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A WITTGENSTEINIAN PHILOSOPHY OF MATHEMATICS

Abstract. Three theses are gleaned from Wittgenstein's writing. First, extra-mathematical uses of mathematical expressions are not referential uses. Second, the senses of the expressions of pure mathematics are to be found in their uses outside of mathematics. Third, mathematical truth is fixed by mathematical proof. These theses are defended. The philosophy of mathematics defined by the three theses is compared with realism, nominalism and formalism.

Keywords: Wittgenstein, Frege, realism, anti-realism, formalism, mathematics

I. Introduction

In the beginning of *Philosophical Investigations* Wittgenstein describes the case of a person who goes to a grocer with a slip marked 'five red apples'. The grocer hands over red apples uttering the numerals up to five and handing over apples as each numeral is uttered.

> Now think of the following use of language: I send someone shopping. I give him a slip marked "five red apples". He takes the slip to the shopkeeper, who opens the drawer marked "apples"; then he looks up the word "red" in a table and finds a colour sample opposite it; then he says the series of cardinal numbers — I assume that he knows them by heart — up to the word "five" and for each number he takes an apple of the same colour as the sample out of the drawer. — It is in this and similar ways that one

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> operates with words.—"But how does he know where and how he is to look up the word 'red' and what he is to do with the word 'five'?"—Well, I assume that he *acts* as I have described. Explanations come to an end somewhere.—But what is the meaning of the word "five"?—No such thing was in question here, only how the word "five" is used.[1]

There is a clear contrast here between the use of 'five', the use of 'red', and the use of 'apples'. It is not plausible to suppose that, in this case at least, the word 'five' designates anything.

A thesis suggested by this case is the following:

Thesis 1 In extra-mathematical statements of number, mathematical expressions do not function as referential expressions.

In the *Notebooks* Wittgenstein writes:

> Now everything turns on the fact that I apply numbers to ordinary things, etc., which in fact says no more than that numbers occur in our quite ordinary sentences.[2]

And in the *Tractatus* he writes:

> 6.211 Indeed in real life a mathematical proposition is never what we want. Rather, we make use of mathematical propositions only in inferences from propositions that do not belong to mathematics to others that likewise do not belong to mathematics. (In philosophy the question 'What do we actually use this word for?' repeatedly leads to valuable insights.)[3]

Finally in *Remarks on the Foundations of Mathematics* Wittgenstein writes:

> I want to say: it is essential to mathematics that its signs are also employed in *mufti*.
>
> It is the use outside, and so the *meaning* of the signs, that makes the sign-game into mathematics.[4]

A second thesis suggested by these remarks is the following:

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[1] Wittgenstein, 1953, pp. 2e–3e.

[2] Wittgenstein, 1961, p. 67.

[4] Wittgenstein, 1956, p. 133e. His emphasis.

Thesis 2 The meaning of mathematical signs is determined by their use in extra-mathematical statements of number.

For a third thesis I go back to *Remarks on the Foundations of Mathematics*. Clearly having in mind Gödel's theorem about the existence of undecidable sentences in arithmetic, Wittgenstein writes:

> I imagine someone asking my advice; he says: "I have constructed a proposition (I will use 'P' to designate it) in Russell's symbolism, and by means of certain definitions and transformations it can be so interpreted that it says: 'P is not provable in Russell's system'. Must I not say that this proposition on the one hand is true, and on the other hand is unprovable? For suppose it were false; then it is true that it is provable. And that surely cannot be! And if it is proved, then it is proved that it is not provable. Thus it can only be true, but unprovable."[5]

Here Wittgenstein is considering a familiar argument that there are arithmetical truths which are unprovable.

To this argument Wittgenstein responds:

> Just as we ask: "'provable' in what system?", so we must also ask: "'true' in what system?" 'True in Russell's system' means, as was said: proved in Russell's system; and 'false in Russell's system' means: the opposite has been proved in Russell's system. — Now what does your "suppose it is false" mean? *In the Russell sense* it means 'suppose the opposite is proved in Russell's system'; *if that is your assumption*, you will now presumably give up the interpretation that it is unprovable. And by 'this interpretation' I understand the translation into this English sentence. — If you assume that the proposition is provable in Russell's system, that means it is true *in the Russell sense*, and the interpretation "P is not provable" again has to be given up. If you assume that the proposition is true in the Russell sense, *the same* thing follows. Further: if the proposition is supposed to be false in some other than the Russell sense, then it does not contradict this for it to be proved in Russell's system. (What is called "losing" in chess may constitute winning in another game.)[6]

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[5] Wittgenstein, 1956, p. 50e.

[6] Wittgenstein, 1956, p. 51e.

Wittgenstein is making two claims:

(i) proof in mathematics is a system-relative notion;

(ii) truth in mathematics is a system-relative notion.

The first claim does not appear to be controversial.
For example, a classical Peano proof is based on the Peano axioms and classical logic. A non-classical Peano proof is based on the Peano axioms and non-classical logic. But Wittgenstein is also saying that truth in mathematics is a system-relative notion. This suggests that truth in mathematics is fixed by proof in mathematics:

Thesis 3. Truth in mathematics is fixed by what is provable in mathematics.

In his later years, at least, Wittgenstein seems to have eschewed all philosophical theories. So calling any philosophical theory, including a philosophical theory about mathematics, 'Wittgensteinian' seems problematic on its face. Nonetheless, each of the three theses set out above is suggested by what Wittgenstein did write. Collectively they do determine a philosophical theory about mathematics. So, perhaps, there is some justification for labeling the theory "a Wittgensteinian philosophy of mathematics". How it is labeled is not as important as how it fares against other philosophies of mathematics. In this paper I contrast the Wittgensteinian theory with other philosophies of mathematics. I defend the Wittgensteinian theory by defending each of the three theses.

II. Realism and Anti-Realism

A sentence of pure mathematics formulates an assertion about how things are, and the sentences we recognize as among the truths of pure mathematics are true in virtue of things being as they assert them to be. This view has been called *mathematical realism*.

There are many ways in which philosophers have developed this view, and many ways in which they have opposed it.

One important *opposing* point of view takes the form of agreeing that the sentences of pure mathematics formulate assertions about how things are, while holding that how things are makes for the falsity, not the truth, of enormously many of the mathematical sentences we commonly accept. (We are like people who misread a topographical map and go on to assert things about mountains in central Nebraska. What they assert is mostly false because there are no mountains there at all. But an occasional sentence may be true — e.g., that there are no more than 25 mountains in Lancaster County, something true since there are no mountains there at all.)

Many philosophers who reject mathematical realism take it that enormously many of the sentences of pure mathematics to which we ordinarily assent and for which we possess what we commonly accept as proofs are true only if numbers really exist, but deny that these sentences actually *are* true on the grounds that there really are no numbers.

These philosophers agree with the mathematical realist that the *content* of the sentences of pure mathematics is such that those sentences formulate assertions about how things are, but deny that how things are makes for mathematical truth as commonly accepted.

Closely akin to this is the view that *if* the sentences of pure mathematics possessed content they would be sentences formulating assertions about how things are, so that, since nothing is as the mathematical sentences we accept would say things are if they had content, enormously many of the sentences of pure mathematics for which we possess proofs would actually be false. But, on the view now being sketched, no such sharp divergence of truth from provability is possible. And so the conclusion is drawn that the sentences of pure mathematics do *not formulate* assertions about how things are and consequently lack content.

On this view the sentences of pure mathematics are logical forms filled with signs without sense and the proof procedures of pure mathematics are nothing but syntactic transformations in accordance with laws of logic making reference to nothing beyond syntactic structure.

This type of anti-realism also has its point of agreement with mathematical realism, for it holds that *if* the sentences of pure mathematics possessed content they then would formulate assertions about how things are. With this the realist entirely agrees, affirms the antecedent and draws the consequent as its conclusion: the sentences of pure mathematics formulate assertions about how things are.

The views thus far sketched agree that the sentences of pure mathematics have a content only if they formulate assertions about how things are.

Their *shared conception* is that the form of a mathematical proposition is such that if it possesses a sense, then it formulates an assertion about how things are.

A quite different kind of "anti-realism" regards this as a shared *misconception.* This type of view does not deny meaning to the signs which occur in the sentences of pure mathematics, but it denies that those sentences, with the mathematical content their signs naturally carry, formulate assertions about how things are.

This sort of view is opposed not only to mathematical realism, but also to those types of anti-realism (formalism included) which share with mathematical realism the conception noted just above.

It is common to formulate the realist position by saying that it holds that the sentences of pure mathematics are about certain entities. The kind of anti-realism here under discussion rejects the assumption that the sentences of pure mathematics have the content they seem to have only if they are about things (the numbers) which may fail to exist, e.g., as some description ("a horse with wings") may be about things which fail to exist.

Rather, it is central to the kind of anti-realism here under discussion to hold that although the sentences of pure mathematics are contentful, they do not constitute an "about something" sort of discourse at all.

To sum up, four general philosophies of mathematics are distinguishable.

The first is mathematical realism whose central idea is that the sentences of pure mathematics say how things are and things really are as those sentences of pure mathematics which we take to be true say they are.

The most natural form of mathematical realism *explains* how the sentences of pure mathematics can say how things are by construing its numerical terms as names of entities external to the spatio-temporal causal order. This is "ontological" platonism. The most natural accompanying view of mathematical knowledge invokes some special source of knowledge fitted to such objects. This is "epistemological" platonism.

Next there are three versions of anti-realism. The first is nominalistic. It agrees with realism that pure mathematics says how things are, but contends that things are not as pure mathematics says they are, since numbers do not exist.

The second is (a version of) formalism. This is the view that the sentences of pure mathematics are devoid of content. Pure mathematics is all logical form, lacking mathematical content. And so it also lacks truth or falsity.

The last anti-realistic doctrine is derived from remarks by Wittgenstein. The idea is that pure mathematics-taken in isolation from the use of its signs in empirical judgement — is an activity for which formalism is correct. Mathematical signs nonetheless have a sense, but only in and through belonging to a system of signs with empirical application, and thus a system of signs other than that of *pure* mathematics.

I shall call the four views realism, nominalism, formalism, and the Wittgensteinian view, respectively.

III. Thesis One

Thesis 1 says:

> In extra-mathematical statements of number, mathematical expressions do not function as referential expressions.

I defend this thesis in two stages: first, I consider the case where the mathematical expression is a numeral ('There are five applies in the box'); second, I consider the case where the mathematical expression is quantificational ('There is an odd number of apples in the box').

3.1 By a statement of number Frege meant any statement which answers a how-many question:[7]
How many apples are there on the table?
There are five apples on the table.
How many students does Bill teach?
Bill teaches fifty students.
How many numbers between 1 and 10 are prime?
Four numbers between 1 and 10 are prime.

Statements of number often (but not always: see the last example) are contingent e.g., the statement that Bill owns 2 suits says something that is but might not have been the case.

Many statements of number are perceptual. To find out how many apples are on a table one usually will visually locate the table and, using sight, count the apples upon it. Statements of number typically purport to convey perceptually achieved factual information.

Frege held that words like 'two' and 'three' function in a manner unlike that of adjectives such as 'red' or 'old'.[8] He argued for the point by such observations as the following: 'I own something red' is a logical consequence of 'I own red cars', but 'I own something two' not only is not a consequence of 'I own two cars', it is not even a well-formed sentence. Also

One man walked down the street.

is well formed, but

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[8]Frege, 1953, p. 40.

Old man walked down the street.

isn't.

Note that the sentence

Angry men are dangerous.

has a close paraphrase in

If a man is angry he is dangerous.

In contrast, the sentence

Some men are dangerous.

is not paraphrased by

If a man is some he is dangerous.

Indeed, the latter sentence is not even well-formed. On this score 'two' is like 'some' and unlike 'angry', for

If a man is two he is dangerous.

also is not well-formed and hence no paraphrase of

Two men are dangerous.

Note that

Old men are wise and old men are not wise.

is a contradiction, whereas

Some men are wise and some men are not wise.

is not. And the same holds for

Two men are wise and two men are not wise.

(imagine a world with just four men just two of whom are wise).

We may also note that a word like 'two' no more sorts with singular terms than it does with adjectives like 'red' and 'old'. Each of the following is ill-formed

Tom owns Bill cars.

Tom owns the star nearest the earth cars.

But what about singular terms for numbers? Well,

Tom owns the number two cars.

is just as ill-formed as the sentences just above.

I may sum up the points thus far urged by saying that in statements of number neither number words nor numerals occur either as adjectives or as names.

How then do they occur? The main point here is made by saying that they occur in the way in which such words as ‘some’, ‘no’, and ‘several’ occur. Borrowing again from Frege, we may say that in their use in statements of number both number words and numerals serve for the expression of generality.

In the symbolism of modern logic generality is expressed through quantification. We have, then, ‘all’ quantifiers, and ‘some’ quantifiers, and ‘no’ quantifiers. Equally, there are ‘most’ and ‘several’ quantifiers and, finally, numerical quantifiers. So, just as we write e.g.,

‘$(x)\mathrm{F}x$’ is true iff all values of ‘x’ satisfy ‘$\mathrm{F}x$’

we may write

‘$(\mathrm{S}x)\mathrm{F}x$’ is true iff several values of ‘x’ satisfy ‘$\mathrm{F}x$’

and

‘$(2x)\mathrm{F}x$’ is true iff two values of ‘x’ satisfy ‘Fx’.

The last form indicates how we might write statements of number in the modern symbolism.

These considerations lead me to say that so long as we consider only extra-mathematical statements of number it would be as erroneous to speak of the entities (objects, properties, relations,...) denoted by number words as it would be erroneous to speak of the entities denoted by such words as ‘some’ or ‘no’ or ‘several’.

Against this it might be said — and said perhaps by Frege[9] — that statements of number do involve numbers as entities because statements of number are strictly equivalent to statements about numbers. For just note, e.g., the equivalence of

Bill owns two cars.

[9] Cf. Frege, 1953, p. 69.

and

THE NUMBER OF CARS OWNED BY BILL = 2.

But now consider the following two sentences:

SOME WHALES ARE MAMMALS,

THE NUMBER OF MAMMALIAN WHALES > 0.

Surely these are also and in the same way equivalent. Shall we say that the first sentence involves a reference to the number zero or to the number of whales which are mammals? Or consider the pair:

ALL WHALES ARE MAMMALS.

THE NUMBER OF MAMMALIAN WHALES = THE NUMBER OF WHALES.

Shall we way that the first of these sentences also involves a reference to the number noted just above?

Still, it might be held that since the transformation into an equation yields an equivalent sentence, *that* marks the positions of numerical expressions as accessible to singular terms and thus as referential.

But here it is enough to note that '=' makes sense quite apart from the apparatus of singular terms and predicates. We can link mass terms like 'ice' and 'frozen water' with the identity sign without loss of sense, as in

ICE = FROZEN WATER.

and even quantify in respect to mass terms without loss of sense as in

ALL SNOW IS WHITE.

3.2 The idea that what fixes the meanings of mathematical signs is their use in empirical judgment might still be taken to show that those meanings must be referential.

One argument to this conclusion runs as follows: Empirical judgments are ones for which a realist conception is correct — for such judgments are true or false in virtue of how things actually are. But among empirical judgments are ones involving numerical quantifications. So, those quantifications must also be one's for which a realist conception is correct. Thus, there are numbers.

The sentence 'For some n, Belle has $n+n$ legs' expresses a true empirical judgment, one whose truth is not due to us. So it is a realist truth. But it quantifies over numbers. Thus, numbers exist.

The second inference is the one I question.

There is no doubt that it is an empirical truth that Hugly's dog, Belle, has an even number of legs *and* that the sentence expressing this truth is (or is equivalent to) an existential numerical quantification. And so there is no doubt that certain numerical quantifications are realist truths. But this is not sufficient to show that an existential number quantification is ontologically akin to an existential dog quantification *even if* both quantifications are realist truths.

The sentence 'For some x and n, x is a dog and x has $n + n$ legs' is a realist truth and is *both* a quantification "over numbers" and a quantification "over dogs". But that alone does not show that the uses of 'Belle' and '1' in the sentence 'Belle has $3 + 1$ legs' resemble one another in, say, the way that the functions of 'Leo' and 'Belle' resemble one another in the sentence 'Leo roars and Belle barks'.

Quantification is simply a method for constructing generalizations. Variables suitable for quantification are not limited to some one category. Quantifiable variables may occur in positions appropriate to names of persons, color predicates, sentences, numerals, etc. It is only if the quantifiable variables occur in referential positions that the quantification has ontological import.

What is important about a variable is that by means of it we produce a *form* (a "prototype") which indicates the type of judgment we generalize in its use.

The "ontology" of this or that species of quantification is fixed by the "ontology" of the type of judgment it generalizes in respect to the position in those judgments marked out by the variable.

That we generalize a certain range of judgments in a particular way *itself* tells us *nothing* about the "ontology" of the quantifications expressing those generalizations. What needs to be examined is not the *form* of the quantification, but the particular ways in which expressions of the type which serve as substituends for the variables function in the sentences in which they occur.

So long as we remain within the domain of extra-mathematical statements of number we shall lack any basis for holding that such statements involve references to numbers.

IV. Thesis Two

Thesis 2 says:

> The meaning of mathematical signs is determined by extra-mathematical statements of number.

I begin my argument for this thesis by imagining a society in which no extra-mathematical statements of number are made. The members of this society do not correlate the numerals with everyday objects as we do. They do not count. But they do pure mathematics. In particular, high priests do pure mathematics. Religious significance is attached to the business of proving and disproving. Religious holidays occur regularly. On these days the high priests pick arithmetical sentences at random from a big box (the contents of which are changed every holiday). Contestants are chosen from the adult population. Each is given a sentence and assigned the task of proving or disproving that sentence. Results are judged by the high priests. Success means admission to the ranks of the high priests (which sure beats working in the fields). Failure means sacrifice to the gods (which does not beat much of anything). This includes those unfortunates who happen to get undecidable sentences — a sure sign of having incurred the gods's displeasure. The whole business is a tremendous boast to the study of mathematics in the schools.

Isn't it clear that formalism is a correct account of this use of mathematics? Strings of symbols are derived from other strings of symbols. Other things happen; for example, definitions are given in which certain strings of symbols are put forth as shorthand for other strings of symbols. What this use of mathematics comes to, however, is nothing other than symbol manipulation done in accord with certain rules. What is there about this symbol manipulation which confers any content to the symbols? Nothing.

If you agree with this, then you should agree with thesis 2. For if formalism is true of this use of mathematics and is false of our actual use of mathematics, then the meaning of the signs of mathematics must be due to their use of extra-mathematical statements of number.

Pure mathematics — taken in isolation from the use of its signs in empirical judgment — is an activity for which a formalist account is roughly correct. Mathematical signs nonetheless have a sense, but only in and through belonging to a system of signs with empirical application — and thus a system of signs other than that of *pure* mathematics.

This is the upshot of thesis two.

V. Thesis Three

If the theorems of pure mathematics function as suppressed premises in sound extra-mathematical arguments, then these theorems must be true statements. But if realism is false, these theorems are not true in virtue of how things are with numbers. So in what does their truth consist?

Their truth consists in their provability. There is nothing else for their truth to consist in if they are not true in virtue of the way things are with numbers.

Outside of mathematics a proof establishes truth. But truth does not consist in proof. To prove that wild elephants still exist you have to search out one that the poachers or hunters or park managers have not yet murdered. That would establish the truth of the assertion. But the truth of the assertion does not consist in its having a proof; it is true in virtue of the way the world is with wild elephants.

Part of the content of the provability thesis (thesis 3) is that provability within mathematics is fundamentally different from provability outside mathematics. Outside of mathematics what establishes a sentence is not what makes it true. But within mathematics being true consists in having a proof. Outside mathematics proof establishes something beyond itself: truth. Within mathematics proof establishes nothing beyond itself.

There are two major objections to the provability theory.

1. The first objection is that Gödel showed that mathematical truth cannot be identified with provability. For example, Richard Jeffrey writes:

> Gödel's theorem dealt a deathblow to the theory which identified mathematical truth with provability.[10]

This theme is echoed in one logic text after another.[11]

The result of Gödel of which Jeffrey speaks is actually pretty simple to understand. The complexities lie on the side of the proof. Let us put that to the side and just think about *what* he proved.

It comes to this: That for any effective and consistent axiomatization of a theory including at least elementary arithmetic there are sentences in

[10] Jeffrey, 1967, p. 196.

[11] See, for example: Robert Stoll, 1961, p. 167; John Pollock, 1969, p. 229; Gerald Massey, 1970, p. 129; Benson Mates, 1972, p. 229.

the language of the theory such that neither they nor their negations are derivable from the axioms.[12]

This is a syntactical result. The notions of truth and falsity do not enter into it at all, either by way of the content of the theorem itself or by way of its proof. In particular, that truth and falsity in mathematics go beyond proof and disproof is no part of what Gödel proves.

2. The second objection goes thus: A proof in mathematics is a derivation from axioms. So, according to the provability theory, the truth of an axiom consists in its being derivable from itself. Is it not just obvious how implausible that is? Consider one of the Peano axioms:

$$\forall x(0 \neq sx)$$

How do we know that is true? The answer that it is derivable from itself is not likely to satisfy anyone. And it should not satisfy anyone since every statement is derivable from itself.

And why is not one consistent set of axioms as good as any other on the account offered by the provability thesis? Suppose that instead of the Peano axioms we had as our only axiom for arithmetic

$$\forall x(x = 0)$$

Relative to this axiom a wholly different set of sentences is true.

In *Remarks on the Foundations of Mathematics* Wittgenstein writes:

> I should like to say mathematics is a *motley* of techniques of proof.[13]

I am sure Wittgenstein would have denied that the motley of techniques of proof all reduce to derivations from axioms.

Consider a proof of '2+3=5' which is not a derivation from axioms. First, put the sentence into the primitive notation of number theory:

$$ss0 + sss0 = sssss0$$

[12]The Gödel result referred to is that if arithmetic is omega-consistent (if, that is, $\neg(x)A(x)$ is unprovable if each $A(n)$ is provable) then it is incomplete (there are sentences such that neither they nor their negations are provable). Rosser extended this: if arithmetic is consistent (if, that is, not every sentence is provable) then it is incomplete.

[13]Ludwig Wittgenstein, 1956, p. 84. His emphasis.

Then a proof of the sentence correlates the number of occurrences of 's' on the left side of the equation with the number of occurrences of 's' on the right. One can do this by simply counting the number of occurrences of 's' on the left and the number of occurrences of 's' on the right; or, one might draw a line from each 's' on the left to a unique 's' on the right.

In the case of a product, say,

$$ss0 \times sss0 = ssssss0$$

a similar proof is available. First, make a correlation of the three occurrences of 's' in '$sss0$' by drawing lower lines to the first three occurrences of 's' in '$ssssss0$'. Second, do it by drawing upper lines to the second three occurrences of 's' in '$ssssss0$'. Third, draw a line from the second 's' in '$ss0$' to the lower group of lines. Fourth, draw a line from the first 's' in '$ss0$' to the upper group of lines. This shows by the indicated correlations that there are twice as many occurrences of 's' in the numeral to the right as there are occurrences of 's' in the second numeral to the left.

This method of correlating 's''s on one side of an equation with 's''s on the other side can be extended to prove any atomic sentence of arithmetic provable from the Peano axioms and to disprove any atomic sentence whose negation is provable from those axioms.

Here is a definition of truth for arithmetic that is in accord with thesis 3. An atomic sentence is true if provable and false if disprovable. A negation is true if true if the negated sentence is false and false if the negated sentence is true. The other connectives are treated similarly. A universal quantification is true if each substitution instance of the quantified formula is true and is false if some instance of the quantified formula is false. Existential quantification is treated similarly.

By this definition truth is not the same as provability. For example, Gödel sentences are true on this definition although they are not provable.

What is true is *fixed* by what is provable since at the atomic level what is true is the same as what is provable. And all other truth is determined by truth at the atomic level.[14]

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