

## THE BORSUK–ULAM PROPERTY FOR HOMOTOPY CLASSES OF MAPS FROM THE TORUS TO THE KLEIN BOTTLE

DACIBERG LIMA GONÇALVES — JOHN GUASCHI  
VINICIUS CASTELUBER LAASS

---

ABSTRACT. Let  $M$  be a topological space that admits a free involution  $\tau$ , and let  $N$  be a topological space. A homotopy class  $\beta \in [M, N]$  is said to have the *Borsuk–Ulam property with respect to  $\tau$*  if for every representative map  $f: M \rightarrow N$  of  $\beta$ , there exists a point  $x \in M$  such that  $f(\tau(x)) = f(x)$ . In this paper, we determine the homotopy classes of maps from the 2-torus  $\mathbb{T}^2$  to the Klein bottle  $\mathbb{K}^2$  that possess the Borsuk–Ulam property with respect to a free involution  $\tau_1$  of  $\mathbb{T}^2$  for which the orbit space is  $\mathbb{T}^2$ . Our results are given in terms of a certain family of homomorphisms involving the fundamental groups of  $\mathbb{T}^2$  and  $\mathbb{K}^2$ .

### 1. Introduction

In the early twentieth century, St. Ulam conjectured that if  $f: \mathbb{S}^n \rightarrow \mathbb{R}^n$  is a continuous map, there exists  $x \in \mathbb{S}^n$  such that  $f(A(x)) = f(x)$ , where  $A: \mathbb{S}^n \rightarrow \mathbb{S}^n$  is the antipodal map. The confirmation of this result by K. Borsuk

---

2020 *Mathematics Subject Classification*. Primary: 55M20, 57M07; Secondary: 20F36.

*Key words and phrases*. Borsuk–Ulam theorem; homotopy class; braid groups; surfaces.

This work is a continuation of part of the Ph.D. thesis [7] of the third author who was supported by CNPq project no. 140836 and Capes/COFECUB project no. 12693/13-8.

The first and second authors wish to thank the ‘Réseau Franco–Brésilien en Mathématiques’ for financial support during their respective visits to the Laboratoire de Mathématiques Nicolas Oresme UMR CNRS 6139, Université de Caen Normandie, from the 9<sup>th</sup> to the 24<sup>th</sup> of November 2019, and to the Instituto de Matemática e Estatística, Universidade de São Paulo, from the 17<sup>th</sup> to the 31<sup>st</sup> of August 2019.