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LOGIC AND SETS

Abstract. The notion of the extension of a concept has been used in logic for a long time. It is usually considered to be closely connected to the intuitive notion of a set and thus seems as though it should be embedded into set theory. However, there are significant differences between this “logical” concept of set and the notion of set (class) as defined via standard axiomatic systems of set theory; it may, therefore, be quite misleading to consider the two concepts as being continuous with each other. When we look at the writings of Gottlob Frege and consider the development of his attitude to extensions, we can see what the differences consist in and which of the two notions is more apt to be used in foundations of logic. Frege himself eventually rejected sets entirely.

Keywords: extensions; sets; Gottlob Frege; unsaturated functions; extensional thesis; set theory

Two conceptions of sets

What is a set?¹ Any *collection* of things, as Cantor’s definition claims?² But the usual sense of “collection” seems to be different from how we understand sets. For example, the collection of John’s suits would contain both the trousers and the jackets which belong to the suits. But the set of John’s suits is something different than the set of John’s suit

¹ I use “set” where very often “class” is used. The reason is to highlight the connection with “set theories”, and to exclude from my consideration “proper classes”, which are not sets.

² “By an ‘aggregate’ (Menge) we are to understand any collection into a whole M of definite and separate objects m of our intuition or our thought. These objects are called the ‘elements’ of M ” [10, p. 85].

trousers and suit jackets — they are even disjoint.³ Thus it seems that it may be more promising to think of sets as extensions of concepts or as collections of all things having a certain property.⁴

In this sense, the concept of set, as it appears in logic, is very old. Indeed, Aristotle's syllogistic can be interpreted as a system dealing with these kinds of "sets"⁵ (this fact becomes apparent in using Euler- or

³ Cf. Quine: "We can say that a class is any aggregate, any collection, any combination of objects of any sort; if this helps, well and good. But even this will be less help than hindrance unless we keep clearly in mind that the aggregating or collecting or combining here is to connote no actual displacement of the objects, and further that the aggregation or collection or combination of say seven given pairs of shoes is not to be identified with the aggregation or collection or combination of those fourteen shoes, nor with that of the twenty-eight soles and uppers. In short, a class may be thought of as an aggregate or collection or combination of objects just so long as 'aggregate' or 'collection' or 'combination' is understood strictly in the sense of 'class'" [33, p. 1].

⁴ Where the property may be understood in the broad sense, and can be, within the context of a formal language, captured by an open formula $\varphi(x)$. Cf. Quine: "We can be more articulate on the function of the notion of class. Imagine a sentence about something. Put a blank or variable where the thing is referred to. You have no longer a sentence about that particular thing, but an open sentence, so-called, that may hold true of each of various things and be false of others. Now the notion of class is such that there is supposed to be, in addition to the various things of which that sentence is true, also a further thing which is the *class* having each of those things and no others as member. It is the class *determined* by the open sentence" [33, p. 1]. Cf. also Cantor's earlier definition: "By a 'manifold' (*Mannigfaltigkeit*) or a 'set' (*Menge*) I namely understand generally every multiplicity (*Viele*) which can be thought as one, i.e., every complex (*Inbegriff*) of definite elements which can be united to a whole by a law, and by this I believe I have defined something that is related to the Platonic eidos or idea [...]" [11, note 1, p. 204].

⁵ It seems legitimate to interpret in such a way, i.e., set-theoretically, formulations like: "That one term should be in another as in a whole is the same as for the other to be predicated of all of the first. And we say that one term is predicated of all of another, whenever nothing can be found of which the other term cannot be asserted." Aristotle, *Prior Analytics* [2, 24b27-30]; "A substance — that which is called a substance most strictly, primarily, and most of all — is that which is neither said of a subject nor in a subject, e.g., the individual man or the individual horse. The species in which the things primarily called substances are, are called secondary substances, as also are the genera of these species. For example, the individual man *belongs in a species*, man, and animal is a genus of the species; so these — both man and animal — are called secondary substances." Aristotle, *Categories* [2, 2a13-18], emphasis mine. But the Aristotelian system, with the distinction between individuals and their accidents (which are also particular but do not exist separately), is in fact more complicated than the standard set-theoretical system of *individuals/sets* (or only *sets*).

Venn-Diagrams in syllogistic reasoning). In traditional logic, it is quite common to speak about “anything falling under a concept” or about “the extension of a concept” [3, p. 49]. These *extensions* have certain features which are standardly assumed to be typical for sets. Extensions are not *collections* of things in the everyday sense, i.e. not mereological sums. Things fall under a concept only as whole units, so it is, for example, whole elephants that fall under the concept *elephant*, not their parts (trunks, tusks, etc.) because the parts are not elephants.

However, we can see that the modern mathematical notion of set as defined (stipulated?) by common axiomatic systems of set theory⁶ is something new and different. The difference does not consist simply in using axiomatic methods, because Aristotle was already using something similar to them. The main and essential difference between modern set theory and (what we can call) the *logic of extensions* of the historical logical conception can be seen in this: in the historical conception, we can say that an individual falls under a concept; we can consider something like the class of all individuals falling under a certain concept or sharing a property; but this class is in no way an individual and therefore it can never fall under another concept (still less the same one!). If we (theoretically) want to speak about second-order properties, we ascribe these properties to first-order *properties*, not to sets or collections.⁷ And there is a gap between properties of individuals and properties of properties, they are something different and not applicable in the same way. In this conception, there cannot be a set of sets; the paradoxes of naive set theory cannot emerge; and also the idea of a complex structure of sets without any genuine individuals, without “urelements”, does not make sense (in contrast to the usual modern set theories).

We can see this attitude quite clearly in George Boole’s conception. He operates explicitly with classes, which represent, in his conception, one of the basic tools for the technical rendering of logical relations and reasoning. But for him, classes are always classes of *individuals*, there is nothing like a class of classes.⁸ He does not even use anything

⁶ Zermelo-Fraenkel or Gödel-Bernays systems, for example.

⁷ Traditionally, we can consider as second-order properties like “genus” or “idea”. See, e.g., [1, pp. 138–149].

⁸ “By a class is usually meant a collection of individuals, to each of which a particular name or description may be applied; but in this work the meaning of the term will be extended so as to include the case in which but a single individual exists, answering to the required name or description, as well as the cases denoted by the

like the membership relation — he does not work with individuals in his logical system, it suffices to operate with intersections of classes and their emptiness or nonemptiness.⁹ Boole’s classes cannot be elements of one another and therefore they do not face the same problems as Cantorian sets do.

On the other hand, for “modern” sets, it is typical that:

1. sets can be members of other sets;
2. sets are logical individuals,¹⁰ all sets are “on the same level”, i.e., we can legitimately ask about any two sets x, y whether $x \in y$ or $y \in x$;
3. the empty set is the only object which does not have members, standard set theories do not operate with non-set objects, i.e., with individuals in the original sense¹¹.

Frege’s attitude

Even Gottlob Frege, the “founding father” of modern logic, tends much more to the “old conception” of the logic of extensions, especially in the philosophical part of his works. Sets in the modern sense are of little importance in his logic (though not in his logicism — see below). As is well known, the predicate parts of sentences (general terms) denote, according to Frege, special kinds of functions whose values are only “the true” or “the false”. Frege calls these kinds of functions *concepts*. It is important to emphasize that Frege’s functions are in no way sets. They are not objects at all, they are a fundamentally different kind of entity. Unlike objects, they are “unsaturated” or “incomplete”. What does this mean? Frege is well aware that this way of expressing is neither exact nor very clear, but he does not see any better way to express it precisely.¹² In his view, functions are kinds of rules, they are something like “laws

terms ‘nothing’ and ‘universe,’ which as ‘classes’ should be understood to comprise respectively ‘no beings,’ ‘all beings’ ” [7, p. 20].

⁹ “Boole’s theory of classes was an extensional version of part-whole analyses of collections. Inclusion was the only relation [...]” [23, p. 42].

¹⁰ Cf.: “This use of ‘individual’ has nothing to do with the distinction between ‘individuals’ and ‘classes’ in logic [...]. In the logical sense the sets are individuals” [6, p. 7].

¹¹ This attitude is common and standard, but not inevitable — there are nonstandard set theories dealing with “urelements”, i.e. non-set individuals.

¹² “‘Complete’ and ‘unsaturated’ are of course only figures of speech; but all that I wish or am able to do here is to give hints” [17, p. 194, German original

of correlation”.¹³ Their nature is revealed, in some sense, by the usual notation we use to represent functions:

[I]t is precisely by the notation that uses ‘ x ’ to indicate a number indefinitely that we are led to the right conception. People call x the argument, and recognize the same function again in ‘ $2 \cdot 1^3 + 1$ ’, ‘ $2 \cdot 4^3 + 4$ ’, ‘ $2 \cdot 5^3 + 5$ ’ only with different arguments, viz. 1, 4, and 5. From this we may discern that it is the common element of these expressions that contains the essential peculiarity of a function; i.e., what is present in ‘ $2 \cdot x^3 + x$ ’ over and above the letter ‘ x ’. We could write this somewhat as follows: ‘ $2 \cdot ()^3 + ()$ ’. [18, p. 140, orig. 6]

So functions, and therefore also concepts, are, according to Frege, something much more like open procedures than fixed collections of outputs-inputs or a set of n -tuples.¹⁴ Frege borrows the mathematical concept of a function to use it for establishing logic in a more precise way, but he considers these functions, unlike sets, to be something open and dynamic.

In such a conception, it is not easy to say anything *about* concepts, because they (being unsaturated) have an essentially “predicative char-

p. 205]. “The peculiarity of functional signs, which we here called ‘unsaturatedness’, naturally has something answering to it in the functions themselves. They too may be called ‘unsaturated’, and in this way we mark them out as fundamentally different from numbers. Of course this is no definition; but likewise none is here possible. I must confine myself to hinting at what I have in mind by means of a metaphorical expression, and here I rely on my reader’s agreeing to meet me half way” [17, p. 292, orig. 665].

¹³ Cf. “Criterion, then, takes place according to a law, and different laws of this sort can be thought of. In that case, the expression ‘ y is a function of x ’ has no sense, unless it is completed by mentioning the law of correlation. [...] [T]he law [...] is really the main thing. [...] Distinctions between laws of correlation will go along with distinctions between functions; and these cannot any longer be regarded as quantitative” [18, p. 289, orig. 662].

¹⁴ Cf. also: “Each of the expressions ‘ $\sin 0$ ’, ‘ $\sin 1$ ’, ‘ $\sin 2$ ’ means some particular number, but we have a common constituent ‘ \sin ’ and here we find a designation for the essential peculiarity of the sine-function. This ‘ \sin ’ perhaps corresponds to the ‘ f ’ that Mr. Czuber says indicates a law; and the transition from ‘ f ’ to ‘ \sin ’, just like that from ‘ a ’ to ‘ 2 ’, is a transition from a sign that indicates to one that designates. In that case what ‘ \sin ’ means would be a law. Of course that is not quite right. The law seems rather to be expressed in the equation ‘ $y = \sin x$ ’; the symbol ‘ \sin ’ is only part of this, but the part that is distinctive for the essential peculiarity of the law. And surely we have here what we were looking for—the function. ‘ f ’ too will then, strictly speaking, indicate a function” [18, p. 290, orig. 663].

acter” and therefore they cannot be subjects of propositions.¹⁵ Not being objects, they cannot become arguments of first-level concepts (functions). Though Frege admits something like “properties of concepts” (second-level concepts), they are different from first-level ones and the nature of their application is not the same — concepts cannot figure as arguments in the ordinary sense:

[...] the behaviour of the concept is essentially predicative, even where something is being said about it; [...] second-level concepts, which concepts fall under, are essentially different from first-level concepts, which objects fall under. [...] I do not want to say it is false to say concerning an object what is said [...] concerning a concept; I want to say it is *impossible, senseless*, to do so.¹⁶

[18, pp. 189–190, orig. 201, emphasis mine]

So according to Frege, it also does not make sense to predicate a predicate of itself¹⁷ — this kind of predication is not simply false, it is *senseless*.

¹⁵ If they are treated as objects their essential character is lost, they are not concepts any more — this is the point of Frege’s famous claim “the concept *horse* is not a concept”: “In logical discussions one quite often needs to say something about a concept, and to express this in the form usual for such predications - viz. to make what is said about the concept into the content of the grammatical predicate. Consequently, one would expect that what is meant by the grammatical subject would be the concept; but the concept as such cannot play this part, in view of its predicative nature it must first be converted into an object, or, more precisely, an object must go proxy for it” [18, p. 186, orig. 197]. Though it seems that we can speak about concepts quite easily, it is so only because “language, with an almost irresistible force, compels me to use an inappropriate expression which obscures — I might almost say falsifies — the thought” [21, p. 119].

¹⁶ Cf. also: “It need not then surprise us that the same sentence may be conceived as saying something about a concept and also as saying something about an object; only we must observe that *what* is being said is different. In the sentence ‘there is at least one square root of 4’ it is impossible to replace the words ‘square root of 4’ by ‘the concept *square root of 4*’; i.e., what is suitably said of the concept does not suit the object. Although our sentence does not present the concept as a subject, it says something about it; it can be regarded as expressing that a concept falls under a higher one. But this does not in any way efface the distinction between object and concept. We see to begin with that in the sentence ‘there is at least one square root of 4’ the predicative nature of the concept is not belied; we could say ‘there is something that has the property of giving the result 4 when multiplied by itself’. Hence what is here said concerning a concept can never be said concerning an object” [18, p. 186, orig. 197].

¹⁷ “[T]he expression ‘A predicate is predicated of itself’ does not seem exact to me. A predicate is as a rule a first-level function which requires an object as argument

However, Frege accepts that we can speak about *extensions* (*Umfänge*) of concepts, and those *are* objects. But they are not intended to play any role in his entire enterprise of formulating the basic laws of logic. The extensions are “something derived, whereas in the concept — as I understand the word — we have something primitive”¹⁸ and “the primitive laws of Logic may contain nothing derived” [20, p. 191, orig. 121].

Sets in Frege’s foundations of arithmetic

The reason why Frege needed to work with extensions as objects (sets) was his attempt to reduce numbers to only logical notions (the thesis that this reduction is possible is called “logicism”). He considered numbers to be connected with the higher-order relation “be equinumerous to” between concepts; but at the same time, numbers must be, according to Frege, objects, not functions, so they are to be established as *extensions* of functions.¹⁹ Frege introduced extensions as objects explicitly by his questionable “Basic Law V” in his *Basic Laws of Arithmetic* (*Grundgesetze der Arithmetik*). It establishes something like the graph of a function, “Wertverlauf” (“course-of-values”) as an object, because it says that the Wertverläufe of any two functions f and g are *equal* (which in Frege’s conception can be said meaningfully only about objects) if and only if f gives for every argument the same value as g does:

(V) Wertverlauf of f = Wertverlauf of g if and only if $\forall x(f(x) = g(x))$.

In the case of Frege’s concepts (i.e. one-place functions correlating arguments with truth-values), the Wertverläufe can be simply seen as ex-

and which cannot therefore have itself as argument (subject)” [20, pp. 132–133, *Frege to Russell*, XV/2, 22. 6. 1902].

¹⁸ Shaughan Lavine even writes: “Whatever Frege’s extensions are, their members are not constitutive of them. The fact that he identified them with classes [...] shows that he had not understood the notion of class” [27, p. 65].

¹⁹ See [19, §68, pp. 79–80]. Cf. also “I myself was long reluctant to recognize ranges of values and hence classes; but I saw no other possibility of placing arithmetic on a logical foundation. But the question is, How do we apprehend logical objects? And I have found no other answer to it than this, We apprehend them as extensions of concepts, or more generally, as ranges of values of functions. I have always been aware that there are difficulties connected with this, and your discovery of the contradiction has added to them; but what other way is there?” [20, pp. 140–141, XXXVI/7].

tensions of the concepts. [16, p. 36, orig. 7–8] However, when Frege introduced the Basic Law V into his logic he admitted that it is perhaps not entirely evident:

A dispute can arise, so far as I can see, only with regard to my Basic Law concerning courses-of-values (V), which logicians perhaps have not yet expressly enunciated, and yet is what people have in mind, for example, where they speak of the extensions of concepts. I hold that it is a law of pure logic. In any event the place is pointed out where the decision must be made. [16, pp. 3–4, orig. VII]

After Bertrand Russell's uncovering of the contradiction in the system of *Basic Laws of Arithmetic*,²⁰ Frege saw the origin of the problems precisely in admitting the possibility of transforming unsaturated concepts into saturated objects, i.e. sets. He formulated it later in this way:

I turn first to the paradoxes of set theory. They arise because a concept, e.g. fixed star, is connected with something that is called the set of fixed stars, which appears to be determined by the concept — and determined as an object. I thus think of the objects falling under the concept fixed star combined into a whole, which I construe as an object and designate by a proper name, 'the set of fixed stars'. This transformation of a concept into an object is *inadmissible*; for the set of fixed stars only seems to be an object; in truth there is *no such object at all*.

[20, Frege to Hönigswald (V/2), p. 54, emphasis mine]

Frege blamed himself for not being cautious enough regarding sets:

Only with difficulty did I resolve to introduce classes (or extents of concepts), because the matter did not appear to me quite secure — and rightly so, as it turned out. [20, p. 191, orig. 121]

He also explained the reasons behind his carelessness and the terrible result of it:

The expressions 'the extension of F' seems naturalized by reason of its manifold employment and certified by science, so that one does not think it necessary to examine it more closely; but experience has shown how easily this can get one into a morass. I am among those who have suffered this fate. When I tried to place number theory on scientific

²⁰ Namely what is called Russell's paradox: we can consider a concept *class that does not belong to itself*. The extension of this concept is class of all classes that do not belong to themselves. Does this class belong to itself? If it does, it should not belong to itself, if it does not, it should belong to itself [see 16, pp. 127–128, orig. 253–254].

foundations, I found such an expression very convenient. While I sometimes had slight doubts during the execution of the work, I paid no attention to them. And so it happened that after the completion of the *Basic Laws of Arithmetic* the whole edifice collapsed around me. Such an event should be a warning not only to oneself but also to others. We must set up a warning sign visible from afar: let no one imagine that he can transform a concept into an object.

[20, Frege to Hönigswald (V/2), p. 55]

So Frege eventually, and quite radically, rejected object-like sets as a tool for founding logic: “there are no such objects at all”²¹. Nevertheless, he believes that the rejection of sets does not touch his logic as such, because it is built without this expedient:

In my fashion of regarding concepts as functions, we can treat the principal parts of Logic without speaking of classes, as I have done in my *Begriffsschrift*, and that difficulty does not then come into consideration. [...] The difficulties which are bound up with the use of classes vanish if we only deal with objects, concepts and relations, and this is possible in the fundamental part of Logic.²² [20, p. 191, orig. 121]

Indeed, in the foundational part of *Begriffsschrift*, which presents Frege’s formulation of his basic logical calculus, Frege nowhere allows concepts to be transformed into objects. Though he permits quantification over functions (or at least over function symbols), he does not use it in the core of his theory, i.e., in the axioms for the logical calculus — he needs it only for the “mathematical” part of his work, for the development of the idea of logicism²³; and, what is important, as he employs substitutional

²¹ See the footnote n. 21; cf. also “A particularly noteworthy example of this is the formation of a proper name after the pattern of ‘the extension of the concept *a*’ [...]. Because of the definite article, this expression appears to designate an object; but there is no object for which this phrase could be a linguistically appropriate designation. From this has arisen the paradoxes of set theory which have dealt the death blow to set theory itself” Frege, *Sources of Knowledge of Mathematics and Natural Sciences* [21, p. 269].

²² Cf. also: “Set theory in ruins. My concept-script in the main not dependent on it” [21, p. 176; *On Schoenflies: Die logischen Paradoxien der Mengenlehre*]. See also [26, p. 37].

²³ “Its [of *Begriffsschrift*] only flaw is some confusion about quantification over functions. Frege reluctantly accepted such quantification because it is needed in his logical definition of sequence, hence of natural number (when the ‘ancestral’ of a relation is introduced)” [24, p. 243]; “Frege allows a functional letter to occur in a quantifier [...]. This license is not a necessary feature of quantification theory, but

quantification²⁴, he does not need any objects representing concepts to quantify over them; it is enough to speak about the substitution of function *symbols*, which can be substituted only so that the sentence does not cease to express a thought.²⁵ Therefore, in the logic of *Begriffsschrift*, it is not possible to substitute function symbols at other places besides where they are supposed to denote concepts (i.e. unsaturated functions).

Was Frege's conception of concepts coherent?

Nevertheless, in the next stage of logic's development, Frege's *concepts* were simply replaced by sets.²⁶ This happened along the lines of Carnap's replacement of Frege's term 'Bedeutung' by his term 'extension' and it became an integral part of modern semantics for predicate calculi and of model theory. Semantical interpretation usually equips every (unary) predicate with a set; an individual is taken to have the property expressed by the predicate if it is a member of the set. And it is modern, axiomatic set theory which is taken to be the theory of these extensions/sets.

Frege has to admit it in his system for the definitions and derivations of the third part of the book" [25, p. 3].

²⁴ See, e.g., [17, §12, p. 27, orig. 23] or [16, I, §1, pp. 33–34, orig. 5–6, or §8, p. 40, orig. 11]: "We considered in §3 the case in which an equation such as ' $\Phi(x) = \Psi(x)$ ' always yields a name of the True, whatever proper name we may substitute for ' x ', provided only that this name actually denotes an object". Cf. also [23, p. 181].

²⁵ "The only restrictions imposed on the meaning of a Gothic letter [i.e. the variable] are the obvious ones: (i) that the complex of symbols following a content-stroke must still remain a possible content of judgment (§2); (ii) that if the Gothic letter occurs as a functional symbol, account must be taken of this circumstance" [17, p. 16, orig. 19]. Cf. also Frege's later explanation: "As one uses letters instead of numerical signs so as to be able to express general thoughts concerning numbers, one will also introduce letters for specific purpose of being able to express general thoughts concerning functions. [...] But now the function's need of supplementation must somehow or other find expression. Now it is appropriate to introduce brackets after every function-letter, which together with that letter are to be regarded as one single sign. The space within the brackets is then the place where the sign that supplements the function-letter is to be inserted" [21, p. 272; Sources of Knowledge of Mathematics and Natural Sciences].

²⁶ Cf., e.g.: "With respect to predicators, Frege does not seem to have explained how his concepts are to be applied; however, I think that Church is in accord with Frege's intentions when he regards a class as the (ordinary) nominatum of a predicator (of degree one) — for instance, a common noun — and a property as its (ordinary) sense" [13, §29, p. 125].

One of the reasons why this step was so easily accepted (despite Frege's late warning) can be the so-called *extensional thesis* which Frege (a bit reluctantly) formulates in his (unpublished) paper:

[I]n any sentence we can substitute *salva veritate* one concept-word for another if they have the same extension, so that it is also the case that in relation to inference, and where the laws of logic are concerned, that concepts differ only in so far as their extensions are different.²⁷

[21, p. 118, Comments on Sense and Meaning]

If it is true, then from the viewpoint of logic we can distinguish between concepts only in so far as they have different extensions. There then seems to be quite a small step from this thesis to the identifying concepts with their extensions.

However, the extensional thesis is not, in fact, in accordance with some of Frege's other claims and attitudes. The extensional thesis seems to be quite trivial for standard first-order sentences. If we define coextensive concepts as those for which the equivalence $\forall x(P(x) \leftrightarrow Q(x))$ is true, then, if $P(a)$ is true, obviously $Q(a)$ is true and vice versa. Therefore, if the concepts P and Q are coextensive, then the signs ' P ' and ' Q ' can be interchanged without altering the truth-value of those kinds of sentences. But in fact, as Frege states, it does not hold for some special kinds of sentences, such as those containing indirect speech, modal claims, propositional attitudes and the like: "in this way of speaking words do not have their customary meaning but designate what is usually their sense" [18, p. 159, orig. 28]. If what is denoted is in some contexts the usual sense, having the same meaning does not guarantee interchangeability *salva veritate*. However, we can now put aside Frege's somewhat weird manoeuvre: logic focuses on truth; the truth of sentences is fully determined by the meaning ["Bedeutung"] of their parts; the sense ["Sinn"] of them is therefore irrelevant for logic; so in logic, we can operate only with meaning; but in some cases it is not so, because the meaning of some parts of certain types of sentences is not ordinary meaning, but ordinary sense; so in some cases, sense is crucial for logic and the extensional thesis does not hold. There is another important feature of Frege's conception which undermines the extensional thesis. According to Frege, concepts can have "properties", and hence they can "fall within" [18, pp. 189–190, orig. 201] second-level

²⁷ See also: Frege, "Review: Husserl, *Philosophy of Arithmetic*", [18, p. 200, orig. 320].

concepts. As was already said, there is a difference between having a first-order property and having a second-order property. We can ascribe a first-order property only to objects, a second-order property only to first-order properties, and so on. So concepts can have second-order properties; we can, in a way, say something *about* concepts (but do not forget that “the behaviour of the concept is essentially predicative, even where something is being said about it”²⁸). For example, according to Frege, concepts can be composed:

Concepts are usually composed of partial concepts, the characteristic marks. *Black silk cloth* has the characteristic marks *black*, *silk* and *cloth*. An object which falls under this concept has these characteristic marks as its properties. [20, p. 92, XXVII/2]

If we say that a concept is composed and we list the partial concepts of which it is composed, we say something *about* the concept. It is important to remember that concepts are *meanings* of general terms, not senses, and therefore they should determine the truth or falsity of the propositions which say something about them. If the extensional thesis is right, we should also be able to replace, e.g., the concept-word “animal with kidneys” with the concept-word “animal with a heart” (denoting the coextensive concept) in the sentence “*Animal with kidneys* is composed from the partial concepts *animal*, *having kidneys*” without changing its truth-value. But is the concept *animal with a heart* really composed from partial concepts *animal*, *having kidneys*? Concepts are for Frege simply properties²⁹ and the property *having kidneys* is undoubtedly different from the property *having a heart*. We can see the obscurity of the extensional thesis more clearly if we consider empty concepts, which are *all* coextensive. The concepts *right-angled equilateral pentagon* and *female president of the USA* are certainly not composed from the same partial concepts; in the sentence “the concept of *right-angled equilateral pentagon* contains a contradiction” (Frege’s own formulation [21, p. 179]), we cannot substitute the concept-word “female president of the USA” (denoting an empty concept) for the concept-word “right-angled equilateral pentagon” (denoting also an empty concept) *salva veritate*, because the sentence would cease to be true. If we say “the concept of *female president of the USA* is a first-order concept” we cannot substitute any

²⁸ See footnote 16.

²⁹ “I call the concepts under which an object falls its properties” [18, p. 190, orig. 201, Concept and Object].

concept-word denoting an empty (i.e. coextensive) second-order concept for the concept-word “female president of the USA” *salva veritate*. So when we are speaking *about* concepts (i.e. properties) the extensional thesis does not seem to hold. But Frege explicitly assumes that we can ascribe second-order properties to concepts, because for him also existence and the magnitude of extensions are in fact properties of concepts, not of individuals.

It seems that the extensional thesis, as put forward by Frege, has two parts:

1. in “standard” contexts, what is decisive for the truth of sentences is the meaning (Bedeutung) of words;
2. though the meaning of a general term is a concept, not its extension, having the same extension means that appropriate concepts are also “the same”, at least from the point of view of interchangeability *salva veritate*.

But the second point does not seem to be in agreement with some other parts of Frege’s conception.

However, it is far from clear how much Frege was aware of those ambiguities. When he speaks about second-order properties, what he often has in mind are properties like “it exists ...” or “there are twelve ...”. These kinds of properties (if they are regarded as second-order) depend only on the cardinality of the extensions of the relevant concepts. So Dummett can be partly right when he claims: “In any case, there is nothing anywhere in Frege’s writings to indicate that he held the principle of extensionality to be false for concepts”; but when he adds: “indeed, his doctrine of indirect reference in effect precluded him from acknowledging as genuine any apparent counter-example to the principle” [15, p. 208], then this claim appears to be based upon Dummett’s conviction that for Frege every non-extensional feature of concepts is a matter of the corresponding *sense*. But this is obviously not true, as we saw: when Frege states that concepts are usually composed of partial concepts, that they can be composed in such a way that the result contains a contradiction, that they are first-order or second-order — in every case Frege expressly speaks about *concepts* as such, not about their “mode of presentation”. In these contexts Frege simply does not consider concepts in a fully extensional manner. So it is an open question whether Frege would be ready to accept all the consequences of the extensional thesis. It is important to have this in mind when we want

to understand Frege's prudent approach and his reluctance to transform concepts into objects (sets) and to the formulating of the extensional thesis (the "concession [*Zugeständnis*] to the extensionalist logicians" as he calls it [21, p. 122]).

Frege's final solution: Refutation of sets

It is precisely this extensional thesis that Frege abandoned in his first attempt to save the whole system of his *Basic Laws of Arithmetic*. He proved that, in his system, there necessarily are concepts with the same extension which do not give the same truth value for all arguments³⁰ (so they do not follow *Basic Law V* and are not interchangeable *salva veritate*). Frege saw that "this simply does away with extensions of concepts in the received sense of the term" [16, Appendix, p. 137, orig. 260]. To preserve consistency he therefore tried to limit Basic Law V (so that it is not longer generally true that if *Wertverlauf* of $f = \text{Wertverlauf}$ of g , then $\forall x(f(x) = g(x))$), but he is far from satisfied with this "solution" to the inconsistency of his system (neither are other logicians; it does not seem to be a very promising approach [see, e.g., 31]). This could be the reason why Frege eventually accepted a more radical solution.

So we shall now return to Frege's ultimate refutation of sets. It can be said that Frege eventually accepts a view perhaps even more cautious than the traditional one: it is possible to speak about things falling under a concept, but extensions of concepts are not only nothing like individuals (i.e., they cannot fall under a concept), but it is even the case that "there are no such objects at all". This means that not only are there no sets in the "modern" sense — sets which can be elements of other sets; but also there are no sets in, e.g., the Boolean sense — sets as collections of *individuals* with which we can operate. Frege saw from the very beginning that his *concepts* are not sets. And after the collapse of his system he withdrew his thesis that for the purposes of logic(ism) it is possible to operate with corresponding extensions instead of with concepts — he eventually rejected sets completely (and tried to find another way how to define the concept of number, namely a geometrical

³⁰ "If it is permissible generally for any first-level concept that we speak of its extension, then the case arises of concepts' having the same extension although not all objects falling under one also fall under the other" [16, Appendix, p. 137, orig. 260; *Basic Laws of Arithmetic II*].

one³¹). But at the same time, Frege considers his system of logic to be independent of the problematic notion of set and to be based only on genuine logical notions, viz. the notions of “objects, concepts and relations”.

Set theory and basic concepts of logic

Is Frege right that the classical logical calculus can work without sets? Does the current standard logic need “Cantorian” sets, i.e. sets regarded as objects? Standard first-order logic allows us to quantify only over individuals and does not need to operate with anything like sets at all. On the metalevel, predicate symbols *can* be interpreted as denoting sets but they can just as well be interpreted in another way (as denoting attributes, for example; Tarski’s classical truth definition is, in fact, based on this conception of predicates: the sentence “Snow is white” is true if and only if snow satisfies the function “ x is white” and this holds if and only if snow is white; nothing is said about any set of white things [35]). The modern model-theoretic investigation of logic certainly uses Cantorian sets, but the question is whether this is necessary and appropriate. As Quine expresses aptly:

The set theorist’s ontological excesses may sometimes escape public notice, we see, disguised as logic. But we must in fairness recognize also an opposite tendency, toward over-acknowledgement: a tendency to speak ostensibly of sets [...] where logic in a narrower sense would have sufficed. [32, p. 68]

Second-order logic (“set theory in sheep’s clothing” [32, p. 66]), which would require some objects in the universe of discourse corresponding to predicate symbols (sets?), is not considered to be a part of standard logic. It is often held that a standard set theory (Zermelo-Fraenkel theory, for

³¹ “So an *a priori* mode of cognition must be involved here. But this cognition does not have to flow from purely logical principles, as I originally assumed. There is the further possibility that it has a geometrical source” [21, p. 277; Numbers and Arithmetic]; see also Frege, Sources of Knowledge of Mathematics and the Mathematical Natural Sciences, Numbers and Arithmetic, A new Attempt at a Foundation for Arithmetic [21, pp. 267–281]. Cf.: “Frege bezeichnet die geometrische Erkenntnisquelle nun als die ‘eigentliche mathematische Erkenntnisquelle’ und sieht allein in ihr den Grund von Arithmetik und Geometrie. [...] Frege glaubte gegen Ende seines Lebens im Unterschied zu allen diesen Wissenschaftlern, daß die Mengenlehre durch die Antinomien ‘vernichtet’ worden sei” [14, pp. 341–342].

example) is the best tool by which not only the whole of mathematics, but also of classical first-order predicate logic can be formulated and investigated. Is it really appropriate to see the “fundamental logical relation”, namely “an objects’s falling under a concept”³² to be properly captured by “an object’s being a member of a set”? Are Cantorian sets good substitutes for properties?

Usually, standard set theories are formulated as first-order ones. This means that if we speak about a model we presuppose some logical individuals which satisfy the axioms and which we want to call “sets”. But unlike other logical individuals, these sets should follow certain very special rules for identity. Standardly, it is required for individuals’ being identical that they satisfy the Leibnizian criterion of identity: $x = y$ if and only if for all properties P , it is true that $P(x) \leftrightarrow P(y)$. In set theory, there is only one predicate (relation, better to say), namely ‘ \in ’, so the Leibnizian criterion of identity can be easily formulated by first-order statement: $\forall x \forall y (x = y \leftrightarrow \forall z ((z \in x \leftrightarrow z \in y) \wedge (x \in z \leftrightarrow y \in z)))$. But in the standard axiomatization of set theories the criterion of identity is limited by the axiom of extensionality only to the first conjunct, i.e. $\forall x \forall y (x = y \leftrightarrow \forall z (z \in x \leftrightarrow z \in y))$. In combination with Leibniz’s criterion, the result is that we postulate, via axioms, the following rule for the membership relation: $\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \rightarrow \forall z (x \in z \leftrightarrow y \in z))$ (if sets x and y have exactly the same members, they are members of exactly the same sets); or written in a way which is common for relations:

$$(R) \quad \forall x \forall y (\forall z (R_{\in}(z, x) \leftrightarrow R_{\in}(z, y)) \rightarrow \forall z (R_{\in}(x, z) \leftrightarrow R_{\in}(y, z))).$$

It is, in fact, a very strong requirement on the relation. Does anything like that hold for the relation “fall under a concept”? (Let us call it “ R_f ”.) Imagine that, by chance, all red things are round and vice versa. We can say that the following statement is true: $\forall z (R_f(z, \text{red}) \leftrightarrow R_f(z, \text{round}))$ (where “red” and “round” denote the concept *red* and *round*, respectively). But from this (possible) fact it would not follow that *red* and *round* cannot fall under various concepts — for example, *red* is a colour, but *round* is not, so $R_f(\text{red}, \text{colour}) \wedge \neg R_f(\text{round}, \text{colour})$. Therefore in this case, the relation *falling under a concept* does not follow rule (R), which defines the membership relation.

³² Frege, Comments on Sense and Meaning [21, p. 118]: “The fundamental logical relation is that of an object’s falling under a concept: all relations between concepts can be reduced to this.”

If we consider sets as objects, we can ascribe to them some attributes. But these attributes are not second-order properties, that is, properties of properties — they are properties of objects. The property *red* has some features: it is for example a colour, a quality, and so on. On the other hand, the set of all red things, if there is anything like that, has other kinds of features: it can be nonempty, it can have n members and so on. Sets are different from concepts and have different properties; sets as (Fregean) objects can have only first-order properties, concepts as first-order properties can have only second-order properties. So though sets may be determined by means of concepts, they “behave” differently and the relations between sets are not the same as relations between concepts. Therefore sets do not seem to be good substitutes for properties.

It seems that extensions taken as logical individuals do follow the set-theoretical rules. But we can see that if we want them to follow the (R) rule then there cannot be extensions of higher-order properties among them. There cannot be sets of properties, only sets of sets or individuals; properties are not individuals and at the same time they do not follow the (R) rule. The (R) rule administrates only extensions of first-order properties and then extensions of properties of extensions, which are again, in Cantorian conception, first-order properties. So among Cantorian sets, there cannot be extensions of higher-order properties, there cannot be any set containing properties.

These limited extensions might be not appropriate substitutes for properties, but can we still operate with them independently? Frege’s last answer is negative: we cannot, because the very notion of objects-like extensions of concepts leads to contradictions.³³ And to say that only some concepts have objects-like extensions or that extensions are some kind of “improper objects” would lead to a confused and ambiguous system [16, pp. 128–130, orig. 254–255]. Russell chose the latter way in his type theory: extensions are not objects of the same kind as individuals, but there is a whole hierarchy of various kinds of (we may well say “improper”) objects — one kind of object for extensions of first-level predicates, another kind of object for extensions of second-level predicates, and so on. The system is then truly complicated. (And it is, in fact, closer to the traditional view of extensions than to the Cantorian

³³ Cf. “However, there is a respectable tradition in the subject that denies the existence of a hierarchical notion of collection and recognizes only fusions. Frege, for instance, drew the distinction between the two notions precisely in order to deny the coherence of the first” [30, p. 33].

conception because it does not presuppose that sets are individuals on one and the same level. Instead, there is a huge amount of various kinds of “objects”.) Zermelo-Fraenkel axiomatics (ZF), on the other hand, chose the former way — sets are object-like extensions of concepts, but there are concepts which do not have extensions in this sense (“be a set”, for example, because there is not a set of all sets in the ZF system).

Does set theory capture any intuitive notion?

It is then questionable how much the modern notion of set as established via axioms is in fact based on *any* intuitive, natural, pretheoretical concept. As we saw in the previous sections, the intuitive notion of “set” is much closer to the traditional notions of extensions of concepts or collections of things sharing a property — including the natural presupposition that members of extensions are only individuals or (in special cases of higher-order concepts) merely attributes.³⁴ This conception is not properly captured via axioms of standard set theory. If we try to describe our intuitive notion of “collections/combinations of anything”,³⁵ it results in the paradoxes of naive set theory. So the axioms of modern set theories do not describe extensions of concepts and they do not capture “collections” — so what are they supposed to capture? The (quite commonly accepted) answer is: nothing. They do not capture anything, they *stipulate*.³⁶ But Frege emphasizes (in his polemic with Hilbert)

³⁴ Russell’s paradox is sometimes presented, despite Frege, as if it is formulable also for properties or concepts, because we can ask whether the concept “not applying to itself” is applying to itself [see, e.g., 22, p. 452]. But Frege’s attitude seems to be more intuitively acceptable: every concept requires a certain kind of argument (first-order concepts require objects, second order concepts require first order concepts etc.) and, without an argument of this kind, asking whether it is applying or not is not false, but “impossible, senseless”. Things sharing the same property must be, intuitively, on the same level, of the same category. Even the concept “applies to” requires some kinds of arguments to be used in a sensible way.

³⁵ Penelope Maddy distinguishes “the mathematical notion of a collection” [29, p. 102]. It refers to the idea that sets are not given primarily as extensions, but simply combinatorially — there exists any combination of objects as an individual object (and it is not necessary to be able to describe each of these combinations via an open formula). But this conception faces the same problems: if every combination exists, there should be a collection of all collections and a collection of every collection not containing itself.

³⁶ “The notion of set is so simple that it is usually introduced informally, and regarded as self-evident. In set theory, however, as is usual in mathematics, sets

that definitions cannot be given via axioms³⁷ and that proving a consistency rests in showing a model, not the reverse.³⁸ So *if there were* sets they might follow these or those rules. But stating any rules (“axioms”) cannot *create* any model, according to Frege.³⁹ As Carnap also explains in this connection: an implicit definition given by axioms only tells us something about the whole structure, but it cannot determine what objects fall under defined concepts — so, in the present case, what object is a set.⁴⁰ Imagine, for example, that we have a classical set-theoretical universe where instead of the empty class we have an individual, let’s say Socrates. Socrates has no members, therefore he “behaves” from the axiomatic point of view exactly like the empty set. The axioms cannot “recognize” that the only object without members is in fact not the empty set, but Socrates. He is (inter alia) a subset of every set, because

are given axiomatically, so their existence and basic properties are postulated by the appropriate formal axioms.” [4].

³⁷ “The other propositions (axioms, fundamental laws, theorems) must not contain a word or sign whose sense and meaning, or whose contribution to the expression of a thought, was not already completely laid down, so that there is no doubt about the sense of the proposition and the thought it expresses. The only question can be whether this thought is true and what its truth rests on. Thus axioms and theorems can never try to lay down the meaning of a sign or word that occurs in them, but it must already be laid down” [20, p. 36, orig. 62–63; Frege to Hilbert XV/3].

³⁸ “[W]e must ask, What means have we of demonstrating that certain properties, requirements (or whatever else one wants to call them) do not contradict one another? The only means I know is this: to point to an object that has all those properties, to give a case where all those requirements are satisfied. It does not seem possible to demonstrate the lack of contradiction in any other way” [20, p. 43, orig. 70–71; Frege to Hilbert XV/5].

³⁹ “Just as the geographer does not create a sea when he draws boundary lines and says: the part of the ocean’s surface bounded by these lines I am going to call the Yellow Sea, so too the mathematician cannot really create anything by his defining. [...] Now suppose one defines, for instance, the number zero, by saying: it is something which yields one when added to one. [...] People frequently seem to fancy that by the definition something has been created that yields one when added to one. A great delusion! The concept defined does not possess this property, nor is the definition any guarantee that the concept is realized — a matter requiring separate investigation. Only when we have proved that there exists at least and at most one object with the required property are we in a position to invest this object with the proper name ‘zero’. To create zero is consequently impossible” [16, pp. 11–12, orig. XIV; *The Basic Laws of Arithmetic*].

⁴⁰ See, e.g., [12]: “Für einen uneigentlichen [implizit definierten] Begriff dagegen ist die Frage, ob ein bestimmter Gegenstand unter ihn falle, trotz aller Kenntnisse über den Gegenstand nicht entscheidbar und hat daher keinen Sinn.”

“for every x which is a member of Socrates it is true that ...” is true in any case. All very well, but an empty set is a set, unlike Socrates!⁴¹ However, there is no property expressible in the language of set theory which Socrates (in the given case) has and the empty set does not. So this (weird) universe containing Socrates is a model of the given axiomatic system. Does the axiomatic system really offer the right definition of sets, if any individual can play the role of the empty set? On the basis of the axioms alone we can never know “whether my pocket watch is a set”.⁴² The axiomatic system can give us hints as to what the whole structure should look like, but it does not tell us what a set in fact is. Set theories stipulate a structure, but is there any “intuitive” model of it?⁴³

Some logicians claim that it is not necessary to have any “intuitive” or “intended” model for establishing a theory as useful. According to some of them, describing the structure is enough as there are no special entities like numbers or sets; there is only the appropriate structure and the “positions” in it.⁴⁴ But it is one thing to accept this kind of structure as useful for mathematics, quite another thing to regard it as appropriately capturing the logical semantics. As we have seen, the hierarchy of attributes is not properly mirrored by the hierarchy of sets. So the structure described by ZF has little resemblance to the structure of “objects, concepts and relations”.

Some proponents of standard set theories think that there *is* an intuitive, pretheoretical notion of set behind set theory. It is considered to be connected with the so-called “cumulative-hierarchy” conception of sets or, more generally, with the iterative conception of set. The conception is based on the idea that we can imagine that sets are formed at stages. At the first stage, we combine individuals into sets; at the second stage we combine individuals and/or sets from the first stage; and so on — at every stage, only individuals and the sets formed so far from the lower stages are available and only they can be members of the newly-created

⁴¹ “Individuals are objects which are not sets, but which share with the empty set the property of not having any elements” [34, p. 133].

⁴² Paraphrase of Frege’s formulation in his polemical exchange with Hilbert: “Given your definitions, I do not know how to decide the question whether my pocket watch is a point” [20, p. 45, orig. 73].

⁴³ The consistency of first-order set theory should assure us that it has a model, but the set theory itself cannot give a proof of its own consistency and it is highly dubious whether there is any other more general theory to provide the proof.

⁴⁴ Cf., e.g., Benacerraf’s well-known article [5].

sets [see, e.g., 8]. (If there are no individuals, at the first stage only the empty set can be formed. This is the ZF system.) The characterisation just described is only metaphorical because sets are not, according to this conception, successively formed. But we have a rough idea how the system of sets “works” or how to imagine it. Because no set is available at the time of its own formation, no set can belong to itself and hence there is also no set of all sets. Therefore there is no place for the paradoxes.

Boolos claims that “the iterative conception of set [...] strikes people as entirely natural, free from artificiality, not at all ad hoc, and one they might perhaps have formulated themselves”. But there can be doubts about this conviction: the conception does not give any, not even an approximate, answer to the pretheoretical question: “What is a set?” The traditional conception of an extension considers sets to be collections of things sharing the same property (falling under the same concept); Cantor’s conception considers sets to be things which are collections (combinations) of things. But the iterative conception of set is, as Boolos admits, “not quite so simple to describe” [8, p. 486]. If the theory is not purely stipulative, we expect some rough concept at the beginning which is to be made more precise via the theory. But what pretheoretical concept of set does the iterative conception describe? “Collections of things which are things and which can be imagined to be formed at stages by combination of things already formed, though they are, in fact, not successively formed”? If this rather clumsy concept⁴⁵ is the right one then there is another doubt: if we are to understand sets as being formed at stages, we can never get, at any stage, to an infinite set (unless there are infinitely many individuals at the first stage). At any stage 1, 2, 3, ... we have only a finite amount of objects which can be combined in a finite amount of ways. We cannot “imagine” forming an infinite set at any stage. Even the magic formula “and so on” does not help us. (In the same way we cannot “imagine” forming an infinite natural number by going step by step from zero.) It is necessary to stipulate the existence of an infinite set by an axiom. So if the “entirely natural” iterative conception rests on the idea of “forming at stages” we can never reach an infinite system.

Therefore, it is highly dubious that e.g. ZF *describes* (or *explicates*)

⁴⁵ Cf. “Nevertheless, it is difficult to see how this particular metaphor helps: to be told that collections are subject of a time-like structure that is not time is not to be told very much” [30, p. 39].

any basic pretheoretical notion, still less a pretheoretical notion fundamental for logic. Why should then ZF be the right theory for the foundations of logic?

Conclusion

By way of a summary: there is the traditional notion of the extension of a concept, which is closely connected with an intuitive notion of set. There is Cantor's notion of set which is inconsistent. And there is the modern "Cantorian" notion of set, which is nowadays considered to be consistently delimited via the standard axiomatic systems of set theory; however, it seems to be much more stipulated than described. It seems that the traditional notion and the modern notion have less in common with each other than is generally held to be true: the important difference between them lies in (not-)considering sets to be individuals and properties to be sets. It seems that for the needs of logic, the traditional notion is sufficient and more adequate. This message is implicit in Frege's (philosophical) logic, though it is far from clear to what extent Frege was always fully aware of this fact. At the end of his career, Frege regretted his acceptance of Cantor's sets in his conception of the foundations of mathematics. He saw that, for the purposes of logic, it is better to operate with *properties* or "concepts" as unsaturated functions than it is with sets.

Acknowledgments. This work was supported by the Czech Science Foundation under Grant GA ČR 17-15645S. I would like to thank my colleagues Jaroslav Peregrin and Vít Punčochář for their very helpful conversations and comments on a previous draft of this paper.

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