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A THREE SOLUTION THEOREM FOR A SINGULAR DIFFERENTIAL EQUATION WITH NONLINEAR BOUNDARY CONDITIONS

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Abstract. We study positive solutions to singular boundary value problems of the form:

$$\begin{cases} -u'' = h(t)\,\dfrac{f(u)}{u^{\alpha}} & \text{for } t \in (0,1),\\ u(0) = 0, \\ u'(1) + c(u(1))u(1) = 0, \end{cases}$$

where $0 < \alpha < 1$, $h\colon (0,1] \to (0,\infty)$ is continuous such that $h(t) \leq d/t^{\beta}$ for some $d > 0$ and $\beta \in [0, 1-\alpha)$ and $c\colon [0,\infty) \to [0,\infty)$ is continuous such that $c(s)s$ is nondecreasing. We assume that $f\colon [0,\infty) \to (0,\infty)$ is continuously differentiable such that $[(f(s) - f(0))/s^{\alpha}] + \tau s$ is strictly increasing for some $\tau \geq 0$ for $s \in (0,\infty)$. When there exists a pair of sub-supersolutions (ψ,ϕ) such that $0 \leq \psi \leq \phi$, we first establish a minimal solution $\underline{u}$ and a maximal solution $\overline{u}$ in $[\psi,\phi]$. When there exist two pairs of sub-supersolutions (ψ_1,ϕ_1) and (ψ_2,ϕ_2) where $0 \leq \psi_1 \leq \psi_2 \leq \phi_1$, $\psi_1 \leq \phi_2 \leq \phi_1$ with $\psi_2 \not\leq \phi_2$, and ψ_2, ϕ_2 are not solutions, we next establish the existence of at least three solutions u_1, u_2 and u_3 satisfying $u_1 \in [\psi_1,\phi_2], u_2 \in [\psi_2,\phi_1]$ and $u_3 \in [\psi_1,\phi_1] \setminus ([\psi_1,\phi_2] \cup [\psi_2,\phi_1])$.

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