimal disjunctions of abnormalities which are derivable from Γ, we only have to verify, for each $\Delta' \subseteq \Omega_\Gamma$, whether $\Gamma \vdash_{CLuN} Dab\,(\Delta')$ holds or not, and this can be done effectively as was noted in the previous section. As a result, we computably obtain the finite set $U\,(\Gamma)$.

Now, according to Corollary 3.3, it remains to check if there exists $\Delta \subseteq \Omega_{\Gamma\cup\{A\}}$ (obviously, $\Omega_{\Gamma\cup\{A\}}$ is finite, just like Ω_Γ, and can also be effectively found) such that $\Gamma \vdash_{CLuN} A \vee Dab(\Delta)$ and $\Delta \cap U(\Gamma) = \varnothing$. ⊣

PROPOSITION 3.6. *The relation*

$$\{(\Gamma, A) \mid \Gamma \cup \{A\} \subseteq_{fin} For_{CL} \text{ and } \Gamma \vdash_{CLuN^{\mathbf{m}}} A\}$$

is decidable.

PROOF. Due to Theorem 2.9 and Corollary 3.4, $\Gamma \vdash_{CLuN^{\mathbf{m}}} A$ is equivalent to $v\,(A) = 1$ for all $v \in \mathcal{K}$ where

$$\mathcal{K} := \{v \mid v(\Gamma) = 1,\ Ab(v) \subseteq \Omega_\Gamma, \\ \text{and there is no } v' \text{ such that } v'(\Gamma) = 1 \text{ and } Ab(v') \subset Ab(v)\}.$$

Let $\mathcal{R}$ be the class of all mappings ρ: $SubF(\Gamma \cup \{A\} \cup \Omega_\Gamma) \to \{0,1\}$ satisfying the conditions 1–4 from the definition of $CLuN$-valuation (restricted to the elements of $SubF(\Gamma \cup \{A\} \cup \Omega_\Gamma)$). Assume the notation $Ab(\rho) := \{A \in \Omega \mid \rho(A) = 1\}$, for $\rho \in \mathcal{R}$. Suppose

$$\mathcal{G} := \{\rho \in \mathcal{R} \mid \rho(\Gamma) = 1,\ Ab(\rho) \subseteq \Omega_\Gamma, \\ \text{and there is no } \rho' \in \mathcal{R} \text{ such that } \rho'(\Gamma) = 1 \text{ and } Ab(\rho') \subset Ab(\rho)\}.$$

To verify the conditions $v(A) = 1$ and $v(\Gamma) = 1$ we only need to know how v acts on the elements of $SubF(\Gamma \cup \{A\})$. Therefore, instead of checking $v(A) = 1$ for all $v \in \mathcal{K}$, it will be enough to examine the equality $\rho(A) = 1$ for all $\rho \in \mathcal{R}$. Since the set $\Gamma \cup \{A\} \cup \Omega_\Gamma$ is finite, $\mathcal{G}$ is also finite and, moreover, can be found in the effective way. This means that we have an algorithm deciding whether $\Gamma \vdash_{CLuN^{\mathbf{m}}} A$ holds or not. ⊣

Now we turn to the upper estimations for the general case. Remark: in the sequel, we often identify Γ with $\gamma(\Gamma) := \{\gamma(B) \mid B \in \Gamma\}$.

THEOREM 3.7. *For every $\Gamma \subseteq For_{CL}$, the set $Cn^{\mathbf{r}}(\Gamma)$ is $\Sigma_3^{0,\Gamma}$.*

PROOF. Having a Gödel numbering of formulas allows us to provide an effective coding for more complex syntactical objects, e.g., finite sequences of formulas, lines of stages of proof, finite stages of proof, finite sets of formulas, finite sets of finite sets of formulas, etc.

Let us consider the following predicates and functions:

- $\mathsf{Proof}(n)$ which is true iff n encodes some finite stage of proof from For_{CL};
- $\mathsf{Proof}_\Gamma(n)$ which is true iff n encodes some finite stage of proof from Γ;
- $\mathsf{len}(n)$ which returns the number of lines in the finite stage of proof encoded by n (i.e., its length) in case $\mathsf{Proof}(n)$ holds, and 0 otherwise;
- $\mathsf{Sub}(n,k)$ which is true iff both $\mathsf{Proof}(n)$ and $\mathsf{Proof}(k)$ hold, and k corresponds to the stage proof which is an extension of the stage of proof encoded by n;
- $\mathsf{Head}(n,i,k)$ which is true iff $\mathsf{Proof}(n)$ holds, $1 \leqslant i \leqslant \mathsf{len}(n)$, and $\gamma^{-1}(k)$ is the (ii)-component of the i-th line of the stage of proof encoded by n;

- $\mathsf{Mrk}^{\mathbf{r}}(i, n)$ which is true iff $\mathsf{Proof}(n)$ holds, $1 \leqslant i \leqslant \mathsf{len}(n)$, and the i-th line of the stage of proof encoded by n appears to be **r**-marked;

Notice, $\mathsf{Proof}_\Gamma(n)$ implies $\mathsf{Proof}(n)$ and, in case $\mathsf{Proof}_\Gamma(n)$ holds, the predicate $\mathsf{Mrk}^{\mathbf{r}}(i, n)$ works correctly as if it was applied to the stages of proof from Γ. In other words, all necessary information is encoded in n and we don't need to know if a stage of proof is from Γ or another set of premisses to provide an appropriate marking.

Lemma 3.8. *The predicates* Proof, Sub, Head, $\mathsf{Mrk}^{\mathbf{r}}$, *and the function* len *are all recursive, while the predicate* Proof_Γ *is recursive w.r.t.* Γ.

Proof. The recursiveness of Proof, Sub, Head, and len is straightforward. Indeed, to verify whether $\mathsf{Proof}(n)$ holds, we need to check that n is a code of a finite sequence of quintuples and for the i-th quintuple of this sequence (encoded by n), that: 1. the first of its components equals to i; 2. the second component is a code of a formula; 3. the third is a code of a finite set of numbers strictly smaller than i; 4. the forth is a code of the name of a rule; 5. the fifth is a code of a finite set of abnormalities; 6. finally, certain 'extra requirements' (they are discussed below) related to the name of the rule used in the forth component should be satisfied. These 'extra requirements' are also easy to check, namely

- if the forth component of the i-th line is RU, the fifth element B, and $A_1, \ldots, A_m$ are the formulas represented by the second elements of lines the numbers of which are sewed in the forth element of i, then $A_1, \ldots, A_m \vdash_{CLuN} B$ and the fifth element of i is the code of the set $\Delta_1 \cup \ldots \cup \Delta_m$ where Δ_k's ($k = 1, \ldots, m$) are the the fifth elements of lines corresponding to A_k's;
- if the forth element of line i is RC, then we have to verify whether $A_1, \ldots, A_m \vdash_{CLuN} B \vee Dab(\Theta)$ or not for some set of abnormalities Θ with the property $\Delta \setminus (\Delta_1 \cup \ldots \cup \Delta_m) \subseteq \Theta \subseteq \Delta$ where A_k's and Δ_k's are as in the previous item, and Δ is the fifth element of i;
- if the forth element of line i is PREM, then the third and the fifth elements are empty (we might reserved a special code for 'empty').

Clearly, all these conditions can be checked computably. Now it follows readily from the recursiveness of Proof that Sub and Head are recursive predicates, whereas len is a recursive function.

Note that $\mathsf{Proof}_\Gamma(n)$ is true iff $\mathsf{Proof}(n)$ holds and, additionally, for lines with the mark 'PREM' in their (iv)-component, their (ii)-

components are some elements of Γ — the latter is recursive w.r.t. the oracle Γ.

Why is $\mathsf{Mrk}^{\mathbf{r}}(i,n)$ recursive? Clearly, having the code n of a stage of proof (call it s, for short) at hands, one is able to find, in the effective way, all minimal *Dab*-formulas at that stage, hence construct the finite set U_s which allows to effectively provide the **r**-marking for all lines in s. $\dashv$

Now we are to complete the proof of the proposition. Using the predicates introduced above, the condition $\Gamma \vdash_{CLuN^{\mathbf{r}}} A$ can be expressed as

$$(\dagger)\quad \exists n\,\exists i\,(\mathsf{Proof}_\Gamma(n) \wedge \mathsf{Head}(n,i,\gamma(A)) \wedge \neg\mathsf{Mrk}^{\mathbf{r}}(i,n) \wedge \forall k\,(\mathsf{Sub}(n,k) \wedge \mathsf{Proof}_\Gamma(k) \wedge \mathsf{Mrk}^{\mathbf{r}}(i,k) \to \exists l\,(\mathsf{Sub}(k,l) \wedge \mathsf{Proof}_\Gamma(l) \wedge \neg\mathsf{Mrk}^{\mathbf{r}}(i,l))))\,,$$

or, equivalently, as

$$\exists n \exists i \forall k \exists l (\mathsf{Proof}_\Gamma(n) \wedge \mathsf{Head}(n,i,\gamma(A)) \wedge \neg\mathsf{Mrk}^{\mathbf{r}}(i,n) \wedge (\mathsf{Sub}(n,k) \wedge \mathsf{Proof}_\Gamma(k) \wedge \mathsf{Mrk}^{\mathbf{r}}(i,k) \to (\mathsf{Sub}(k,l) \wedge \mathsf{Proof}_\Gamma(l) \wedge \neg\mathsf{Mrk}^{\mathbf{r}}(i,l)))).$$

Obviously, the latter represents a $\Sigma_3^{0,\Gamma}$-relation. $\dashv$

Corollary 3.9. *Let $\Gamma \subseteq For_{CL}$. If Γ is Π^0_m, then $Cn^{\mathbf{r}}(\Gamma)$ is Σ^0_{m+3}, and if Γ is Σ^0_{m+1}, then $Cn^{\mathbf{r}}(\Gamma)$ is Σ^0_{m+3}.*

Proof. First, remark that the predicate Proof_Γ has the same complexity as Γ.

If Γ is in Σ^0_{m+1}, then $(\dagger)$ (from the proof of Theorem 3.7) can be represented as

$$\exists n\,\exists i\ (\mathsf{A} \wedge \forall k\ (\neg\mathsf{B} \vee \exists l\,\mathsf{C}))$$

where A, B and C are Σ_{m+1}-formulas[3]. Since $\neg\mathsf{B}$ is equivalent to a Π_{m+1}-formula, it can be transformed into $\forall\overline{s}\,\mathsf{D}$ with D being a Σ_m-formula. Hence we get the chain of equivalences:

$$(\dagger) \iff \exists n\,\exists i\ (\mathsf{A} \wedge \forall k\ (\forall\overline{s}\,\mathsf{D} \vee \exists l\,\mathsf{C})) \iff \exists n\,\exists i\ (\mathsf{A} \wedge \forall k\,\forall\overline{s}\ (\mathsf{D} \vee \exists l\,\mathsf{C}))$$
$$\iff \exists n\,\exists i\ (\mathsf{A} \wedge \forall k\,\forall\overline{s}\,\exists l\ (\mathsf{D} \vee \mathsf{C})) \iff \exists n\,\exists i\,\forall k\,\forall\overline{s}\,\exists l\ (\mathsf{A} \wedge (\mathsf{D} \vee \mathsf{C}))$$

[3] Obviously, one may assume that k, $\overline{s}$ and l does not occur in A, l does not occur in D, and $\overline{s}$ does not occur in C.

where $\mathsf{A} \wedge (\mathsf{D} \vee \mathsf{C})$ may be expressed by a Σ_{m+1}-formula. Thus, the condition $\Gamma \vdash_{CLuN^{\mathbf{r}}} A$ is specified by a Σ_{m+3}-formula, whence the result follows.

Clearly, if the set Γ is Π^0_m, then it is Σ^0_{m+1} as well. So, by the previous argument, $Cn^{\mathbf{r}}(\Gamma)$ will be in Σ^0_{m+3}. ⊣

In particular, the special case of Corollary 3.9 is

Corollary 3.10. *For every r.e.* $\Gamma \subseteq For_{CL}$, *the set* $Cn^{\mathbf{r}}(\Gamma)$ *is* Σ^0_3.

This statement can be reformulated in a uniform way. Let W_n, $n \in \omega$, be an effective enumeration of all r.e. subsets of ω (here the 'effectiveness' means that the set $\{\langle n, m\rangle \mid n \in W_m\}$ is again r.e.).

Corollary 3.11. *The set*

$$\{\langle n, \gamma(A)\rangle \mid \Gamma \cup \{A\} \subseteq For_{CL}, W_n = \gamma(\Gamma) \text{ and } \Gamma \vdash_{CLuN^{\mathbf{r}}} \varphi\}$$

is Σ^0_3.

Notice that Corollary 3.10 looks like a generalization of the result on the complexity upper bound for the set of $CLuN^{\mathbf{r}}$-consequences of a recursive set of premisses (namely the result stated in [5]). Actually, these statements are equivalent due to the fact that every r.e. $CLuN^{\mathbf{r}}(CLuN^{\mathbf{m}})$-theory can be recursively axiomatized.

Proposition 3.12. *For every r.e.* $\Gamma \subseteq For_{CL}$, *there is a recursive* $\Gamma' \subseteq For_{CL}$ *such that*

$$Cn^{\mathbf{r}}(\Gamma) = Cn^{\mathbf{r}}(\Gamma') \quad \text{and} \quad Cn^{\mathbf{m}}(\Gamma) = Cn^{\mathbf{m}}(\Gamma').$$

Proof. Let $\varphi_0, \varphi_1, \ldots$ be an effective enumeration of all elements of Γ. Consider the sequence of formulas $\psi_n := \varphi_0 \wedge \ldots \wedge \varphi_n$, $n \in \omega$. Due to the requirements on the Gödel numbering, if $n < m$ then $\gamma(\psi_n) < \gamma(\psi_m)$, because in this case ψ_n is a proper subformula of ψ_m. Thus, $\Gamma' = \{\psi_n \mid n \in \omega\}$ can be enumerated by means of a monotonic recursive function and hence is recursive. Trivially, $Cn_{CLuN}(\Gamma) = Cn_{CLuN}(\Gamma')$.

Since Γ and Γ' are syntactically (and so semantically) equivalent, they have the same models and $U(\Gamma) = U(\Gamma')$. By definitions, this immediately implies the desired conclusions. ⊣

In effect, the last statement can be generalized to every lower limit logic **LLL** the language of which contains a fusion connective $*$ such that for any formulas $A_1, \ldots, A_n$ (in the language of **LLL**), we have

$$Cn_{\mathbf{LLL}}(\{A_1, \ldots, A_n\}) = Cn_{\mathbf{LLL}}(\{A_1 * \cdots * A_n\}) .$$

In case of *CLuN*, the conjunction plays the role of fusion. Moreover, the transformation $\Gamma \mapsto \Gamma'$ (cf. the proof) can be viewed effectively in the sense that given a number of some r.e. set Γ (i.e., n satisfying $W_n = \gamma(\Gamma)$) we computably get a Kleene number of an appropriate recursive set Γ'.

Finally, note that the lower bound proof (for the reliability strategy) from [5] can be adapted to obtain

PROPOSITION 3.13. *For each $m \geqslant 0$, there exists a $\Pi^0_m(\Sigma^0_{m+1})$-set $\Gamma \subseteq For_{CL}$ such that $Cn^{\mathbf{r}}(\Gamma)$ is Σ^0_{m+3}-hard.*

SKETCH OF PROOF. Let $A(v)$ be an arithmetical Σ_{m+3}-formula with the property: the set $\{n \in \omega \mid \mathfrak{N} \vDash A(n)\}$ is Σ^0_{m+3}-complete. Clearly, using the usual coding techniques, $A(v)$ can be translated into the form

$$\exists x\, \forall y\, \exists z\, B(x, y, z, v)$$

where $B(x, y, z, v)$ is a Π_m-formula.

Assume that $\Gamma \subseteq For_{CL}$ is obtained by applying the scheme:

- for any n, i, k and l, the set Γ contains the formulas

$$s^n_{i,k,l}\,, \quad \left(q^n_{i,k} \wedge \neg q^n_{i,k}\right) \vee (r^n_i \wedge \neg r^n_i) \quad \text{and} \quad p_n \vee (r^n_i \wedge \neg r^n_i)\,;$$

- for any n, i, k and l, if $B(i, k, l, n)$ holds in $\mathfrak{N}$, then Γ includes

$$s^n_{i,k,l} \to q^n_{i,k} \wedge \neg q^n_{i,k}\,.$$

Trivially, we have that (the set of codes of formulas in) Γ is Π^0_m. By a routine argument, one is able to demonstrate the equivalence

$$\Gamma \vDash_{CLuN^{\mathbf{r}}} p_n \quad \Longleftrightarrow \quad \mathfrak{N} \vDash A(n)\ ,$$

whence the first part of the result follows.

For the second part, remark that if we already have a Π^0_m-set $\Gamma \subseteq For_{CL}$ with Σ^0_{m+3}-hard set of $CLuN^{\mathbf{r}}$-consequences (see the previous case), then Γ is obviously a Σ^0_{m+1}-set with the same consequences. $\dashv$

Therefore the estimations from Corollary 3.9 are exact, namely

Corollary 3.14. *For each* $m \geqslant 0$*, there exists a* $\Pi^0_m (\Sigma^0_{m+1})$*-set* $\Gamma \subseteq$ For_{CL} *such that* $Cn^{\mathbf{r}}(\Gamma)$ *is* Σ^0_{m+3}*-complete.*

Further, we discuss the algorithmic complexity of $CLuN^{\mathbf{m}}$-consequence relation. In [7] P. Verdee constructed the recursive set of premisses such that the set of its $CLuN^{\mathbf{m}}$-consequences is Π^1_1-hard. It follows from the next statement that Π^1_1 appears to be the upper bound for the complexity of the set of $CLuN^{\mathbf{m}}$-consequences from any (fixed) arithmetical Γ.

Theorem 3.15. *For every* $\Gamma \subseteq For_{CL}$*, the set* $Cn^{\mathbf{m}}(\Gamma)$ *is* $\Pi^{1,\Gamma}_1$.

Proof. Let us consider the following predicates and functions: $\mathsf{Seq}(n)$ which is true iff n is a code of a non-empty finite sequence of numbers; $\mathsf{lh}(n)$ which returns the length of n in case $\mathsf{Seq}(n)$ holds, and 0 otherwise; $(n)_i$ which returns the i-th component of n in case $\mathsf{Seq}(n)$ holds, and 0 otherwise.

Obviously, all these are primitive recursive ones, and so representable via the formulas of the first order arithmetic with restricted quantifies. Hence we can introduce the corresponding predicate and functions into the language of arithmetic with no harm in expressiveness (cf. [6] for the details). For simplicity, suppose we use the same notation $\mathsf{Seq}(x)$, $\mathsf{lh}(x)$ and $(x)_i$ for them in the formal language (a similar technique is to be applied to other recursive predicates and functions needed below). So the formula

$$\mathsf{Sbset}(x,y) := \mathsf{Seq}(x) \wedge \mathsf{Seq}(y) \wedge \forall i \leq \mathsf{lh}(x)\, \exists j \leq \mathsf{lh}(y)\, ((x)_i = (y)_j)$$

expresses the fact that all elements of (the finite sequence) x occur in (the finite sequence) y. Now if $\Omega(x)$ is a primitive recursive predicate checking that x is a code of some abnormality, then

$$\mathsf{Fsa}(x) := \mathsf{Seq}(x) \wedge \forall i \leq \mathsf{lh}(x)\, \Omega((x)_i)$$

says that x is a finite sequence of abnormalities. Analogously, let $\mathsf{dab}(x)$ be a function returning the code of the disjunction of all elements of x in case $\mathsf{Fsa}(x)$ holds, and 0 otherwise (trivially, it is primitive recursive).

Naturally, one is able to write down a $\Sigma^{0,\Gamma}_1$-predicate $\mathsf{Pr}^{\Gamma}_{CLuN}(x)$ which verifies if a formula codified by x is provable from Γ in $CLuN$. Thus, the following

$$\begin{aligned}\mathsf{Mdab}^{\Gamma}(x) := {} & \mathsf{Fsa}(x) \wedge \mathsf{Pr}^{\Gamma}_{CLuN}(\mathsf{dab}(x)) \wedge \\ & \forall z\,((\mathsf{Fsa}(z) \wedge \mathsf{Pr}^{\Gamma}_{CLuN}(\mathsf{dab}(z)) \wedge \mathsf{Sbset}(z,x)) \to \mathsf{Sbset}(x,z))\end{aligned}$$

means that $\mathsf{dab}(x)$ is a minimal Dab-consequence of Γ.

Next, if P is an unary predicate variable, then the second order formula

$$\mathsf{Choice}^{\Gamma}(P) := \forall x\,(\mathsf{M}Dab^{\Gamma}(x) \to \exists i \leq \mathsf{lh}(x)\, P((x)_i))$$

stands for "P is a choice set for the set of all minimal Dab-consequences of Γ". In view of Proposition 2.4, each minimal choice set for the set of minimal Dab-consequences of Γ can be distinguished by the property

$$\begin{aligned}\mathsf{Mchoice}^{\Gamma}(P) := {} & \mathsf{Choice}^{\Gamma}(P) \wedge \forall x\,(P(x) \to \\ & \exists y\,(\mathsf{Mdab}^{\Gamma}(y) \wedge \forall i \leq \mathsf{lh}(y)\,(P((y)_i) \to (y)_i = x)))\,.\end{aligned}$$

And then we use Theorem 2.11 to express the fact that A is finally $CLuN^m$-derivable from Γ, namely

$$\begin{aligned}\forall P\Big(&\mathsf{Mchoice}^{\Gamma}(P) \to \exists x(\mathsf{Fsa}(x) \wedge \\ & \forall i \leqslant \mathsf{lh}(x)\ (\neg P((x)_i)) \wedge \mathsf{Pr}^{\Gamma}_{CLuN}(\gamma(A) \vee \mathsf{dab}(x)))\Big),\end{aligned}$$

where $\gamma(A) \vee \mathsf{dab}(x)$ is a shorthand for $\vee(\gamma(A), \mathsf{dab}(x))$ (here $\vee$ is a function which returns the code of the disjunction of formulas represented by its arguments). Obviously, we have obtained a $\Pi^{1,\Gamma}_1$-formula. ⊣

Corollary 3.16. *For every arithmetical $\Gamma \subseteq For_{CL}$, the set $Cn^{\mathbf{m}}(\Gamma)$ is Π^1_1.*

The complexity of the set of $CLuN^{\mathbf{m}}$-consequences of a given Γ can be essentially reduced if we additionally presuppose that the set of formulas unreliable w.r.t. the premiss set Γ is finite.

Proposition 3.17. *For each $\Gamma \subseteq For_{CL}$, if the set $U(\Gamma)$ is finite, then the set $Cn^{\mathbf{m}}(\Gamma)$ is $\Sigma^{0,\Gamma}_1$.*

Proof. We will use the notation from the proof of Theorem 3.15. Since every minimally abnormal model of Γ is reliable (remember Corollary 2.12), the finiteness of $U(\Gamma)$ implies that both the set (of sets) $\Phi(\Gamma)$ and all of its elements are finite. To check whether $\Gamma \vdash_{CLuN^{\mathbf{m}}} A$ holds or not, one has to verify, for every finite $\varphi \in \Phi(\Gamma)$, the condition

$$\exists x \left(\mathsf{Fsa}(x) \wedge \forall i \leqslant \mathsf{lh}(x)\, \forall j \leqslant \mathsf{lh}(\gamma(\varphi))\, ((x)_i \neq (\gamma(\varphi))_j) \wedge \right.$$
$$\left. \mathsf{Pr}^{\Gamma}_{CLuN}(\gamma(A) \vee \mathsf{dab}(x))\right),$$

where $\gamma(\varphi)$ is the code of some finite sequence consisting of the codes of elements in φ (one may choose an arbitrary sequence with this property). Since $\mathsf{Pr}^{\Gamma}_{CLuN}(x)$ is a $\Sigma^{0,\Gamma}_1$-formula, the above condition can be given by a $\Sigma^{0,\Gamma}_1$-formula. And the finite conjunction of all such formulas is, of course, a $\Sigma^{0,\Gamma}_1$-formula as well. $\dashv$

References

[1] Batens, D., "Inconsistency-adaptive logics';, pages 445–472 in: E. Orłowska (ed.), *Logic at Work. Essays dedicated to the memory of Helena Rasiowa*, Springer, Heidelberg, New York, 1999,

[2] Batens, D.,"A general characterization of adaptive logics", *Logique et Analyse* 173–174–175 (2001): 45–68.

[3] Batens, D., "A procedural criterion for final derivability in inconsistency-adaptive logics", *Journal of Applied Logic* 3 (2005): 221–250.

[4] Batens, D. "Adaptive logics and dynamic proofs", manuscript, available at http://logica.ugent.be/adlog/book.html

[5] Horsten, L., and P. Welch, "The undecidability of propositional adaptive logic", *Synthese* 158 (2007): 41–60.

[6] Smorynski, C., *Self-reference and Modal Logic*, Springer, Berlin, 1985.

[7] Verdee, P.,"Adaptive logics using the minimal abnormality strategy are Π^1_1-complex", *Synthese* 167 (2009): 93–104.

[8] Verdee, P., "A proof procedure for adaptive logics", to appear in *Logical Journal of IGPL*.

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