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ON THREE AXIOM SYSTEMS
FOR CLASSICAL MEREOLOGY

Abstract. We correct an error and expose two redundancies in the axiom systems presented by Paul Hovda in his 2009 influential paper, 'What is classical mereology?'.

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Paul Hovda's excellent paper 'What is classical mereology?' [2] has fruitfully reshaped the debate concerning the axiomatic foundations of so-called Classical Mereology, the formal theory of parts and wholes stemming from the work of Leśniewski [4] and of Leonard and Goodman [3]. Precisely because of the importance of Hovda's work and its usefulness as a reference tool, we note here that one of the five axiom systems presented in [2], corresponding to the 'Third Way' to Classical Mereology, is defective and must be amended. In addition, we note that two other axiom systems, corresponding to the 'First Way' and to the 'Fifth Way', involve redundancies.

1. Definitions

For easy reference, we recall some basic definitions. The language is a standard first-order language with identity supplied with a distinguished binary predicate constant, $\leq$, to be interpreted as the parthood relation; the underlying logic is the classical predicate calculus.

$$x \ll y \quad := \quad x \leq y \land y \neq x$$
$$x \circ y \quad := \quad \exists z(z \leq x \land z \leq y)$$
$$x \wr y \quad := \quad \neg\, x \circ y$$

$$Mub(z, \phi_x) \quad := \quad \forall x(\phi_x \to x \leq z) \wedge \forall w(\forall x(\phi_x \to x \leq w) \to z \leq w)$$
$$Fu_1(z, \phi_x) \quad := \quad \forall y(y \circ z \leftrightarrow \exists x(\phi_x \wedge y \circ x))$$
$$Fu_2(z, \phi_x) \quad := \quad \forall x(\phi_x \to x \leq z) \wedge \forall y(y \leq z \to \exists x(\phi_x \wedge y \circ x))$$

Intuitively, $Mub(z, \phi_x)$ says that z is a minimal upper bound of the ϕs relative to $\leq$, whereas $Fu_1(z, \phi_x)$ and $Fu_2(z, \phi_x)$ say that z is a fusion of the ϕs in Leonard and Goodman's sense (as revised by Goodman [1]) and in Leśniewski's sense (as revisited by Tarski [7]), respectively.

2. The 'third way' axiom system

Hovda's 'Third Way' to Classical Mereology has the following four axioms.[1]

Transitivity $\forall x \forall y \forall z((x \leq y \wedge y \leq z) \to x \leq z)$
WeakSup $\forall x \forall y(x \ll y \to \exists z(z \leq y \wedge x \wr z)$
Filtration $\forall y \forall z((y \leq z \wedge Mub(z, \phi_x)) \to \exists x(\phi_x \wedge y \circ x))$
MubE $\exists x \phi_x \to \exists z Mub(z, \phi_x)$

The interesting idea, here, is that the conjunction of Filtration and MubE is strong enough to imply

Fusion2E $\exists x \phi_x \to \exists z Fu_2(z, \phi_x)$

(see [2, §3.1]). Since it is well known that Transitivity + WeakSup + Fusion2E amounts to an axiomatization of Classical Mereology, corresponding to Hovda's 'Second Way' (see e.g. [5, §IV.5]), replacing Fusion2E with Filtration + MubE should do just as well.

This, however, is not quite correct. Filtration is meant to make up for the relative weakness of MubE in comparison to Fusion2E, but in fact it goes further, admitting instances that are false in some models of Classical Mereology. Specifically, a one-element model in which the only object of the domain fails to satisfy ϕ_x falsifies the corresponding instance of Filtration.[2] In such a model the antecedent is true (since classical parthood is reflexive and '$Mub(z, \phi_x)$' consists of two conjuncts

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[1] MubE is actually an axiom schema if ϕ_x is read schematically; see [2, p. 60] for details. The same applies to Fusion2E and Fusion1E below.

[2] A similar problem affects the 'strong overlapping principle' SA25 of Simons [6, p. 37], which is the counterpart of Filtration when it comes to making up for the leeway between MubE and the Fusion1E principle listed below. The present solution applies *mutatis mutandis*.

the first of which holds vacuously and the second, again, by reflexivity); yet the consequent is false (since nothing in the model satisfies ϕ_x).

To obtain a correct axiomatization of Classical Mereology, Filtration must be amended by adding the relevant existential antecedent:

FiltrationE $\exists x \phi_x \to \forall y \forall z((y \leq z \land Mub(z, \phi_x)) \to \exists x(\phi_x \land y \circ x))$

Clearly this is enough to take care of the one-element model. More generally, it is easy to see that FiltrationE is in fact a theorem of Classical Mereology. For, as Hovda notes [2, p. 67], Classical Mereology proves the conditional

MubFu2 $\exists x \phi_x \to \forall z(Mub(z, \phi_x) \leftrightarrow Fu_2(z, \phi_x))$

and from this conditional FiltrationE follows immediately by the second conjunct in the definition of Fu_2. On the other hand, since the antecedent of FiltrationE coincides with the antecedent of Fusion2E, it's clear that the amendment will not affect Hovda's original argument, i.e., Fusion2E will follow from FiltrationE + MubE just as it follows from Filtration + MubE. (Whenever a conditional $\psi \to \xi$ is derivable from a set of axioms containing a certain formula φ, it remains derivable when φ is replaced by $\psi \to \varphi$.) Since MubE is itself provable in the axiomatization of Classical Mereology based on Fusion2E (again by MubFu2), we may therefore conclude that the amendment achieves the desired result. Adding FiltrationE + MubE to Transitivity + WeakSup yields a system strictly equivalent to the 'Second Way'.

3. The 'fifth way' axiom system

Hovda's 'Fifth Way' to Classical Mereology has the following five axioms.

Anti-symmetry	$\forall x \forall y((x \leq y \land y \leq x) \to x = y)$
Transitivity	$\forall x \forall y \forall z((x \leq y \land y \leq z) \to x \leq z)$
MubE	$\exists x \phi_x \to \exists z Mub(z, \phi_x)$
Strong Complement	$\forall x(\exists y\ y \not\leq x \to$
	$\exists z(z \wr x \land \forall y((y \wr x \to y \leq z) \land (y \wr z \to y \leq x)))))$
NoZero	$\exists x \exists y\ x \neq y \to \neg \exists x \forall y\ x \leq y$

This is indeed a valid and complete axiomatization. The defect, in this case, is simply that the last axiom, which plays a key role in Hovda's discussion, is redundant.

To see this, suppose we have a and b such that $a \neq b$. Then, by Anti-symmetry, either $a \not\leq b$ or $b \not\leq a$. Hence, by Strong Complement, either $\exists z\ z \wr b$ or $\exists z\ z \wr a$, which means that either $\exists z \neg \exists w (w \leq z \wedge w \leq b)$ or $\exists z \neg \exists w (w \leq z \wedge w \leq a)$. In both cases, it follows immediately that $\neg \exists x \forall y\ x \leq y$, whence NoZero.

4. The 'first way' axiom system

Finally, it is worth pointing out that there is a redundancy also in Hovda's 'First Way', which corresponds to the following (popular) axiom system:

Reflexivity	$\forall x\ x \leq x$
Anti-symmetry	$\forall x \forall y ((x \leq y \wedge y \leq x) \to x = y)$
Transitivity	$\forall x \forall y \forall z ((x \leq y \wedge y \leq z) \to x \leq z)$
StrongSup	$\forall z \forall y (\forall x (x \leq y \to x \circ z) \to y \leq z)$
Fusion1E	$\exists x \phi_x \to \exists z Fu_1(z, \phi_x)$

In this case, the redundancy lies in the fact that Reflexivity is derivable from Transitivity and StrongSup, as already shown by Pietruszczak [5, §IV.5].

References

[1] Goodman, N., *The Structure of Appearance*, Cambridge (MA), Harvard University Press, 1951.

[2] Hovda, P., "What is classical mereology?", *Journal of Philosophical Logic* 38, 1 (2009): 55–82. DOI: 10.1007/s10992-008-9092-4

[3] Leonard, H.S., and Goodman, N. "The calculus of individuals and its uses", *Journal of Symbolic Logic* 5, 2 (1940): 45–55. DOI: 10.2307/2266169

[4] Leśniewski, S., *Podstawy ogólnej teoryi mnogości. I*, Moskow, Prace Polskiego Koła Naukowego w Moskwie, 1916. Eng. trans.: "Foundations of the general theory of sets. I", pages 129–173 in S. Leśniewski, *Collected Works*, vol. 1, ed. by S. J. Surma *et al.*, Dordrecht, Kluwer, 1991.

[5] Pietruszczak, A., *Metamereologia*, Toruń, Wydawnictwo Naukowe Uniwersytetu Mikołaja Kopernika, 2000. Revised and extended Eng. trans.: *Metamereology*, Toruń, Nicolaus Copernicus University Scientific Publishing House, 2018. DOI: 10.12775/3961-4

[6] Simons, P. M., *Parts. A Study in Ontology*, Oxford, Clarendon, 1987. DOI: 10.1093/acprof:oso/9780199241460.001.0001

[7] Tarski, A., "Les fondements de la géométrie des corps", *Księga Pamiątkowa Pierwszkego Polskiego Zjazdu Matematycznego*, suppl. to *Annales de la Société Polonaise de Mathématique*, 7 (1929), pp. 29–33. Extended Eng. trans: "Foundations of the geometry of solids", pages 24–29 in A. Tarski, *Logic, Semantics, Metamathematics. Papers from 1923 to 1938*, Oxford, Clarendon, 1956.

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