

GENERALIZED TONNETZ AND DISCRETE ABEL–JACOBI MAP

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ABSTRACT. Motivated by classical Euler’s *Tonnetz*, we introduce and study the combinatorics and topology of more general simplicial complexes $\text{Tonn}^{n,k}(L)$ of *Tonnetz type*. Our main result is that for a sufficiently generic choice of parameters the generalized Tonnetz $\text{Tonn}^{n,k}(L)$ is a triangulation of a $(k-1)$ -dimensional torus T^{k-1} . In the proof we construct and use the properties of a *discrete Abel–Jacobi map*, which takes values in the torus $T^{k-1} \cong \mathbb{R}^{k-1}/\Lambda$ where $\Lambda \cong \mathbb{A}_{k-1}^*$ is the permutohedral lattice.

1. Introduction

In his seminal work on music theory “*Tentamen novae theoriae musicae ex certissimis harmoniae principiis dilucide expositae*” (1739), Leonhard Euler introduced a lattice diagram – *Tonnetz* – representing the classical tonal space. In more recent interpretations this diagram is identified as a triangulation of a torus with 24 triangles representing all the major and minor chords. If the equal tempered scale is identified with \mathbb{Z}_{12} , the Tonnetz can be described as

$$\text{Tonnetz} = \{\{x, x+3, x+7\} \mid x \in \mathbb{Z}_{12}\} \cup \{\{x, x+4, x+7\} \mid x \in \mathbb{Z}_{12}\}.$$

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