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TOWARDS A BRIDGE OVER TWO APPROACHES IN CONNEXIVE LOGIC

Abstract. The present note aims at bridging two approaches to connexive logic: one approach suggested by Heinrich Wansing, and another approach suggested by Paul Egré and Guy Politzer. To this end, a variant of **FDE**-based modal logic, developed by Sergei Odintsov and Heinrich Wansing, is introduced and some basic results including soundness and completeness results are established.

Keywords: connexive logic; contra-classical logic; Belnap-Dunn logic; modal logic; experimental philosophy

1. Introduction

Connexive logics are characterized by having so-called Aristotle's theses and Boethius' theses as derivable/valid formulas:
Aristotle's theses $\sim(A \to \sim A), \sim(\sim A \to A)$
Boethius' theses $(A \to B) \to \sim(A \to \sim B), (A \to \sim B) \to \sim(A \to B)$
More recently, there are some discussions casting some doubts on the very notion of connexivity [e.g., 5, 6], but in this note I will stick to the more conservative account of connexive logic.

In what follows, I will briefly revisit two approaches in connexive logics discussed in this note.

1.1. Background I

One of the traditions in connexive logic founded by Heinrich Wansing captures connexivity through a nonstandard falsity condition for the conditional. More specifically, Wansing suggests to take the condition

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of the form "if A is true then B is false" rather than the condition of the form "A is true and B is false" as the falsity condition for the conditional of the form "if A then B", where truth and falsity are not necessarily exclusive. This simple idea was first suggested in [19] in which the connexive logic **C** was formulated as a variant of Nelson's logic **N4** [cf. 4, 18].[1] Some later developments observed that the central idea of Wansing does not rely on **N4**. Indeed, Wansing's idea works in the context of a four-valued logic [cf. 20], a three-valued logic [cf. 14], and even in the context of weak relevant logics, namely the basic relevant logic **BD** of Graham Priest and Richard Sylvan [cf. 13], as well as in the context of conditional logics [cf. 7, 22].

It should be noted that as a byproduct, connexive logics formulated à la Wansing will include the converse direction of Boethius' theses as valid/derivable theses. Of course, these formulas are *not* required for connexive logics in general. In fact, these formulas are sometimes criticized.[2] However, as Priest claims in [16, p. 178], Wansing's system is most likely to be "one of the simplest and most natural."

1.2. Background II

Another approach to connexivity in the literature is the one through experimental philosophy. In [15], Niki Pfeifer marked the first contribution towards this direction with a more general aim to "extend the domain of experimental philosophy to conditionals" [15, p. 223].[3] The particular focus is on Aristotle's theses, and Pfeifer proposes an interpretation of Aristotle's theses based on coherence based probability logic and offers a justification for Aristotle's theses.

The present note focuses on another paper [3] by Paul Egré and Guy Politzer who carried out an experiment related to the negation of indicative conditionals.[4] In particular, they consider *weak* conjunctive and

[1] Note that in [19], not only the propositional logic **C**, but also the first-order logic **QC** (quantified **C**) and the propositional modal logic **CK** (connexive analogue of the modal logic **K**) are introduced.

[2] See, e.g., [8, p. 446]. For a counter-argument by Wansing and Skurt, see [21].

[3] It deserves to be highlighted that the focus in [15] is on *indicative* conditionals, not subjunctive conditionals which collected the attention of the modern founder of connexive logic, namely Richard Angell, in [1].

[4] It is noted in [3] that there will be an extended version, but to the best of my knowledge, it is not in print yet.

conditional formulas of the form $A \wedge \Diamond \sim B$ and $A \rightarrow \Diamond \sim B$ respectively, beside the more well-discussed *strong* conjunctive and conditional formulas of the form $A \wedge \sim B$ and $A \rightarrow \sim B$ respectively, as formulas equivalent to $\sim(A \rightarrow B)$. Many of the debates on the negation of conditionals focused on the strong forms and discussed whether the conjunctive formula is appropriate or the conditional formula is appropriate. However, Egré and Politzer challenge the debate by suggesting that we should also take into account of the *weak* forms, not only the strong forms.

1.3. Aim

Based on these backgrounds, the general aim behind this note is to see if we can bridge the above traditions in connexive logics. The more specific aim of this note is to observe that the formulas considered by Egré and Politzer can be formalized in a rather natural manner by following the idea of Wansing to consider falsity conditions of the conditional. To this end, we make use of a modal logic that expands the four-valued logic $\mathbf{N4}_p^{\perp}$ (or $\mathbf{B}_4^{\rightarrow}$ in the terminology of [9])[5], developed by Sergei Odintsov and Heinrich Wansing in [10].[6]

2. Semantics and proof theory

The languages $\mathcal{L}$ and $\mathcal{L}^m$ consist of finite sets $\{\perp, \sim, \wedge, \vee, \rightarrow\}$ and $\{\perp, \sim, \wedge, \vee, \rightarrow, \Box, \Diamond\}$ of propositional connectives respectively and a countable set Prop of propositional variables which we denote by p, q, etc. Furthermore, we denote by Form and Form^m the set of formulas defined as usual in $\mathcal{L}$ and $\mathcal{L}^m$ respectively. We denote a formula of both languages by A, B, C, etc. and a set of formulas of both languages by Γ, Δ, Σ, etc.

2.1. Semantics

The following semantics is obtained by making a simple change to the semantics for the modal logic **BK** of Odintsov and Wansing.

Definition 1. A **WBK**-*model*[7] for the language $\mathcal{L}^m$ is a triple $\langle W, R, V\rangle$, where W is a non-empty set (of states); R is a binary relation on W; and

[5] The bottom element free fragment is known as **HBe** since [2].

[6] See [11] for an overview of modal logics based on Belnap-Dunn logic that may also serve well for our purposes.

[7] **WBK** stands for weak **BK**.

$V : W \times \mathsf{Prop} \longrightarrow \{\emptyset, \{0\}, \{1\}, \{0,1\}\}$ is an assignment of truth values to state-variable pairs. Valuations V are then extended to interpretations I to state-formula pairs by the following conditions:

- $I(w, p) = V(w, p)$,
- $1 \notin I(w, \bot)$,
- $0 \in I(w, \bot)$,
- $1 \in I(w, {\sim} A)$ iff $0 \in I(w, A)$,
- $0 \in I(w, {\sim} A)$ iff $1 \in I(w, A)$,
- $1 \in I(w, A \wedge B)$ iff $1 \in I(w, A)$ and $1 \in I(w, B)$,
- $0 \in I(w, A \wedge B)$ iff $0 \in I(w, A)$ or $0 \in I(w, B)$,
- $1 \in I(w, A \vee B)$ iff $1 \in I(w, A)$ or $1 \in I(w, B)$,
- $0 \in I(w, A \vee B)$ iff $0 \in I(w, A)$ and $0 \in I(w, B)$,
- $1 \in I(w, A \to B)$ iff $1 \notin I(w, A)$ or $1 \in I(w, B)$,
- $0 \in I(w, A \to B)$ iff $1 \notin I(w, A)$ or for some $x \in W$: (wRx and $0 \in I(x, B)$),
- $1 \in I(w, \Box A)$ iff for all $x \in W$: (wRx only if $1 \in I(x, A)$),
- $0 \in I(w, \Box A)$ iff for some $x \in W$: (wRx and $0 \in I(x, A)$),
- $1 \in I(w, \Diamond A)$ iff for some $x \in W$: (wRx and $1 \in I(x, A)$),
- $0 \in I(w, \Diamond A)$ iff for all $x \in W$: (wRx only if $0 \in I(x, A)$).

Finally, the semantic consequence is now defined as follows: $\Sigma \models A$ iff for all **WBK**-models $\langle W, R, I \rangle$, and for all $w \in W$: $1 \in I(w, A)$ if $1 \in I(w, B)$ for all $B \in \Sigma$.

Remark 1. Consider the language $\mathcal{L}$. Then, note that the extension $\mathbf{N4}^{\bot}_{p}$ of Nelson's logic $\mathbf{N4}^{\bot}$ by Peirce's law is obtained by replacing the falsity condition for implication by the following condition.

$$0 \in I(w, A \to B) \text{ iff } 1 \in I(w, A) \text{ and } 0 \in I(w, B).$$

This reflects the strong conjunctive formula in Egré and Politzer's terminology.

Moreover, Wansing's four-valued connexive logic **MC** is obtained by replacing the falsity condition for implication by the following condition.

$$0 \in I(w, A \to B) \text{ iff } 1 \notin I(w, A) \text{ or } 0 \in I(w, B).$$

This reflects the strong conditional formula in Egré and Politzer's terminology.

2.2. Proof Theory

We now turn to the proof theory. Since the modal logic **BK** is presented in terms of a Hilbert-style calculus, we follow the same strategy.

Definition 2. Consider the following axioms and rules, where:

- $\neg A$, $A \leftrightarrow B$ and $A \Leftrightarrow B$

abbreviate, respectively:

- $A \to \bot$, $(A \to B) \wedge (B \to A)$ and $(A \leftrightarrow B) \wedge ({\sim} A \leftrightarrow {\sim} B)$:

$$A \to (B \to A) \tag{Ax1}$$
$$(A \to (B \to C)) \to ((A \to B) \to (A \to C)) \tag{Ax2}$$
$$((A \to B) \to A) \to A \tag{Ax3}$$
$$(A \wedge B) \to A \tag{Ax4}$$
$$(A \wedge B) \to B \tag{Ax5}$$
$$(C \to A) \to ((C \to B) \to (C \to (A \wedge B))) \tag{Ax6}$$
$$A \to (A \vee B) \tag{Ax7}$$
$$B \to (A \vee B) \tag{Ax8}$$
$$(A \to C) \to ((B \to C) \to ((A \vee B) \to C)) \tag{Ax9}$$
$$\bot \to A \tag{Ax10}$$
$$A \to {\sim}\bot \tag{Ax11}$$
$${\sim}{\sim} A \leftrightarrow A \tag{Ax12}$$
$${\sim}(A \wedge B) \leftrightarrow ({\sim} A \vee {\sim} B) \tag{Ax13}$$
$${\sim}(A \vee B) \leftrightarrow ({\sim} A \wedge {\sim} B) \tag{Ax14}$$
$${\sim}(A \to B) \leftrightarrow (A \to \Diamond {\sim} B) \tag{Ax15}$$
$$(\Box A \wedge \Box B) \to \Box(A \wedge B) \tag{Ax16}$$
$$\Box(A \to A) \tag{Ax17}$$
$$\Diamond A \leftrightarrow \neg\Box\neg A \tag{Ax18}$$
$${\sim}\Box A \Leftrightarrow \Diamond {\sim} A \tag{Ax19}$$
$${\sim}\Diamond A \Leftrightarrow \Box {\sim} A \tag{Ax20}$$
$$\frac{A \quad A \to B}{B} \tag{MP}$$
$$\frac{A \to B}{\Box A \to \Box B} \tag{M$\Box$}$$

$$\frac{A \to B}{\Diamond A \to \Diamond B} \tag{M$\Diamond$}$$

The logic **WBK** is defined as the deductive closure of axioms (Ax1)–(Ax20) under the rules (MP), (M$\Box$) and (M$\Diamond$). We write $\Gamma \vdash A$ iff A belongs to the closure of $WBK \cup \Gamma$ under (MP).

Remark 2. Consider again the language $\mathcal{L}$. Then, if we eliminate axioms (Ax16)–(Ax20) together with rules (M$\Box$) and (M$\Diamond$), and replace (Ax15) by the following formula, then we obtain an axiomatization of the extension of Nelson's logic known as $\mathbf{N4}_p^{\perp}$:

$$\sim(A \to B) \leftrightarrow (A \wedge \sim B),$$

Moreover, if we replace (Ax15) by the following formula, then we obtain an axiomatization of the system **MC** of Wansing:

$$\sim(A \to B) \leftrightarrow (A \to \sim B).$$

Remark 3. As one of the referees correctly pointed out, note that the presence of $\Box$ is simply for the sake of presentation, especially the axiomatic proof system.

3. Soundness and completeness

As usual, the soundness part is rather straightforward. By induction on the length of the proof we obtain:

Theorem 1 (Soundness). *For $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, if $\Gamma \vdash A$ then $\Gamma \models A$.*

For the completeness proof, we first introduce some standard notions.

Definition 3. A set of formulas, Σ, is a *prime* **WBK**-*theory* iff (i) $WBK \subseteq \Sigma$, (ii) it is closed under (MP), (iii) $A \vee B \in \Sigma$ implies $A \in \Sigma$ or $B \in \Sigma$ and (iv) it is *non-trivial*, namely if $A \notin \Sigma$ for some A.

The following lemmas are well-known, and thus the proofs are omitted.

Lemma 1. *If $\Sigma \nvdash A$ then there is a prime* ***WBK****-theory, Δ, such that $\Sigma \subseteq \Delta$ and $\Delta \nvdash A$.*

Lemma 2. *If $\Sigma \nvdash \Box A$ then there is a prime* ***WBK****-theory, Δ, such that $\Sigma_{\Box} \subseteq \Delta$ and $\Delta \nvdash A$ where $\Sigma_{\Box} = \{A : \Box A \in \Sigma\}$.*

Now, we are ready to prove the completeness.

THEOREM 2 (Completeness). *For* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, *if* $\Gamma \models A$ *then* $\Gamma \vdash A$.

PROOF. Suppose that $\Gamma \nvdash A$. Then by Lemma 1, there is a $\Pi \supseteq \Gamma$ such that Π is a prime **WBK**-theory and $A \notin \Pi$. Define the model $\mathfrak{A} = \langle X, R, I\rangle$, where $X = \{\Delta : \Delta$ is a prime **WBK**-theory$\}$, $\Delta R \Sigma$ iff $\Delta_{\Box} \subseteq \Sigma$ (recall that $\Delta_{\Box} = \{A : \Box A \in \Delta\}$) and I is defined thus. For every state, Σ and propositional parameter, p:

$$1 \in I(\Sigma, p) \text{ iff } p \in \Sigma \text{ and } 0 \in I(\Sigma, p) \text{ iff } {\sim} p \in \Sigma$$

We show that this condition holds for any arbitrary formula, B:

$$1 \in I(\Sigma, B) \text{ iff } B \in \Sigma \text{ and } 0 \in I(\Sigma, B) \text{ iff } {\sim} B \in \Sigma \qquad (*)$$

It then follows that $\mathfrak{A}$ is a counter-model for the inference, and hence that $\Gamma \not\models A$. The proof of $(*)$ is by a simultaneous induction on the complexity of B with respect to the positive and the negative clause.

For bottom. For the positive clause, note that the semantic clause is $1 \notin I(\Sigma, \bot)$ and that (Ax10) together with the non-triviality of Σ gives us $\bot \notin \Sigma$. Therefore, we obviously have $1 \notin I(\Sigma, \bot)$ iff $\bot \notin \Sigma$, and so, by contraposition, the desired result is proved. For the negative clause, we have the semantic clause $0 \in I(\Sigma, \bot)$. Moreover, since Σ is nonempty, let D be an element of Σ. In view of (Ax11), we have $\vdash D \to {\sim}\bot$, and this together with $D \in \Sigma$ implies ${\sim}\bot \in \Sigma$ since Σ is a prime **WBK**-theory. Therefore, we obtain $0 \in I(\Sigma, \bot)$ iff ${\sim}\bot \in \Sigma$.

For negation. We begin with the positive clause: $1 \in I(\Sigma, {\sim} C)$ iff $0 \in I(\Sigma, C)$ iff ${\sim} C \in \Sigma$ (IH). The negative clause is also straightforward: $0 \in I(\Sigma, {\sim} C)$ iff $1 \in I(\Sigma, C)$ iff $C \in \Sigma$ (IH) iff ${\sim}{\sim} C \in \Sigma$ (Ax12).

For necessity. We begin with the positive clause: $1 \in I(\Sigma, \Box C)$ iff for all $\Delta \in X$: $\Sigma R \Delta$ only if $1 \in I(\Delta, C)$ iff for all $\Delta \in X$: $\Sigma_{\Box} \subseteq \Delta$ only if $C \in \Delta$ (IH) iff $\Box C \in \Sigma$ $(*)$. For the last equivalence, assume $\Box C \in \Sigma$, and that for all $\Delta \in X$: $\Sigma_{\Box} \subseteq \Delta$. Then, $\Box C \in \Sigma$ is equivalent to $C \in \Sigma_{\Box}$, so we obtain $C \in \Delta$, as desired. For the other direction, we use Lemma 2. The negative clause is also straightforward: $0 \in I(\Sigma, \Box C)$ iff for some $\Delta \in X$: ($\Sigma R \Delta$ and $0 \in I(\Delta, C)$) iff for some $\Delta \in X$: ($\Sigma_{\Box} \subseteq \Delta$ and ${\sim} C \in \Delta$) (IH) iff $\Diamond {\sim} C \in \Sigma$ $(**)$ iff ${\sim}\Box C \in \Sigma$ (Ax19).

For the equivalence $(**)$, the proof runs exactly the same with the equivalence $(***)$ in the positive case for possibility.

For possibility. We begin with the positive clause: $1 \in I(\Sigma, \Diamond C)$ iff for some $\Delta \in X$: $\Sigma R \Delta$ and $1 \in I(\Delta, C)$ iff for some $\Delta \in X$: $\Sigma_\Box \subseteq \Delta$ and $C \in \Delta$ (IH) iff $\Diamond C \in \Sigma$ $(***)$.

For the equivalence $(***)$, note first that we obtain the following by the same argument for $(*)$ above.

$$\Box\neg C \in \Sigma \text{ iff for all } \Delta \in X := \Sigma_\Box \subseteq \Delta \text{ only if } C \notin \Delta.$$

By taking the contraposition, and noting that Σ is a prime **WBK**-theory, we obtain:

$$\neg\Box\neg C \in \Sigma \text{ iff for some } \Delta \in X := \Sigma_\Box \subseteq \Delta \text{ and } C \in \Delta.$$

Therefore, in view of (Ax18), the desired equivalence is established.

The negative clause is also straightforward: $0 \in I(\Sigma, \Diamond C)$ iff for all $\Delta \in X$: $\Sigma R \Delta$ only if $0 \in I(\Delta, C)$ iff for all $\Delta \in X$: $\Sigma_\Box \subseteq \Delta$ only if ${\sim} C \in \Delta$ (IH) iff $\Box {\sim} C \in \Sigma$ $(****)$ iff ${\sim}\Diamond C \in \Sigma$ (Ax20).

For the equivalence $(****)$, the proof runs exactly the same with the equivalence $(*)$ in the positive case for necessity.

For disjunction. We begin with the positive clause: $1 \in I(\Sigma, C \vee D)$ iff $1 \in I(\Sigma, C)$ or $1 \in I(\Sigma, D)$ iff $C \in \Sigma$ or $D \in \Sigma$ (IH) iff $C \vee D \in \Sigma$ (since Σ is a prime theory). The negative clause is also straightforward: $0 \in I(\Sigma, C \vee D)$ iff $0 \in I(\Sigma, C)$ and $0 \in I(\Sigma, D)$ iff ${\sim} C \in \Sigma$ and ${\sim} D \in \Sigma$ (IH) iff ${\sim} C \wedge {\sim} D \in \Sigma$ (since Σ is a theory) iff ${\sim}(C \vee D) \in \Sigma$ (Ax14).

For conjunction. Similar to the case for disjunction.

For implication. We begin with the positive clause: $1 \in I(\Sigma, C \to D)$ iff $1 \notin I(\Sigma, C)$ or $1 \in I(\Sigma, D)$ iff $C \notin \Sigma$ or $D \in \Sigma$ (IH) iff $C \to D \in \Sigma$ $(\star)$.

For the last equivalence $(\star)$, assume $C \to D \in \Sigma$ and, for reductio, that $C \in \Sigma$ and $D \notin \Sigma$. Then since Σ is a prime **WBK**-theory, Σ is closed under (MP). Therefore, we have $D \in \Sigma$ but this contradicts to $D \notin \Sigma$. For the other way around, just again note that Σ is a prime **WBK**-theory, and that $C \vee (C \to D) \in \Sigma$ and $D \to (C \to D) \in \Sigma$.

As for the negative clause, it is similar to the positive case: $0 \in I(\Sigma, C \to D)$ iff $1 \notin I(\Sigma, C)$ or for some $\Delta \in X$: $\Sigma R \Delta$ and $0 \in I(\Delta, D)$ iff $C \notin \Sigma$ or for some $\Delta \in X$: $\Sigma_\Box \subseteq \Delta$ and ${\sim} D \in \Delta$ (IH) iff $C \notin \Sigma$ or $\Diamond {\sim} D \in \Delta$ $(\dagger)$ iff $C \to \Diamond {\sim} D \in \Sigma$ $(\ddagger)$ iff ${\sim}(C \to D) \in \Sigma$ (Ax15).

For the equivalences $(\dagger)$ and $(\ddagger)$, the proof runs exactly the same with the equivalences $(***)$ and $(\star)$ in the positive cases for possibility and implication. Thus, we obtain the desired result. $\dashv$

Remark 4. Note that we may also formulate a semantics along the weak conjunctive formula in Egré and Politzer's terminology, and establish soundness and completeness results. More specifically, we may replace the falsity condition by the following condition for the semantics:

$$0 \in I(w, A \to B) \text{ iff } 1 \in I(x, A) \text{ and for some } x \in W\colon\ 0 \in I(x, B).$$

For the proof theory, we replace the axiom (Ax15) by the following formula:

$$\sim(A \to B) \leftrightarrow (A \wedge \Diamond \sim B).$$

Once these changes are made, soundness and completeness results can be established by some obvious modifications. The details are safely left for the interested readers.

4. Basic observations

We now turn to observe a few basic results on **WBK**. First, we observe that **WBK** is *not* connexive.

PROPOSITION 1. *The following holds for* **WBK**:

- $\vdash \sim(A \to \sim A) \leftrightarrow (A \to \Diamond A)$
- $\vdash (A \to B) \to \sim(A \to \sim B)$ *iff* $\vdash B \to \Diamond B$

PROOF. For the first item, the proof runs as follows.

1.	$\sim(A \to \sim A) \leftrightarrow (A \to \Diamond \sim\sim A)$	(Ax15)
2.	$(A \to \Diamond \sim\sim A) \leftrightarrow (A \to \Diamond A)$	(Ax12), (M$\Diamond$)
3.	$\sim(A \to \sim A) \leftrightarrow (A \to \Diamond A)$	1, 2

For the second item, it suffices to prove $\vdash (A \to B) \to (A \to \Diamond B)$ iff $\vdash B \to \Diamond B$ in view of (Ax12), (Ax15) and (M$\Diamond$). For the left-to-right direction, we only need to consider the special case $\vdash (B \to B) \to (B \to \Diamond B)$ and make use of $\vdash B \to B$. For the right-to-left direction, we only need to use $\vdash (B \to C) \to ((A \to B) \to (A \to C))$, a thesis of classical positive calculus. This completes the proof. ⊣

COROLLARY 1. **WBK** *is not connexive.*

PROOF. Just take a model with one element in which the accessibility relation is not reflexive. ⊣

However, connexivity can be characterized in a very simple manner in view of the above proposition.

COROLLARY 2. *The extensions of* **WBK** *are connexive iff* $\vdash A \to \Diamond A$.

Note also that the following restricted form of Aristotle's thesis is derivable in **WBK**.

COROLLARY 3. $\vdash \Diamond A \to \sim(A \to \sim A)$

In other words, this shows that Egré-Politzer-style logics have a loose connection to humble connexivity discussed by Kapsner in [6].

Second, even though **WBK** is not connexive, the system is contradictory which is a general feature of connexive logics that implement Wansing's idea.

PROPOSITION 2. **WBK** *is contradictory. In particular* $\vdash (A \wedge \Diamond \sim A) \to A$ *and* $\vdash \sim((A \wedge \Diamond \sim A) \to A)$.

PROOF. The first item is obvious in view of (Ax4). For the second item, just note that, by (Ax15), the formula is equivalent to $(A \wedge \Diamond \sim A) \to \Diamond \sim A$ which is derivable again in view of (Ax5). $\dashv$

COROLLARY 4. *The paracomplete extension of* **WBK** *obtained by the schema* $(A \wedge \sim A) \to B$ *is trivial.*

Remark 5. Note that the above result requires the following theses and *modus ponens*:

- $(A \wedge B) \to A$,
- $(A \wedge B) \to B$,
- $(A \to \Diamond \sim B) \to \sim(A \to B)$.

Thus, if one is happy to defend these, then the consequence relation needs to be paraconsistent. Of course, this is not an entirely new topic. Indeed, in the discussion of connexive logics, there is a tradition in which conjunction elimination is given up in pain of triviality.[8]

Third, we observe that one of the Wansing-style connexive logics can be seen as a special case of Egré-Politzer-style logics.

PROPOSITION 3. *The following holds for* **WBK**:

- $\vdash (A \to \sim B) \to \sim(A \to B)$ *iff* $\vdash B \to \Diamond B$
- $\vdash \sim(A \to B) \to (A \to \sim B)$ *iff* $\vdash \Diamond B \to B$

[8] For a recent discussion, see [21].

Proof. For the first item, the proof is very similar to the second item of Proposition 1. For the second item, it suffices to prove $\vdash (A \to \Diamond \sim B) \to (A \to \sim B)$ iff $\vdash \Diamond B \to B$ in view of (Ax15), and moreover, in view of (Ax12) and (M$\Diamond$), it suffices to prove $\vdash (A \to \Diamond B) \to (A \to B)$ iff $\vdash \Diamond B \to B$. For the left-to-right direction, we only need to consider the special case $\vdash ((A \to A) \to \Diamond B) \to ((A \to A) \to B)$ and make use of $\vdash ((A \to A) \to B) \leftrightarrow B$. For the right-to-left direction, we only need to use $\vdash (B \to C) \to ((A \to B) \to (A \to C))$, a thesis of classical positive calculus. This completes the proof. $\dashv$

Corollary 5. $\vdash {\sim}(A \to B) \leftrightarrow (A \to \sim B)$ *iff* $\vdash \Diamond B \leftrightarrow B$. *That is, one of the Wansing-style connexive logics, namely the system* **MC**, *is obtained iff the possibility operator is trivialized in* **WBK**.

5. Concluding remarks

What I hope to have established in this note is that a variation of Wansing's idea, inspired by Egré and Politzer, can be smoothly formalized in a simple setting almost for free in view of the formulation of the modal logic **BK** of Odintsov and Wansing. Moreover, I observed that Egré-Politzer-style systems can be seen as a generalization of the Wansing-style connexive logic **MC**.

There are some future topics that seem to be interesting as well as promising. I will only note two of them. First, it is interesting to see if we can export some of the observations in this note and see if some experiments will support connexivity or not, and if contradictoriness gets supported or not along the suggestion made by Egré and Politzer. Second, it is a challenging task to explore variants of **WBK** by having different underlying modal logics such as those based on $\mathbf{N4}^{\perp}$, instead of $\mathbf{N4}^{\perp}_{p}$. Philosophically, this will be of some interest for those who prefer indicative conditionals to be captured by the constructive conditional rather than the classical material conditional. Technically, this will require some careful discussion on the relation between two accessibility relations as in intuitionistic modal logics. I only note in relation to the technical point that we can also think of introducing a modality on top of the semantics of Nelson logics without any additional accessibility relation beside the one for the constructive conditional [cf. 12, 17]. This will be more simple technically, but it remains to be discussed if the modality so defined will be philosophically interesting in the present context.

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