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REMOVING ISOLATED ZEROES BY HOMOTOPY

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Abstract. Suppose that the inverse image of the zero vector by a continuous map $f\colon \mathbb{R}^n \to \mathbb{R}^q$ has an isolated point P. The existence of a continuous map g which approximates f but is nonvanishing near P is equivalent to a topological property we call "local inessentiality of zeros", generalizing the notion of index zero for vector fields, the $q = n$ case. We consider the problem of constructing such an approximation g and a continuous homotopy $F(x,t)$ from f to g through locally nonvanishing maps. If f is a semialgebraic map, then there exists F also semialgebraic. If $q = 2$ and f is real analytic with a locally inessential zero, then there exists a Hölder continuous homotopy $F(x,t)$ which, for $(x,t) \neq (P,0)$, is real analytic and nonvanishing. The existence of a smooth homotopy, given a smooth map f, is stated as an open question.

1. Introduction

For a continuous vector field on a manifold, it is well-known that an isolated zero can be removed by a small, local perturbation if and only if that zero has an "index" equal to 0. That is, for a vector field $\mathbf{f}$ vanishing with index 0 at $\overrightarrow{p}$, and any small neighbourhood of $\overrightarrow{p}$, there is another vector field $\mathbf{g}$ agreeing with $\mathbf{f}$ outside that neighbourhood, and arbitrarily $\mathcal{C}^0$-close to $\mathbf{f}$ but nonvanishing inside it. In fact, the zero is removable in the following stronger, but less well-known, sense ([11]): not only such a perturbation $\mathbf{g}$ exists, but there also exists

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