Logic and Logical Philosophy
Volume 17 (2008), 321–327
DOI: 10.12775/LLP.2008.018

Robert Sochacki

# REJECTED AXIOMS FOR THE "NONSENSE-LOGIC" W AND THE K-VALUED LOGIC OF SOBOCIŃSKI

**Abstract.** In this paper rejection systems for the "nonsense-logic" **W** and the $k$-valued implicational-negational sentential calculi of Sobociński are given. Considered systems consist of computable sets of rejected axioms and only one rejection rule: the rejection version of detachment rule.

*Keywords*: rejected axioms, the logic W, the $k$-valued sentential calculi of Sobociński.

## 1. The logic W

The logic W which is considered in [1] is one of the so called "nonsense-logics" systems. The primitive terms of this logic are: implication '$\to$', conjunction '$\wedge$', disjunction '$\vee$' and negation '$\neg$'. The set **W** of theses of this logic is the content of the following matrix

$$\mathfrak{M}_{\mathbf{W}} = (\{0, \tfrac{1}{2}, 1\}, \{1\}, \{c, k, a, n\}),$$

where functions $c, k, a, n\colon \{0, \frac{1}{2}, 1\} \longrightarrow \{0, \frac{1}{2}, 1\}$ for '$\to$', '$\wedge$', '$\vee$' and '$\neg$', respectively, are defined as follows:

$$c(x, y) = \begin{cases} 0, & \text{if } x = 1 \text{ and } y \neq 1 \\ 1, & \text{otherwise} \end{cases}$$

*Received* August 1, 2008; *Revised* September 30, 2008

ISSN: 1425-3305

$$k(x,y) = \begin{cases} \min(x,y) & \text{if } x \neq \frac{1}{2} \text{ and } y \neq \frac{1}{2} \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

$$a(x,y) = \begin{cases} \max(x,y) & \text{if } x \neq \frac{1}{2} \text{ and } y \neq \frac{1}{2} \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

$$n(x) = 1 - x$$

i.e. $\mathbf{W} = \mathrm{E}(\mathfrak{M}_{\mathbf{W}})$, i.e. $\alpha \in \mathbf{W}$ iff $h^e(\alpha) = 1$, for any valuation $e\colon \mathrm{At} \longrightarrow \{0, \frac{1}{2}, 1\}$, where At is the set of all propositional variables, while $h^e$ is the standard homomorphic extension of $e$ to the set of all formulas.

Of course, if $\ulcorner \alpha \to \beta \urcorner \in \mathbf{W}$ and $\alpha \in \mathbf{W}$, then $\beta \in \mathbf{W}$.

Now, we introduce new functors as follows:

$$F_0(p,q) = (p \to q) \to [(p \vee q) \to (p \wedge q)],$$

$$F_{\frac{1}{2}}(p,q) = [(\neg p \to (p \vee q)) \to (p \vee q)] \vee (p \to q),$$

$$F_1(p,q) = F_0(q,p).$$

To this functors there correspond in the matrix $\mathfrak{M}_{\mathbf{W}}$ the following functions:

$$f_0(x,y) = \begin{cases} 0 & \text{if } x = 0 \text{ and } y = 1 \\ 1 & \text{if } x \neq 0 \text{ or } y \neq 1 \end{cases}$$

$$f_{\frac{1}{2}}(x,y) = \begin{cases} 0 & \text{if } x = 1 \text{ and } y = \frac{1}{2} \\ 1 & \text{if } x \neq 1 \text{ or } y \neq \frac{1}{2} \end{cases}$$

$$f_1(x,y) = \begin{cases} 0 & \text{if } x = 1 \text{ and } y = 0 \\ 1 & \text{if } x \neq 1 \text{ or } y \neq 0 \end{cases}$$

The rejected axioms for the logic **W** are assumed to be the formulas: $F_0(p,q)$, $F_{\frac{1}{2}}(s,r)$, $F_1(t,u)$ or generalized disjunctions of these formulas, i.e. the expressions of the form:

$$F_i(p,q) \vee F_j(r,s) \vee \cdots \vee F_l(t,u),$$

where $i, j, \ldots, l \in \{0, \frac{1}{2}, 1\}$. It is easy to see that the set of these axioms is computable.

Let $\mathbf{W}^*$ be the smallest set of formulae which contains all rejected axioms and is closed under the rejection version of detachment rule (*modus ponens*):

$$\text{if } \ulcorner\alpha \to \beta\urcorner \in \mathbf{W} \text{ and } \beta \in \mathbf{W}^*, \text{ then } \alpha \in \mathbf{W}^*. \tag{RMP}$$

Theorem 1. *For any formula $\alpha$: $\alpha \notin \mathbf{W}$ iff $\alpha \in \mathbf{W}^*$.*

Proof. “$\Rightarrow$” Suppose that $\alpha \notin \mathbf{W}$, i.e. $\alpha \notin \mathrm{E}(\mathfrak{M}_{\mathbf{W}})$, where $\alpha = \alpha(p_{i_1}, p_{i_2}, \ldots, p_{i_n})$, for $i_1, \ldots, i_n \in \mathbb{N}^+$. This means that there is a valuation $e_0$ such that $h^{e_0}(\alpha) \leqslant \frac{1}{2}$. Let us assume that $e_0(p_{i_1}) = l_1, \ldots, e_0(p_{i_n}) = l_n$, where $l_1, \ldots, l_n \in \{0, \frac{1}{2}, 1\}$. In order to reject the formula $\alpha$ we consider the following rejected axiom:

$$\chi_0 := F_{l_1}(p_{i_1}, q) \vee F_{l_2}(r, p_{i_2}) \vee \cdots \vee F_{l_n}(p_{i_n}, s),$$

where the formula $F_{l_k}(r, p_{i_k})$, $k \in \{1, 2, \ldots, n\}$ occurs in $\chi_0$ only if $l_k = \frac{1}{2}$.

It is easy to see that $h^{e_0}(\chi_0) = 0$. Moreover, $\ulcorner\alpha \to \chi_0\urcorner \in \mathrm{E}(\mathfrak{M}_{\mathbf{W}})$, i.e. $\ulcorner\alpha \to \chi_0\urcorner \in \mathbf{W}$. Thus, $\alpha \in \mathbf{W}^*$, by the rejection rule (RMP).

„$\Leftarrow$” It is easy to prove by induction on the length of a proof. If $\alpha$ is a rejected axiom, then $\alpha \notin \mathrm{E}(\mathfrak{M}_{\mathbf{W}})$, i.e., $\alpha \notin \mathbf{W}$. Suppose that for some $\beta \in \mathbf{W}^*$ we have $\ulcorner\alpha \to \beta\urcorner \in \mathbf{W}$. Then by the inductive hypothesis we have that $\beta \notin \mathbf{W}$. So also $\beta \notin \mathbf{W}$. □

*Example* 1. Let us consider the formula $\alpha =$ ‘$p_1 \to [(p_1 \vee p_2) \wedge (p_3 \wedge p_1)]$’. Under the valuation $e$ such that $e_0(p_1) = 1$, $e_0(p_2) = \frac{1}{2}$, $e_0(p_3) = 0$, we have $h^{e_0}(\alpha) = 0$. In order to reject the formula $\alpha$ we consider the rejected axiom $\chi_0$ of the form:

$$F_0(p_3, q) \vee F_{\frac{1}{2}}(r, p_2) \vee F_1(p_1, s).$$

We have $h^{e_0}(\chi_0) = 0$ and $\ulcorner\alpha \to \chi_0\urcorner \in \mathrm{E}(\mathfrak{M}_{\mathbf{W}})$, i.e. $\ulcorner\alpha \to \chi_0\urcorner \in \mathbf{W}$. Now, using the rule (RMP), we obtain $\alpha \in \mathbf{W}^*$.

## 2. The *k*-valued implicational-negational sentential calculus of Sobociński

Let us consider the $k$-valued ($k \geqslant 3$) implicational-negational (‘$\to$’, ‘$\neg$’) sentential calculus of Sobociński [2]. The set $\mathbf{S}_k$ of theses of this calculus is the content of the following matrix

$$\mathfrak{M}_{\mathbf{S}_k} = (\{0, 1, \ldots, k-1\}, \{1, \ldots, k-1\}, \{c, n\}),$$

where functions $c, n\colon \{0,\ldots,k-1\} \longrightarrow \{0,\ldots,k-1\}$ for '$\rightarrow$' and '$\neg$', respectively, are defined as follows:

$$c(x,y) = \begin{cases} y & \text{if } x \neq y \\ k-1 & \text{if } x = y \end{cases}$$

$$n(x) = \begin{cases} x+1 & \text{if } x < k-1 \\ 0 & \text{if } x = k-1 \end{cases}$$

The axiomatization of this calculus is given in [2]. Similarly, as in the case of the logic W, we shall show that for this calculus, any formula which is not a thesis is rejected.

Since $\mathbf{S}_k = \mathrm{E}(\mathfrak{M}_{\mathbf{S}_k})$, we have: if $\ulcorner \alpha \rightarrow \beta \urcorner \in \mathbf{S}_k$ and $\alpha \in \mathbf{S}_k$, then $\beta \in \mathbf{S}_k$.

We adopt the following new functors:

$$\begin{aligned} G_0(p,q) &= p \rightarrow \neg(q \rightarrow q), \\ G_1(p,q) &= p \rightarrow \neg^2(q \rightarrow q), \\ &\vdots \\ G_{k-2}(p,q) &= p \rightarrow \neg^{k-1}(q \rightarrow q) \end{aligned} \tag{$\dagger$}$$

where the symbol $\neg^i$ (for $i \in \mathbb{N}^+$) is defined as follows: $\neg^1 = \neg$ and $\neg^{i+1} = \neg\neg^i$. The following functions correspond in $\mathfrak{M}_{\mathbf{S}_k}$ to functors listed in ($\dagger$):

$$g_0(x,y) = \begin{cases} 0 & \text{if } x \neq 0 \\ k-1 & \text{if } x = 0 \end{cases}$$

$$g_1(x,y) = \begin{cases} 1 & \text{if } x \neq 1 \\ k-1 & \text{if } x = 1 \end{cases}$$

$$\vdots$$

$$g_{k-2}(x,y) = \begin{cases} k-2 & \text{if } x \neq k-2 \\ k-1 & \text{if } x = k-2 \end{cases}$$

Moreover, on the basis of the function $n$ we have:

$$n(g_0(x,y)) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$$n(g_1(x,y)) = \begin{cases} 2 & \text{if } x \neq 1 \\ 0 & \text{if } x = 1 \end{cases}$$

$$\vdots$$

$$n(g_{k-2}(x,y)) = \begin{cases} k-1 & \text{if } x \neq k-2 \\ 0 & \text{if } x = k-2 \end{cases}$$

We shall define the next new functors:

$$\begin{aligned}
F_{k-1}(p,q) &= \neg G_0(q,p) \to (\neg G_1(q,p) \to (\dots \\
&\quad (\neg G_{k-3}(q,p) \to (\neg G_{k-2} \to \neg G_{k-2}(p,q))))), \\
F_{k-2}(p,q) &= F_{k-1}(p,q) \to (\neg G_0(q,p) \to (\neg G_1(q,p) \to (\dots \\
&\quad (\neg G_{k-3}(q,p) \to \neg G_{k-2}(p,q))))), \\
F_{k-3}(p,q) &= F_{k-1}(p,q) \to (F_{k-2}(p,q) \to (\neg G_0(q,p) \to (\dots \\
&\quad (\neg G_{k-4}(q,p) \to \neg G_{k-2}(p,q))))), \\
&\vdots \\
F_1(p,q) &= F_{k-1}(p,q) \to (F_{k-2}(p,q) \to (\dots (F_2(p,q) \\
&\quad \to (\neg G_0(q,p) \to \neg G_{k-2}(p,q))))), \\
F_0(p,q) &= F_{k-1}(p,q) \to (F_{k-2}(p,q) \to (\dots (F_1(p,q) \to \neg G_{k-2}(p,q))))
\end{aligned} \tag{‡}$$

The following functions correspond in the matrix $\mathfrak{M}_{\mathbf{S}_k}$ to these functors:

$$f_l(x,y) = \begin{cases} 0 & \text{for } x = k-2 \text{ and } y = l \\ k-1 & \text{for } x \neq k-2 \text{ or } y \neq l \end{cases}$$

where $0 \leqslant l \leqslant k-1$.

Now, we shall define the very useful functor $A_{\mathrm{S}}$:

$$A_{\mathrm{S}}(p,q) = \neg^2(p \to p) \to [(q \to p) \to \neg G_0(p,q)].$$

It is easy to verify that the following function $a_{\mathrm{S}}$ correspond in the matrix $\mathfrak{M}_{\mathbf{S}_k}$ to the functor $A_{\mathrm{S}}$. This function has a special property:

$$a_{\mathrm{S}}(x,y) = \max\{x,y\}, \text{ for } x,y \in \{0, k-1\}.$$

The rejected axioms are assumed to be the formulas of the form (‡) and expressions formed by the functor $A_{\mathrm{S}}$, i.e.:

$$F_i(p, q) \quad \text{or} \quad A_{\mathrm{S}}((F_i(r, p), F_j(q, s), \ldots, F_t(u, v)),$$

for $i$, $j$ $\ldots$, $t \in \{0, 1, \ldots, k-1\}$, where

$$A_{\mathrm{S}}(\alpha) = \alpha,$$

$$A_{\mathrm{S}}(\alpha_1, \alpha_2, \ldots, \alpha_n) = A_{\mathrm{S}}(A_{\mathrm{S}}(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}), \alpha_n), \text{ for } n \geqslant 2.$$

Let $\mathbf{S}_k^*$ be the smallest set of formulae which contains all rejected axioms and is closed under the rejection version of detachment rule (*modus ponens*):

$$\text{if } \ulcorner \alpha \to \beta \urcorner \in \mathbf{S}_k \text{ and } \beta \in \mathbf{S}_k^*, \text{ then } \alpha \in \mathbf{S}_k^*. \tag{RMP}$$

Theorem 2. *For any formula* $\alpha$: $\alpha \notin \mathbf{S}_k$ *iff* $\alpha \in \mathbf{S}_k^*$.

The proof of this theorem is very analogous to the proof of Theorem 1, so it will be omitted.

*Example* 2. (i) Let $k \geqslant 3$. Consider $\alpha =$ '$(p_1 \to p_2) \to (p_3 \to p_1)$'. The following valuation $e_0$ falsifies the formula $\alpha$: $e_0(p_1) = 0$, $e_0(p_2) = e_0(p_3) = 1$. Under this valuation we have $h^{e_0}(\alpha) = 0$. In order to reject the formula $\alpha$ we adopt the following rejected axiom:

$$\chi_0 := A_{\mathrm{S}}(F_0(q, p_1), F_1(r, p_2), F_1(s, p_3)).$$

For any valuation $e\colon \mathrm{At} \longrightarrow \{0, 1, \ldots, k-1\}$ we have $h^e(\alpha \to \chi_0) = k - 1$. So $\ulcorner \alpha \to \chi_0 \urcorner \in \mathbf{S}_k$. Using (RMP), we obtain that $\alpha \in \mathbf{S}_k^*$.

(ii) Let us notice that for $k \geqslant 5$ the following valuation $e_1$ falsifies the formula $\alpha$ from (i): $e_1(p_1) = 0$, $e_1(p_2) = 3$, and $e_1(p_3) = 4$. We have $h^{e_1}(\alpha) = 0$. Thus, in order to reject the formula $\alpha$ we can adopt the following rejected axiom:

$$\chi_1 := A_{\mathrm{S}}(F_0(q, p_1), F_3(r, p_2), F_4(s, p_3)).$$

## References

[1] Piróg-Rzepecka, K., *Nonsense-Logics Systems* (in Polish), OTPN–PWN, Warszawa–Wroclaw 1977.

[2] Sobociński, B., "Axiomatization of some many-valued systems of deductive theory" (in Polish), *Roczniki Prac Naukowych Zrzeszenia Asystentów Uniwersytetu Józefa Piłsudskiego w Warszawie* I (1936).

Robert Sochacki
Institute of Mathematics and Informatics
Opole University
ul. Oleska 48
52-052 Opole, Poland
sochacki@math.uni.opole.pl