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RELATIVE ENTROPY METHOD FOR MEASURE-VALUED SOLUTIONS IN NATURAL SCIENCES

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Abstract. We describe the applications of the relative entropy framework introduced in [10]. In particular the uniqueness of an entropy solution is proven for a scalar conservation law, using the notion of measure-valued entropy solutions. Further we survey recent results concerning measure-valued-strong uniqueness for a number of physical systems — incompressible and compressible Euler equations, compressible Navier–Stokes, polyconvex elastodynamics and general hyperbolic conservation laws, as well as long-time asymptotics of the McKendrick–Von Foerster equation.

1. Introduction

The origins of the relative entropy method can be traced back to physics. The underlying principle behind it is the simple idea to measure in a certain way how much two evolutions of a given physical system, whose initial states

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are "close", differ and to investigate how this "distance" evolves in time. This framework, closely related to the second law of thermodynamics, is a useful tool in obtaining a variety of interesting analytical results. For instance it can be used to show the uniqueness of solutions to a conservation law in the scalar case, while for many systems of equations it provides the so-called weak-strong uniqueness property, i.e. establishes uniqueness of classical solutions in a wider class of weak solutions. This application, first described by Dafermos [10], [11], will be highlighted in this article.

Other areas where relative entropy method is found useful include stability studies, asymptotic limits and dimension reduction problems (e.g. [9], [23], [17], [2]). The method is also applied to problems arising from biology, cf. [32], [33], [37], [28], known in this context as General Relative Entropy (GRE). It is essentially used for showing asymptotic convergence of solutions to steady-state solutions.

On the level of weak solutions for various physical systems, including Navier–Stokes and Euler, the story seems quite complete. However recent years have delivered many new results on the level of measure-valued (mv) solutions (e.g. [4], [13], [26], [18]). This shows that even though mv solutions are considered a very weak notion of solution, not carrying much information, they do play an important role in the analysis of physical systems. We begin our discussion on the level of the scalar conservation law

$$\begin{aligned} \partial_t u(x,t) + \operatorname{div}_x f(u(x,t)) &= 0 \qquad \text{in } \mathbb{T}^d \times \mathbb{R}_+, \\ u(x,0) &= u_0(x) \quad \text{in } \mathbb{T}^d. \end{aligned} \tag{1.1}$$

in the framework of measure-valued solutions. Here $\mathbb{R}_+ = [0,+\infty)$, $\mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d$ and u_0 is a given initial datum. The main ideas come from Tartar [42] and DiPerna [15], who defined entropy mv solutions in the language of classical Young measures, see also [35].

Definition 1.1 (Measure-valued solution). A measurable measure-valued map $\nu\colon (x,t) \to \nu_{(x,t)} \in \operatorname{Prob}(\mathbb{R})$ from $\mathbb{T}^d \times \mathbb{R}_+$ to the space of probability measures on $\mathbb{R}$ is a *measure-valued solution* of (1.1) if

$$\partial_t \langle \nu_{(x,t)}, \lambda \rangle + \operatorname{div}_x \langle \nu_{(x,t)}, f(\lambda) \rangle = 0 \tag{1.2}$$

in the sense of distributions, that is

$$\int_{\mathbb{R}_+} \int_{\mathbb{T}^d} \{ \langle \nu_{(x,t)}, \lambda \rangle \partial_t \phi + \langle \nu_{(x,t)}, f(\lambda) \rangle \nabla_x \phi \} \, dx \, dt = 0$$

for all $\phi \in \mathrm{C}^1_c(\mathbb{T}^d \times \mathbb{R}_+)$.

Here $\langle \cdot\, , \cdot \rangle$ denotes the duality pairing between $\mathrm{C}_0(\mathbb{R})$, the closure with respect to the supremum norm of the space of continuous functions on $\mathbb{R}$ with compact support, and $\mathcal{M}(\mathbb{R})$, the space of signed Radon measures on $\mathbb{R}$, i.e.

$$\langle \mu, g(\lambda) \rangle := \int_{\mathbb{R}} g(\lambda)\, d\mu(\lambda).$$

By measurability of a measure ν we mean the weak*-measurability of the measures $\nu_{(x,t)}$, i.e. measurability of the map

$$(x,t) \mapsto \langle \nu_{(x,t)}, g(\lambda) \rangle$$

for each $g \in \mathrm{C}_0(\mathbb{R})$. Such a measure-valued solution often arises from a weakly convergent approximating sequence. The framework of Young measures is in a sense a way of immersing the initial problem into a wider space — in this way one gains linearity at the cost of having to deal with measure spaces rather than function spaces. In other words Young measures allow to deal with the non-commutativity of weak limits with nonlinearities. Indeed, it can be shown that if $\{u_k\}$ is a sequence uniformly bounded in L^p, then, along a non-relabeled subsequence, the weak limit of $\{f(u_k)\}$ can be represented by means of a parameterized family of measures, a Young measure.

LEMMA 1.2 (Fundamental Lemma of Classical Young Measures). *Let* $\{u_k\}\colon \mathbb{T}^d \times \mathbb{R}_+ \to \mathbb{R}$ *be a sequence uniformly bounded in* $L^\infty(\mathbb{R}_+; L^p(\mathbb{T}^d))$*, i.e.*

$$\sup_k \|u_k(\,\cdot\,,t)\|_{L^p(\mathbb{T}^d)} \le C, \quad \text{for almost every } t \in \mathbb{R}_+. \tag{1.3}$$

Then there exists a subsequence, still denoted $\{u_k\}$ *and a measurable measure-valued mapping* $\nu\colon \mathbb{T}^d \times \mathbb{R}_+ \to \mathrm{Prob}(\mathbb{R})$*, such that for each* $g \in \mathrm{C}(\mathbb{R}^d)$*, satisfying the growth condition*

$$|g(\lambda)| \le C(1 + |\lambda|^q) \quad \text{for } 1 \le q < p,$$

the weak limit of $g(u_k(x,t))$ *exists and is represented by* $\langle \nu_{(x,t)}, g(\lambda) \rangle$*, i.e.*

$$\lim_{k\to\infty} \int_{\mathbb{T}^d \times \mathbb{R}_+} g(u_k(x,t))\phi(x,t)\, dx\, dt = \int_{\mathbb{T}^d \times \mathbb{R}_+} \langle \nu_{(x,t)}, g(\lambda) \rangle \phi(x,t)\, dx\, dt, \tag{1.4}$$

for all $\phi \in L^\infty(\mathbb{T}^d \times \mathbb{R}_+)$.

DiPerna [15] and later Szepessy [41], under different assumptions on the continuous flux, showed the so-called averaged contraction principle, which is a crucial estimate in showing uniqueness of mv solutions, and is essentially a form of relative entropy inequality. Importantly the uniqueness result is proven under the assumption that the initial data is a Dirac delta measure. For non-atomic initial data uniqueness might fail in the class of measure-valued solutions, even in the scalar case — and even under an entropy inequality, as it provides information only on certain moments of the solution, cf. [20]

To show the existence of a measure-valued entropy solution to (1.1) a parabolic approximate problem is considered. It generates a sequence of approximate solutions, which can be shown to be uniformly integrable. Thus one can see that there is going to be no concentration effect. However, other approximation schemes can be considered, which will not posses sufficient integrability. In fact we prove uniqueness in a wider class of mv solutions not necessarily corresponding to any approximation scheme.

Further, the situation differs substantially in the case of hyperbolic systems. The result introducing measure-valued solutions for the incompressible Euler describes both oscillations and concentrations, cf. [16], [1]. This is because, contrary to the scalar case, for systems there is usually only one entropy-entropy flux pair forming a companion law. The corresponding entropy inequality lacks the symmetry which is present in the scalar case.

One then considers the so-called generalized Young measures. Here by a generalized Young measure we mean a triple (ν, m, ν^∞) describing oscillations, concentrations and concentration-angle respectively. The following representation result, proven in [1] is then true.

Lemma 1.3. *Let* $\{u_k\}\colon \mathbb{T}^d \times \mathbb{R}_+ \to \mathbb{R}^n$ *be a sequence bounded in* $L^\infty(\mathbb{R}_+; L^1(\mathbb{T}^d))$, *i.e.*

$$\sup_k \|u_k(\,\cdot\,,t)\|_{L^1(\mathbb{T}^d)} \leq C, \quad \textit{for almost every } t \in \mathbb{R}_+. \tag{1.5}$$

Then there exists a subsequence, still denoted $\{u_k\}$, *a measurable measure-valued mapping* $\nu\colon \mathbb{T}^d \times \mathbb{R}_+ \to \operatorname{Prob}(\mathbb{R})$, *a non-negative measure* m *on* $\mathbb{T}^d \times \mathbb{R}_+$ *and a parameterized probability measure* $\nu^\infty \in L^\infty_w(\mathbb{T}^d \times \mathbb{R}_+, m, \operatorname{Prob}(\mathbb{S}^{n-1}))$ *such that for each Carathéodory function* $g\colon \mathbb{T}^d \times \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$

$$g(x,t,u_k(x,t)) \overset{*}{\rightharpoonup} \langle \nu_{(x,t)}, g(x,t,\lambda)\rangle + \langle \nu^\infty_{(x,t)}, g^\infty\rangle m \tag{1.6}$$

weak in the sense of measures, provided the function* g^∞, *defined to be*

$$g^\infty(x,t,\beta) := \lim_{s\to\infty} \lim_{(x',t',\beta')\to(x,t,\beta)} \frac{g(x',t',s\beta')}{s}, \quad \textit{where } \beta \in \mathbb{S}^{n-1}.$$

is of class $\mathrm{C}(\mathbb{T}^d \times \mathbb{R}_+ \times \mathbb{S}^{n-1})$.

In this article we want to show, in a way as technically simple as possible, the main idea of this framework. To this end we provide in Section 2 the full proof of an averaged contraction principle for the scalar conservation law under the artificial assumption that concentration effects cannot be excluded. This highlights an interesting phenomenon that the concentration measure indeed vanishes thus concluding uniqueness on the basis of averaged contraction principle. The same observation transfers to systems, of course on the level of proving only weak(measure-valued)-strong uniqueness, cf. [3], [26]. Interestingly, even more can be claimed, namely that the information about concentration angle does not give any essential information, cf. [18]. Indeed it is enough to know that the concentration measure appearing in the weak formulation is dominated by the concentration measure coming from the energy/entropy inequality.

Then in Sections 3–5 we survey the weak-strong uniqueness results obtained using the relative entropy method for equations of fluid dynamics (Section 3), polyconvex elastodynamics (Section 4) and general systems of conservation laws. Finally we display how the relative entropy method is used for the renewal equation of mathematical biology (Section 6).

2. Measure-valued entropy solutions for scalar conservation laws

We consider the Cauchy problem for the scalar conservation law

$$\begin{aligned} \partial_t u + \operatorname{div}_x f(u) &= 0, \\ u(x,0) &= u_0(x), \end{aligned} \tag{2.1}$$

where $f\colon \mathbb{R} \to \mathbb{R}^d$ represents the flux of the quantity $u\colon \mathbb{T}^d \times \mathbb{R}_+ \to \mathbb{R}$. We choose to work in the spatially periodic setting to avoid inessential technical issues.

We introduce now the concept of entropy measure-valued solutions, which will then be used to show existence of a unique entropy solution to the problem (2.1). The following will be the standing assumptions on the flux function f and the initial data u_0:

$$\begin{gathered} f \in \mathrm{C}(\mathbb{R}) \quad \text{and} \quad u_0 \in L^1(\mathbb{T}^d), \\ |f(\lambda)| \le C(1+|\lambda|) \quad \text{for some } C > 0. \end{gathered} \tag{2.2}$$

The definition of a measure-valued entropy solution to (2.1) consists of a classical Young measure ν as well as two concentration measures m_1 and m_2. We will assume that

$$|m_2|(\mathbb{T}^d \times A) \le C\, m_1(\mathbb{T}^d \times A) \tag{2.3}$$

for any Borel set $A \subset \mathbb{R}_+$. We will later see that for a solution arising as a limit of an approximating sequence this is guaranteed by characterization of the corresponding concentration measures.

DEFINITION 2.1 (Measure-valued entropy solution). The triple (ν, m_1, m_2) generated by a sequence which satisfies (1.5) is called a *measure-valued entropy solution with concentration* of conservation law (2.1) if

$$\partial_t(\langle \nu, |\lambda - k|\rangle + m_1) + \operatorname{div}_x(\langle \nu, \operatorname{sgn}(\lambda - k)(f(\lambda) - f(k))\rangle + m_2) \le 0 \tag{2.4}$$

in the sense of distributions on $\mathbb{T}^d \times \mathbb{R}_+$ for all $k \in \mathbb{R}$, and if

$$\begin{aligned} \lim_{T\to 0^+} \bigg\{ \frac{1}{T} \int_0^T \int_{\mathbb{T}^d} \langle \nu_{(x,t)}, |\lambda - u_0(x)|\rangle \,dx\,dt \\ + \frac{1}{T} \int_{\mathbb{T}^d \times [0,T]} m_1(dx\,dt) \bigg\} = 0. \end{aligned} \tag{2.5}$$

Notice that the entropy inequality (2.4) is sufficient to guarantee that the triple (ν, m_1, m_2) satisfies also the weak formulation. To see this assume, for simplicity, that the measure ν has a bounded support $\operatorname{supp}\nu_{(x,t)} \in (-A, A)$. Then, taking $k < -A$, we have $\lambda - k > 0$ and (2.4) becomes for any $\phi \geq 0$

$$\begin{aligned} 0 &\leq \int_{\mathbb{T}^d\times\mathbb{R}_+} \langle \nu_{(x,t)}, \lambda - k\rangle \partial_t\phi \, dx\, dt + \int_{\mathbb{T}^d\times\mathbb{R}_+} \partial_t\phi \; m_1(dx\, dt) \\ &\quad + \int_{\mathbb{T}^d\times\mathbb{R}_+} \langle \nu_{(x,t)}, f(\lambda) - f(k)\rangle \nabla\phi \, dx\, dt + \int_{\mathbb{T}^d\times\mathbb{R}_+} \nabla\phi \; m_2(dx\, dt) \\ &= \int_{\mathbb{T}^d\times\mathbb{R}_+} \langle \nu_{(x,t)}, \lambda\rangle \partial_t\phi \, dx\, dt + \int_{\mathbb{T}^d\times\mathbb{R}_+} \partial_t\phi \; m_1(dx\, dt) \\ &\quad + \int_{\mathbb{T}^d\times\mathbb{R}_+} \langle \nu_{(x,t)}, f(\lambda)\rangle \nabla\phi \, dx\, dt + \int_{\mathbb{T}^d\times\mathbb{R}_+} \nabla\phi \; m_2(dx\, dt). \end{aligned}$$

Taking in turn $k > A$ one obtains the reversed inequality. It follows that

$$\partial_t(\langle \nu_{(x,t)}, \lambda\rangle + m_1) + \operatorname{div}_x(\langle \nu_{(x,t)}, f(\lambda)\rangle + m_2) = 0.$$

Now we prove a uniqueness result concerning mv entropy solutions.

Theorem 2.2. *Suppose (ν, m_1, m_2) and $(\sigma, \overline{m}_1, \overline{m}_2)$ are Young measure-concentration measure triples satisfying* (2.4) *and $\nu_{(x,0)} = \sigma_{(x,0)} = \delta_{\{u_0(x)\}}$. Then there exists a function*

$$w \in L^\infty(\mathbb{R}_+; L^1(\mathbb{T}^d)) \tag{2.6}$$

such that

$$\nu_{(x,t)} = \sigma_{(x,t)} = \delta_{\{w(x,t)\}} \tag{2.7}$$

for almost each $(x,t) \in \mathbb{T}^d \times \mathbb{R}_+$ and $m_i = \overline{m}_i = 0$, $i = 1, 2$.

Proof. We begin by mollifying the measures ν, σ, m_i and $\overline{m}_i$, $i = 1, 2$. Let η be a non-negative, symmetric smooth function on $\mathbb{T}^d \times \mathbb{R}_+$ compactly supported in the open unit ball in $\mathbb{R}^d \times \mathbb{R}$ and such that $\int_{\mathbb{R}^d\times\mathbb{R}} \eta(x,t) = 1$, and let $\eta^\varepsilon(x,t) = \varepsilon^{-(d+1)}\eta(x/\varepsilon, t/\varepsilon)$. By ν^ε we denote the parameterised measure satisfying for all $g \in \mathrm{C}(\mathbb{R})$

$$\langle \nu^\varepsilon_{(x,t)}, g\rangle = \int_{\mathbb{T}^d\times\mathbb{R}_+} \eta^\varepsilon(x - x', t - t')\langle \nu_{(x',t')}, g\rangle \, dx'\, dt' \in \mathrm{C}^\infty(K; \mathcal{M}(\mathbb{R}))$$

where $K \subset \mathbb{T}^d \times \mathbb{R}_+$ is a set whose ε-neighborhood is entirely contained in $\mathbb{T}^d \times \mathbb{R}_+$. Furthermore, we observe that by virtue of the Riesz representation theorem, for each $(x,t) \in \mathbb{T}^d \times \mathbb{R}_+$ there are bounded measures $\partial_x\nu^\varepsilon_{(x,t)}$ and $\partial_t\nu^\varepsilon_{(x,t)}$ such that, for all $g \in \mathrm{C}(\mathbb{R})$,

$$\langle \partial_\alpha\nu^\varepsilon_{(x,t)}, g\rangle = \partial_\alpha\langle \nu^\varepsilon_{(x,t)}, g\rangle, \quad \text{for } \alpha \in \{t, x\},$$

and the map $(x,t) \mapsto \langle \partial_\alpha\nu^\varepsilon_{(x,t)}, g\rangle$ is continuous.

Observe that modifying measures m_i and $\overline{m}_i$ with a regular kernel gives smooth functions $m_1^\varepsilon, \overline{m}_1^\varepsilon \in \mathrm{C}^\infty(K;\mathbb{R}_+)$ and $m_2^\varepsilon, \overline{m}_2^\varepsilon \in \mathrm{C}^\infty(K;\mathbb{R})$. Observe also that ν and σ have finite first moments.

We will now show that the regularized measures satisfy the entropy inequality. Let $V \subset \mathbb{T}^d \times \mathbb{R}_+$ be an open bounded set. For sufficiently small $\varepsilon > 0$ we have, for each non-negative $\phi \in \mathrm{C}_c^\infty(V)$ and $g \in \mathrm{C}(\mathbb{R})$,

$$\begin{aligned}\int_{\mathbb{T}^d\times\mathbb{R}_+} (\langle \nu_{(x,t)}, g\rangle)\partial_t(\phi * \eta^\varepsilon)\,dx\,dt + \int_{\mathbb{T}^d\times\mathbb{R}_+} \partial_t(\phi*\eta^\varepsilon)\,m_1(dx\,dt)&\\ = -\int_{\mathbb{T}^d\times\mathbb{R}_+} \big(\langle \partial_t \nu^\varepsilon_{(x,t)}, g\rangle + \partial_t m_1^\varepsilon\big)\phi(x,t)\,dx\,dt,&\end{aligned}$$

and, analogously, for the spatial derivative. Hence, choosing in turn $g(\lambda) = |\lambda-\mu|$ and $g(\lambda) = \operatorname{sgn}(\lambda-\mu)(f(\lambda)-f(\mu))$ and using (2.4)

$$\begin{aligned}&\int_{\mathbb{T}^d\times\mathbb{R}_+} (\langle \partial_t\nu^\varepsilon_{(x,t)}, |\lambda-\mu|\rangle + \partial_t m_1^\varepsilon\\ &\qquad + \langle \partial_x \nu^\varepsilon_{(x,t)}, \operatorname{sgn}(\lambda-\mu)(f(\lambda)-f(\mu))\rangle + \partial_x m_2^\varepsilon)\phi(x,t)\,dx\,dt\\ &= -\int_{\mathbb{T}^d\times\mathbb{R}_+} [\langle \nu_{(x,t)}, |\lambda-\mu|\rangle\partial_t(\phi*\eta^\varepsilon) + \langle \nu_{(x,t)}, q(\lambda,\mu)\rangle \nabla(\phi*\eta^\varepsilon)]\,dx\,dt\\ &\quad - \int_{\mathbb{T}^d\times\mathbb{R}_+} \partial_t(\phi*\eta^\varepsilon)\ m_1(dx\,dt) - \int_{\mathbb{T}^d\times\mathbb{R}_+} \nabla(\phi*\eta^\varepsilon)\ m_2(dx\,dt) \le 0,\end{aligned}$$

where we denote $q(\lambda,\mu) := \operatorname{sgn}(\lambda-\mu)(f(\lambda)-f(\mu))$ for brevity. Therefore

$$\langle \partial_t \nu^\varepsilon_{(x,t)}, |\lambda-\mu|\rangle + \partial_t m_1^\varepsilon + \langle \partial_x\nu^\varepsilon_{(x,t)}, \operatorname{sgn}(\lambda-\mu)(f(\lambda)-f(\mu))\rangle + \partial_x m_2^\varepsilon \le 0 \tag{2.8}$$

for all $\mu \in \mathbb{R}$ and $(x,t) \in V$. Symmetrically it can be seen that the triple $(\sigma^\varepsilon, \overline{m}_1^\varepsilon, \overline{m}_2^\varepsilon)$ satisfies (2.4) for all $\lambda \in \mathbb{R}$. Next we observe that the function q is continuous on $\mathbb{R}^2$ and it has sublinear growth due to growth conditions (2.2) on f. It follows that the maps $\mu \mapsto \int q(\lambda,\mu)\,d\nu^\varepsilon_{(x,t)}(\lambda)$ and $\lambda \mapsto \int q(\lambda,\mu)\,d\sigma^\varepsilon_{(x,t)}(\mu)$ are continuous (by virtue of the dominated convergence theorem). Let now ε_1 correspond to the smoothing of (ν, m_1, m_2) and ε_2 correspond to smoothing of $(\sigma, \overline{m}_1, \overline{m}_2)$. We can then compute

$$\begin{aligned}\operatorname{div}_x &\big(\big\langle \nu^{\varepsilon_1}_{(x,t)} \otimes \sigma^{\varepsilon_2}_{(x,t)}, q(\lambda,\mu)\big\rangle + m_2^{\varepsilon_1} + \overline{m}_2^{\varepsilon_2}\big)\\ &= \int_{\mathbb{R}} \operatorname{div}_x\left(\int_{\mathbb{R}} q(\lambda,\mu)\,d\nu^{\varepsilon_1}_{(x,t)}\right) d\sigma^{\varepsilon_2}_{(x,t)}\\ &\quad + \int_{\mathbb{R}}\int_{\mathbb{R}} q(\lambda,\mu)\,d\nu^{\varepsilon_1}_{(x,t)}\,d(\partial_x\sigma^{\varepsilon_2}_{(x,t)}) + \partial_x m_2^{\varepsilon_1} + \partial_x\overline{m}_2^{\varepsilon_2}\\ &= \int_{\mathbb{R}} \big\langle \partial_x \nu^{\varepsilon_1}_{(x,t)}, q(\lambda,\mu)\big\rangle\,d\sigma^{\varepsilon_2}_{(x,t)}\\ &\quad + \int_{\mathbb{R}} \big\langle \partial_x\sigma^{\varepsilon_2}_{(x,t)}, q(\lambda,\mu)\big\rangle\,d\nu^{\varepsilon_1}_{(x,t)} + \partial_x m_2^{\varepsilon_1} + \partial_x \overline{m}_2^{\varepsilon_2},\end{aligned}$$

where we have used Fubini's theorem in the last line. The tensor product $\nu_{(x,t)} \otimes \sigma_{(x,t)}$ is defined as the product measure $d\nu_{(x,t)}(\lambda)\, d\sigma_{(x,t)}(\mu)$. Similarly,

$$\begin{aligned}&\partial_t\big(\big\langle \nu^{\varepsilon_1}_{(x,t)} \otimes \sigma^{\varepsilon_2}_{(x,t)}, |\lambda-\mu|\big\rangle + m_1^{\varepsilon_1} + \overline{m}_1^{\varepsilon_2}\big)\\ &= \int_{\mathbb{R}} \big\langle \partial_t \nu^{\varepsilon_1}_{(x,t)}, |\lambda-\mu|\big\rangle\, d\sigma^{\varepsilon_2}_{(x,t)} + \int_{\mathbb{R}} \big\langle \partial_t \sigma^{\varepsilon_2}_{(x,t)}, |\lambda-\mu|\big\rangle\, d\nu^{\varepsilon_1}_{(x,t)} + \partial_t m_1^{\varepsilon_1} + \partial_t \overline{m}_1^{\varepsilon_2}.\end{aligned}$$

Consequently, using (2.8),

$$\begin{aligned}&\int_{\mathbb{T}^d\times\mathbb{R}_+} \big(\big\langle \nu^{\varepsilon_1}_{(x,t)} \otimes \sigma^{\varepsilon_2}_{(x,t)}, |\lambda-\mu|\big\rangle + m_1^{\varepsilon_1} + \overline{m}_1^{\varepsilon_2}\big)\partial_t\phi\, dx\, dt\\ &\qquad + \int_{\mathbb{T}^d\times\mathbb{R}_+} \big(\big\langle \nu^{\varepsilon_1}_{(x,t)} \otimes \sigma^{\varepsilon_2}_{(x,t)}, q(\lambda,\mu)\big\rangle + m_2^{\varepsilon_1} + \overline{m}_2^{\varepsilon_2}\big)\nabla\phi\, dx\, dt\\ &\qquad = -\int_{\mathbb{T}^d\times\mathbb{R}_+} \phi \int_{\mathbb{R}} \big\{\big(\big\langle \partial_t \nu^{\varepsilon_1}_{(x,t)}, |\lambda-\mu|\big\rangle\\ &\qquad\qquad + \big\langle \partial_x \nu^{\varepsilon_1}_{(x,t)}, q(\lambda,\mu)\big\rangle\big)\, d\sigma^{\varepsilon_2}_{(x,t)} - \partial_t m_1^{\varepsilon_1} - \partial_x m_2^{\varepsilon_1}\big\}\, dx\, dt\\ &\qquad\quad - \int_{\mathbb{T}^d\times\mathbb{R}_+} \phi \int_{\mathbb{R}} \big\{\big(\big\langle \partial_t \sigma^{\varepsilon_2}_{(x,t)}, |\lambda-\mu|\big\rangle\\ &\qquad\qquad + \big\langle \partial_x \sigma^{\varepsilon_2}_{(x,t)}, q(\lambda,\mu)\big\rangle\big)\, d\nu^{\varepsilon_1}_{(x,t)} - \partial_t \overline{m}_1^{\varepsilon_2} - \partial_x \overline{m}_2^{\varepsilon_2}\big\}\, dx\, dt \ge 0.\end{aligned}$$

This holds for any $\phi \in \mathrm{C}_c^\infty(V)$, thus establishing the following inequality in $\mathcal{D}'(\mathbb{T}^d\times\mathbb{R}_+)$

$$\begin{aligned}\partial_t\big(\big\langle \nu^{\varepsilon_1}_{(x,t)} \otimes \sigma^{\varepsilon_2}_{(x,t)}, |\lambda-\mu|\big\rangle + m_1^{\varepsilon_1} + \overline{m}_1^{\varepsilon_2}\big)&\\ + \operatorname{div}_x \big(\big\langle \nu^{\varepsilon_1}_{(x,t)} \otimes \sigma^{\varepsilon_2}_{(x,t)}, q(\lambda,\mu)\big\rangle + m_2^{\varepsilon_1} + \overline{m}_2^{\varepsilon_2}\big) &\le 0.\end{aligned} \tag{2.9}$$

Observe that the function $\varphi(x,t) = \psi(x)\phi(t)$ with $\psi \equiv 1$ and $\phi \in \mathrm{C}_c^\infty(\mathbb{R}_+)$ is an admissible test function, since $\mathbb{T}^d$ is a compact manifold without boundary. Testing (2.9) with such a function yields

$$\int_{\mathbb{T}^d\times\mathbb{R}_+} \big(\big\langle \nu^{\varepsilon_1}_{(x,t)} \otimes \sigma^{\varepsilon_2}_{(x,t)}, |\lambda-\mu|\big\rangle + m_1^{\varepsilon_1} + \overline{m}_1^{\varepsilon_2}\big)\partial_t\phi\, dx\, dt \ge 0. \tag{2.10}$$

The argument presented in [41] establishes the limit

$$\begin{aligned}\lim_{\varepsilon_1\to 0}\lim_{\varepsilon_2\to 0} \int_{\mathbb{T}^d\times\mathbb{R}_+} \big\langle \nu^{\varepsilon_1}_{(x,t)} \otimes \sigma^{\varepsilon_2}_{(x,t)}, |\lambda-\mu|\big\rangle\, \partial_t\phi\, dx\, dt&\\ = \int_{\mathbb{T}^d\times\mathbb{R}_+} \big\langle \nu_{(x,t)} \otimes \sigma_{(x,t)}, |\lambda-\mu|\big\rangle\, dx\, dt&.\end{aligned}$$

Clearly we have the convergence

$$\begin{aligned}&\int_{\mathbb{T}^d\times\mathbb{R}_+} m_1^{\varepsilon_1}\partial_t\phi\, dx\, dt\\ &\qquad = \int_{\mathbb{T}^d\times\mathbb{R}_+} \partial_t(\phi * \eta^{\varepsilon_1})\ m_1(dx\, dt) \xrightarrow{\varepsilon_1\to 0} \int_{\mathbb{T}^d\times\mathbb{R}_+} \partial_t\phi\ m_1(dx\, dt).\end{aligned}$$

Similarly

$$\int \overline{m}_1^{\varepsilon_2}\, \partial_t \phi \to \int \partial_t \phi\, \overline{m}_1(dx\, dt).$$

We thus have, for any non-negative $\phi \in \mathrm{C}_c^\infty(\mathbb{R}_+)$,

$$\int_{\mathbb{T}^d \times \mathbb{R}_+} \langle \nu_{(x,t)} \otimes \sigma_{(x,t)}, |\lambda - \mu| \rangle \partial_t \phi(t)\, dx\, dt + \int_{\mathbb{T}^d \times \mathbb{R}_+} \partial_t \phi\; m_1(dx\, dt) + \int_{\mathbb{T}^d \times \mathbb{R}_+} \partial_t \phi\; \overline{m}_1(dx\, dt) \geq 0. \tag{2.11}$$

Let now

$$A(t) := \int_{\mathbb{T}^d} \langle \nu_{(x,t)} \otimes \sigma_{(x,t)}, |\lambda - \mu| \rangle\, dx. \tag{2.12}$$

Then A is a non-negative locally integrable function on $\mathbb{R}_+$. Let $\tau > 0$ be a fixed Lebesgue point of A and define $\phi \colon \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\phi(t) = \Big(\frac{t}{\varepsilon} - 1\Big)\chi_{(\varepsilon, 2\varepsilon)}(t) + \chi_{(2\varepsilon, \tau - \varepsilon)}(t) + \Big(-\frac{t}{2\varepsilon} + \frac{\tau + \varepsilon}{2\varepsilon}\Big)\chi_{(\tau - \varepsilon, \tau + \varepsilon)}(t).$$

We denote by ϕ^δ, $\delta < \varepsilon$, the mollification of ϕ by a smooth kernel. Likewise m_1^δ and $\overline{m}_1^\delta$ denote mollifications of measures m_1 and $\overline{m}_1$ with respect to the time variable. Notice that $\partial_t \phi$ is supported only in the intervals $(\varepsilon, 2\varepsilon)$ and $(\tau - \varepsilon, \tau + \varepsilon)$, where it is equal to $1/\varepsilon$ and $-1/(2\varepsilon)$, respectively. Hence using the following inequalities

$$m_1(\mathbb{T}^d \times (a,b)) \leq \liminf_{\delta \to 0^+} \int_{\mathbb{R}_+} \Phi\; m_1^\delta(dx\, dt) \leq \limsup_{\delta \to 0^+} \int_{\mathbb{R}_+} \Phi\; m_1^\delta(dx\, dt) \leq m_1(\mathbb{T}^d \times [a,b]),$$

where $\Phi = \chi_{(a,b)}$ almost everywhere with respect to the Lebesgue measure, we obtain using also (2.11)

$$\begin{aligned} &\frac{1}{\varepsilon}\int_{\varepsilon}^{2\varepsilon} A(t)\, dt + \frac{1}{\varepsilon}\int_{\mathbb{T}^d \times [\varepsilon, 2\varepsilon]} (m_1(dx\, dt) + \overline{m}_1(dx\, dt)) \\ &\quad - \frac{1}{2\varepsilon}\int_{\tau-\varepsilon}^{\tau+\varepsilon} A(t)\, dt - \frac{1}{2\varepsilon}\int_{\mathbb{T}^d \times (\tau - \varepsilon, \tau + \varepsilon)} (m_1(dx\, dt) + \overline{m}_1(dx\, dt)) \\ &\geq \lim_{\delta \to 0} \int_{\mathbb{T}^d \times \mathbb{R}_+} A(t) \partial_t \phi^\delta\, dt + \int_{\mathbb{T}^d \times \mathbb{R}_+} \partial_t \phi^\delta (m_1(dx\, dt) + \overline{m}_1(dx\, dt)) \geq 0. \end{aligned} \tag{2.13}$$

This implies that

$$\begin{aligned} &\frac{1}{2\varepsilon}\int_{\tau-\varepsilon}^{\tau+\varepsilon} A(t)\, dt + \frac{1}{2\varepsilon}\int_{\mathbb{T}^d \times (\tau - \varepsilon, \tau + \varepsilon)} (m_1(dx\, dt) + \overline{m}_1(dx\, dt)) \\ &\leq \frac{1}{\varepsilon}\int_{\varepsilon}^{2\varepsilon} A(t)\, dt + \frac{1}{\varepsilon}\int_{\mathbb{T}^d \times [\varepsilon, 2\varepsilon]} (m_1(dx\, dt) + \overline{m}_1(dx\, dt)) \end{aligned} \tag{2.14}$$

$$\leq \frac{1}{\varepsilon}\int_0^{2\varepsilon} A(t)\,dt + \frac{1}{\varepsilon}\int_{\mathbb{T}^d\times[0,2\varepsilon]} (m_1(dx\,dt) + \overline{m}_1(dx\,dt)).$$

Furthermore,

$$\begin{aligned} A(t) &\leq \int_{\mathbb{T}^d} \langle \nu_{(x,t)} \otimes \sigma_{(x,t)}, |\lambda - u_0(x)|\rangle\,dx + \int_{\mathbb{T}^d} \langle \nu_{(x,t)} \otimes \sigma_{(x,t)}, |\mu - u_0(x)|\rangle\,dx \\ &= \int_{\mathbb{T}^d} \langle \nu_{(x,t)}, |\lambda - u_0(x)|\rangle\,dx + \int_{\mathbb{T}^d} \langle \sigma_{(x,t)}, |\mu - u_0(x)|\rangle\,dx. \end{aligned}$$

Consequently, by the initial condition (2.5), we have

$$\lim_{T\to 0^+} \bigg\{ \frac{1}{T}\int_0^T A(t)\,dt + \frac{1}{T}\int_{\mathbb{T}^d\times[0,T]} m_1(dx\,dt) + \frac{1}{T}\int_{\mathbb{T}^d\times[0,T]} \overline{m}_1(dx\,dt)\bigg\} = 0.$$

It follows that the right hand side of (2.14) converges to zero as $\varepsilon \to 0$. Therefore, since A and the measures m_1, $\overline{m}_1$ are non-negative, we see that

$$\begin{gathered} \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon}\int_{\tau-\varepsilon}^{\tau+\varepsilon} A(t)\,dt = 0, \\ \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon}\int_{\mathbb{T}^d\times(\tau-\varepsilon,\tau+\varepsilon)} (m_1(dx\,dt) + \overline{m}_1(dx\,dt)) = 0. \end{gathered} \tag{2.15}$$

Hence, by Lebesgue differentiation theorem, it follows that $A(\tau) = 0$ for almost all $\tau \in \mathbb{R}_+$. This implies that

$$\int_{\mathbb{R}\times\mathbb{R}} |\lambda - \mu|\,d\nu_{(x,t)}(\lambda)\,d\sigma_{(x,t)}(\mu) = 0. \tag{2.16}$$

It can be easily seen from (2.16) that the measures $\nu_{(x,t)}$ and $\sigma_{(x,t)}$ have a common support consisting of a point $w(x,t)$ for almost every $(x,t) \in \mathbb{T}^d \times \mathbb{R}_+$. Further, the second limit in (2.15) implies that $m_1 = \overline{m}_1 = 0$. To see this consider an arbitrary time interval $[a,b] \subset \mathbb{R}_+$. It can be covered with a finite number of overlapping open intervals of radius ε, denoted $B(t_i,\varepsilon)$, so that

$$\sum_{i\in J} \mathcal{L}^1(B(t_i,\varepsilon)) \leq 2(b-a),$$

where $\mathcal{L}^1$ denotes the 1-dimensional Lebesgue measure. From(2.15) we have that

$$m_1(\mathbb{T}^d \times B(\tau,\varepsilon)) \leq C(\varepsilon)\mathcal{L}^{d+1}(\mathbb{T}^d \times B(\tau,\varepsilon))$$

for any ball, where $C(\varepsilon) \to 0$ as $\varepsilon \to 0$. Hence

$$m_1(\mathbb{T}^d \times [a,b]) \leq C(\varepsilon)\sum_{i\in J} \mathcal{L}^{d+1}(\mathbb{T}^d \times B(t_i,\varepsilon)) \leq 2C(\varepsilon)\mathcal{L}^d(\mathbb{T}^d)(b-a) \xrightarrow{\varepsilon\to 0} 0.$$

Hence $m_1(\mathbb{T}^d \times [a,b]) = 0$ for any $a < b$. Similarly $\overline{m}_1 = 0$. This in turn implies $m_2 = \overline{m}_2 = 0$ as a consequence of (2.3).

Finally, the claimed regularity of w follows from the convergence

$$\int_{\mathbb{T}^d\times\mathbb{R}_+} g(w_k(x,t))\cdot\phi(x,t)\,dx\,dt \xrightarrow{k\to\infty} \int_{\mathbb{T}^d\times\mathbb{R}_+} \underbrace{\langle \nu_{(x,t)}, g\rangle}_{g(w(x,t))} \phi(x,t)\,dx\,dt$$

and the boundedness of the sequence u_k. $\square$

REMARK 2.3. One can also pass to the limit in the divergence term of (2.9) to obtain the following averaged contraction principle

$$\begin{aligned}\partial_t(\langle \nu_{(x,t)} \otimes \sigma_{(x,t)}, |\lambda - \mu|\rangle + m_1 + \overline{m}_1) \\ + \operatorname{div}_x(\langle \nu_{(x,t)} \otimes \sigma_{(x,t)}, q(\lambda, \mu)\rangle + m_2 + \overline{m}_2) \leq 0.\end{aligned} \tag{2.17}$$

The theorem is extendible to the problem defined on $\mathbb{R}^d \times \mathbb{R}_+$ rather than $\mathbb{T}^d \times \mathbb{R}_+$. An approximation argument is needed in that case since the constant unit function no longer belongs to the class $\mathrm{C}_c^\infty(\mathbb{R}^d)$.

We have thus established uniqueness in the class of mv entropy solutions. We conclude this section with proving existence of a unique entropy solution. First we observe that recession functions for $\eta = |\lambda - \mu|$ and $q = \operatorname{sgn}(\lambda - \mu)(f(\lambda) - f(\mu))$ can be easily described as follows:

$$\eta^\infty = \lim_{\lambda \to \pm\infty} \frac{|\lambda v - k|}{|\lambda|} = 1 \tag{2.18}$$

where $v \in \mathbb{S}^0$, and

$$q^\infty(v) = \begin{cases} f^\infty(1) & \text{for } v = 1, \\ -f^\infty(-1) & \text{for } v = -1, \end{cases} \tag{2.19}$$

where we define

$$f^\infty(1) := \lim_{\lambda \to +\infty} \frac{f(\lambda)}{\lambda} \quad \text{and} \quad f^\infty(-1) := \lim_{\lambda \to -\infty} \frac{f(-\lambda)}{\lambda}. \tag{2.20}$$

Notice that in fact the assumption on existence of the limits in (2.20) together with continuity of the flux implies condition (2.2).

THEOREM 2.4. *There exists a unique entropy solution w to* (2.1) *such that*

$$\|w(\,\cdot\,, t)\|_{L^1(\mathbb{T}^d)} \leq \|u_0\|_{L^1(\mathbb{T}^d)} \tag{2.21}$$

for almost every $t \in \mathbb{R}_+$.

PROOF. We adapt the proof presented in [41]. First we prove existence of a mv entropy solution (ν, m_1, m_2) — to this end we consider the following parabolic regularization

$$\begin{aligned} \partial_t w_n + \operatorname{div}_x f_n(w_n) &= \frac{1}{n}\Delta w_n \quad \text{on } \mathbb{T}^d \times \mathbb{R}_+, \\ w_n(x, 0) &= u_0^n(x) \quad \text{on } \mathbb{T}^d, \end{aligned} \tag{2.22}$$

where $u_0^n \in \mathrm{C}_c^\infty(\mathbb{T}^d)$ and $u_0^n \to u_0$ in $L^1(\mathbb{T}^d)$ and $f_n := f * \eta^{\varepsilon_n}$. The sequence ε_n is chosen so that, for $|z| \leq \|u_0\|_{L^\infty}$, we have

$$\sup_{|h| \leq \varepsilon_n} |f(z + h) - f(z)| \leq \frac{1}{n}. \tag{2.23}$$

This is possible because of the assumptions on asymptotic behaviour (2.20) of the flux. Problem (2.22) has a unique smooth solution w_n satisfying the bound $\|w_n(\,\cdot\,,t)\|_{L^1(\mathbb{T}^d)} \leq \|u_0\|_{L^1(\mathbb{T}^d)}$ for almost every t. Since the sequence $\{w_n\}$ is bounded in $L^\infty(\mathbb{R}_+;L^1(\mathbb{T}^d))$, there is, after passing to a subsequence, a triple (ν, m_1, m_2) of associated Young measure and concentration measures. We then have by (2.23)

$$
\begin{aligned}
&\lim_{n\to\infty}\int_{\mathbb{T}^d\times\mathbb{R}_+} \operatorname{sgn}(w_n-k)(f_n(w_n)-f_n(k))\nabla\phi\,dx\,dt\\
&= \lim_{n\to\infty}\int_{\mathbb{T}^d\times\mathbb{R}_+} \operatorname{sgn}(w_n-k)\\
&\qquad\qquad\cdot[(f(w_n)-f(k))+(f_n(w_n)-f(w_n))+(f_n(k)-f(k))]\nabla\phi\,dx\,dt\\
&= \lim_{n\to\infty}\int_{\mathbb{T}^d\times\mathbb{R}_+} \operatorname{sgn}(w_n-k)(f(w_n)-f(k))\nabla\phi\,dx\,dt\\
&= \int_{\mathbb{T}^d\times\mathbb{R}_+} (\langle\nu_{(x,t)}, \operatorname{sgn}(\lambda-k)(f(\lambda)-f(k))\rangle + m_2)\nabla\phi\,dx\,dt,
\end{aligned}
$$

where the last equality follows from Lemma 1.3. Moreover, in virtue of the same lemma, we have

$$
\lim_{n\to\infty}\int_{\mathbb{T}^d\times\mathbb{R}_+} |w_n-k|\partial_t\phi\,dx\,dt = \int_{\mathbb{T}^d\times\mathbb{R}_+} (\langle\nu_{(x,t)}, |\lambda-k|\rangle + m_1)\partial_t\phi\,dx\,dt.
$$

We observe that from Lemma 1.3 and (2.18)–(2.20), the following characterization of the concentration measures holds true

$$
m_2 = \bigg(\int_{\mathbb{S}^0} q^\infty(v)\,d\nu^\infty_{(x,t)}\bigg) m_1.
$$

Clearly this implies the bound (2.3). We remark that the measures m_1 and m_2 are common for each choice of entropy-entropy flux pair (i.e. for each choice of $k\in\mathbb{R}$), because the associated recession functions do not depend on k.

Let now $\operatorname{sgn}_\delta$, θ_δ and q_δ be regularizations of the functions $z\mapsto\operatorname{sgn}(z)$, $|z-k|$ and $q(z,k)$ respectively, for $k\in\mathbb{R}$. Multiplying (2.22) by $\phi\operatorname{sgn}_\delta(w_n-k)$, integrating in time and space and integrating by parts yields

$$
\begin{aligned}
\text{(2.24)}\qquad &\int_{\mathbb{T}^d\times\mathbb{R}_+} (\theta_\delta(w_n,k)\partial_t\phi + q_\delta(w_n,k)\nabla\phi)\,dx\,dt\\
&= \frac{1}{n}\int_{\mathbb{T}^d\times\mathbb{R}_+} |\nabla w_n|^2\operatorname{sgn}'_\delta(w_n-k)\phi\,dx\,dt - \frac{1}{n}\int_{\mathbb{T}^d\times\mathbb{R}_+} \theta_\delta(w_n,k)\Delta\phi\,dx\,dt.
\end{aligned}
$$

The right hand side of the last identity can be bounded from below by

$$
-\frac{C(\phi)}{n}(\|w_n\|_{L^\infty(R_+;L^1(\mathbb{T}^d))}+1). \tag{2.25}
$$

Combining (2.24) and (2.25) and passing with δ to zero we see that

$$\int_{\mathbb{T}^d\times\mathbb{R}_+} |w_n - k|\partial_t\phi + \operatorname{sgn}(w_n - k)(f_n(w_n) - f_n(k))\nabla\phi\,dx\,dt \geq -\frac{C}{n}.$$

Then, passing to infinity with n, we get

$$\partial_t(\langle\nu_{(x,t)}, |\lambda - k|\rangle + m_1) + \operatorname{div}_x(\langle\nu_{(x,t)}, q(\lambda, k) + m_2) \leq 0.$$

Therefore the generalized Young measure generated by the sequence $\{w_n\}$ satisfies the entropy inequlity (2.4). Standard methods of analysis for PDEs can be employed to show that the initial condition (2.5) is satisfied as well. Therefore (ν, m_1, m_2) is a mv entropy solution of (2.1). By the previous theorem there is a function w such that $\nu_{(x,t)} = \delta_{\{w(x,t)\}}$. This function is then the unique entropy solution. □

Remark 2.5. Other approximation schemes can be used rather than the viscosity approximation employed here. Indeed from the point of view of numerics other schemes are favourable. In [31], [38], [39] and [6] a kinetic approximation (i.e. an approximation by a Boltzmann type equation) is used, while in [8] a hyperbolic conservation law is realized as a limit of the attracitve zero range process (ZRP). In the latter paper a discontinuous flux is considered. An extension of the averaged contraction principle to the case of discontinuous flux (both in x and u) was considered in a series of papers [5]–[7], [27].

3. Relative entropy method for equations of fluid dynamics

In this section we survey selected weak-strong uniqueness results for equations of fluid dynamics. The global existence of measure-valued solutions to the incompressible Euler system was proven in [16] for any finite-energy initial data. Later, the existence was also shown for the compressible Euler and Navier–Stokes systems, cf. [36], [29]. The first weak-strong uniqueness result was proven for incompressible Euler equation in [3]. Note however that the existence of a strong solution is needed — otherwise uniqueness for admissible solutions might not hold, cf. [12] and [40].

3.1. Euler equations. First we consider the *incompressible Euler equations*

$$\begin{aligned} \partial_t u + \operatorname{div}(u\otimes u) + \nabla p &= 0,\\ \operatorname{div} u &= 0, \end{aligned} \tag{3.1}$$

where $u\colon \mathbb{T}^d\times\mathbb{R}_+\to\mathbb{R}^d$ is the velocity of a fluid and p is the scalar pressure.

Definition 3.1. Let ν be a Young measure, m a matrix-valued measure on $\mathbb{T}^d\times[0,T]$ satisfying $m(dx\,dt) = m_t(dx)\otimes dt$ for some family $\{m_t\}_{t\in(0,T)}$ of uniformly bounded measures on $\mathbb{R}^d$. Further, let $D\in L^\infty(0,T)$ with $D\geq 0$ such that $|m_t|(\mathbb{T}^d) \leq CD(t)$ for some constant $C > 0$ and almost every $t \in [0,T]$. The triple (ν, m, D) is called a *dissipative measure-valued solution* to (3.1) with initial datum u_0 if it satisfies the following conditions:

(a) for any divergence-free $\phi \in \mathrm{C}^1(\mathbb{T}^d \times [0,T]; \mathbb{R}^d)$ the equation

$$\int_0^T \int_{\mathbb{T}^d} \partial_t \phi(x,t) \cdot \langle \nu_{(x,t)}, \mathrm{id} \rangle + \nabla \phi(x,t) : (\langle \nu_{(x,t)}, \mathrm{id} \otimes \mathrm{id} \rangle + m_t)\, dx\, dt = \int_{\mathbb{T}^d} \langle \nu_{(x,\tau)}, \mathrm{id} \rangle \cdot \phi(x,\tau) - u_0(x) \cdot \phi(x,0)\, dx$$

holds for almost every $\tau \in (0,T)$;

(b) the divergence free condition

$$\int_{\mathbb{T}^d} \langle \nu_{(x,t)}, \mathrm{id} \rangle \cdot \nabla \psi(x)\, dx = 0$$

holds for every $\psi \in \mathrm{C}^1(\mathbb{T}^d)$ and almost every $t \in (0,T)$;

(c) the admissibility condition

$$E(\tau) \leq \frac{1}{2} \int_{\mathbb{T}^d} |u_0(x,\tau)|^2\, dx$$

is satisfied for almost every $\tau \in (0,T)$, where the measure-valued energy is defined by

$$E(\tau) := \frac{1}{2} \int_{\mathbb{T}^d} \langle \nu_{(x,\tau)}, |u|^2 \rangle\, dx + D(\tau).$$

We will now show how measuring the relative entropy between a dissipative mvs and a strong solution leads to a uniqueness result. We repeat the proof presented in [43].

THEOREM 3.2. *Let (ν, m, D) be a dissipative measure-valued solution and $U \in \mathrm{C}^1(\mathbb{T}^d \times [0,T])$ a strong solution to (3.1) with the same initial datum u_0. Then $\nu_{(x,\tau)} = \delta_{\{U(x,\tau)\}}$ for almost every $(x,\tau) \in \mathbb{T}^d \times (0,T)$, $m = 0$ and $D = 0$.*

PROOF. We begin by defining the relative entropy E_{rel} as

$$E_{\mathrm{rel}}(\tau) := \frac{1}{2} \int_{\mathbb{T}^d} \langle \nu_{(x,\tau)}, |\mathrm{id} - U(x,\tau)|^2 \rangle\, dx + D(\tau).$$

This quantity can be estimated as follows.

$$\begin{aligned}
E_{\mathrm{rel}}(\tau) &= \frac{1}{2} \int_{\mathbb{T}^d} |U(x,\tau)|^2\, dx + \frac{1}{2} \int_{\mathbb{T}^d} \langle \nu_{(x,\tau)}, |\mathrm{id}|^2 \rangle\, dx \\
&\quad - \int_{\mathbb{T}^d} \langle \nu_{(x,\tau)}, \mathrm{id} \rangle \cdot U(x,\tau)\, dx + D(\tau) \\
&\leq \frac{1}{2} \int_{\mathbb{T}^d} |u_0|^2\, dx + \frac{1}{2} \int_{\mathbb{T}^d} |u_0|^2\, dx - \int_{\mathbb{T}^d} \langle \nu_{(x,\tau)}, \mathrm{id} \rangle \cdot U(x,\tau)\, dx \\
&= \int_{\mathbb{T}^d} |u_0|^2\, dx - \int_{\mathbb{T}^d} u_0(x) \cdot u_0(x)\, dx
\end{aligned}$$

$$
\begin{aligned}
&-\int_0^\tau \int_{\mathbb{T}^d} \partial_t U(x,t)\cdot\langle \nu_{(x,t)}, \mathrm{id}\rangle + \nabla U(x,t) : \langle \nu_{(x,t)}, \mathrm{id}\otimes\mathrm{id}\rangle \,dx\,dt\\
&-\int_0^\tau \int_{\mathbb{T}^d} \nabla U(x,t)\,dm(x,t)\\
&=\int_0^\tau \int_{\mathbb{T}^d} \operatorname{div}(U\otimes U)(x,t)\cdot\langle \nu_{(x,t)}, \mathrm{id}\rangle - \nabla U(x,t)\langle \nu_{(x,t)}, \mathrm{id}\otimes\mathrm{id}\rangle \,dx\,dt\\
&-\int_0^\tau \int_{\mathbb{T}^d} \nabla_{\mathrm{sym}} U(x,t)\,dm(x,t)\\
&=\int_0^\tau \int_{\mathbb{T}^d} \langle \nu_{(x,t)}, (U(x,t)-\mathrm{id})\cdot\nabla_{\mathrm{sym}} U(x,t)(\mathrm{id}-U(x,t))\rangle\,dx\,dt\\
&-\int_0^\tau \int_{\mathbb{T}^d} \nabla_{\mathrm{sym}} U(x,t)\,dm(x,t)\\
&\le \int_0^\tau \|\nabla_{\mathrm{sym}} U(t)\|_{L^\infty} E_{\mathrm{rel}}(t)\,dt.
\end{aligned}
$$

It now follows from Gronwall's inequality that the relative entropy is zero almost everywhere. This in turn implies that $\nu_{(x,\tau)} = \delta_{\{U(x,\tau)\}}$ and $D(\tau)=0$. □

Now consider the *isentropic compressible Euler system*

$$
\begin{aligned}
\partial_t \rho + \operatorname{div}(\rho u) &= 0,\\
\partial_t(\rho u) + \operatorname{div}(\rho u\otimes u) + \nabla(\rho^\gamma) &= 0,
\end{aligned}
\tag{3.2}
$$

where $\gamma > 1$ is the adiabatic coefficient. The definition of a measure-valued solution to the above system requires a slight refinement of the Alibert–Bouchitté framework, which we ignore here; see Section 3 in [26] for details. The following notation is used for brevity

$$
\overline{f}(\,dx\,dt) = \langle \nu_{(x,t)}, f\rangle\,dx\,dt + \langle \nu^\infty_{(x,t)}, f^\infty\rangle\, m(\,dx\,dt).
$$

Definition 3.3. The triple (ν, m, ν^∞) is called a *measure-valued solution* of (3.2) with initial data (ρ_0, u_0) such that ρ_0 and $\rho_0 u_0$ are integrable if for every $\tau\in[0,T]$, $\psi\in \mathrm{C}^1(\mathbb{T}^d\times[0,T])$ and $\phi\in\mathrm{C}^1(\mathbb{T}^d\times[0,T];\mathbb{R}^n)$

$$
\int_0^T\int_{\mathbb{T}^d} \partial_t\psi\overline{\rho} + \nabla\psi\cdot\overline{\rho u}\,dx\,dt + \int_{\mathbb{T}^d}\psi(x,0)\rho_0 - \psi(x,T)\overline{\rho}(x,T)\,dx = 0
$$

and

$$
\begin{aligned}
\int_0^T\int_{\mathbb{T}^d} &\partial_t\phi\cdot\overline{\rho u} + \nabla\phi : \overline{\rho u\otimes u} + \operatorname{div}\phi\overline{\rho^\gamma}\,dx\,dt\\
&+\int_{\mathbb{T}^d}\phi(x,0)\cdot\rho_0 u_0 - \phi(x,T)\cdot\overline{\rho u}(x,T)\,dx = 0.
\end{aligned}
$$

We then define the entropy of such a measure-valued solution

$$
E_{\mathrm{mvs}}(t) := \int_{\mathbb{T}^d} \frac{1}{2}\overline{\rho|u|^2}(x,t) + \frac{1}{\gamma-1}\overline{\rho^\gamma}(x,t)\,dx
$$

and

$$E_0 := \int_{\mathbb{T}^d} \frac{1}{2}\,\rho_0|u_0|^2(x) + \frac{1}{\gamma-1}\,\rho_0^\gamma(x)\,dx.$$

An admissibility criterion is then posed as $E_{\mathrm{mvs}} \le E_0$. It was shown in [26] that one can use the relative entropy method to prove the following weak-strong uniqueness result.

THEOREM 3.4. *Suppose* $(P,U) \in W^{1,\infty}(\mathbb{T}^d \times [0,T]) \times C^1(\mathbb{T}^d \times [0,T])$ *is a strong solution of* (3.2) *with initial data* (ρ_0, u_0) *such that*

$$\rho_0 \ge c > 0, \qquad \rho_0 \in L^r(\mathbb{T}^d), \qquad \rho_0 u_0 \in L^1(\mathbb{T}^d) \quad \textit{and} \quad P \ge c > 0.$$

Then, if (ν, m, ν^∞) *is an admissible mv solution with the same initial data, then*

$$\nu_{(x,t)} = \delta_{\left(P(x,t),\sqrt{P(x,t)}U(x,t)\right)} \quad \textit{for a.e. } (x,t)$$

and $m = 0$.

The relative entropy functional used in this case has the form

$$E_{\mathrm{rel}}(t) := \frac{1}{2}\int_{\mathbb{T}^d} \overline{\rho|u-U|^2} + \frac{1}{\gamma-1}\,\overline{\rho^\gamma} - \frac{\gamma}{\gamma-1}\,\overline{P^{\gamma-1}\rho} + \overline{P^\gamma}\,dx.$$

3.2. Navier–Stokes equations. We now consider the barotropic Navier–Stokes system

$$\partial_t \varrho + \operatorname{div}_x(\varrho u) = 0, \tag{3.3}$$

$$\partial_t(\varrho u) + \operatorname{div}_x(\varrho u \otimes u) + \nabla_x p(\varrho) = \operatorname{div}_x \mathbb{S}(\nabla_x u), \tag{3.4}$$

$$u|_{\partial\Omega} = 0. \tag{3.5}$$

Here Ω is a regular bounded domain in $\mathbb{R}^2$ or $\mathbb{R}^3$ and $\mathbb{S}$ is the Newtonian viscous stress. The following definition of a dissipative measure-valued solution and subsequent results are taken from [18].

DEFINITION 3.5. We say that a parameterized measure $\{\nu_{(x,t)}\}_{(x,t)\in\Omega\times(0,T)}$,

$$\nu \in L^\infty_{\mathrm{weak}}(\Omega \times (0,T); \mathcal{P}(\mathbb{R}^N \times [0,\infty))), \quad \langle \nu_{(x,t)}; s\rangle \equiv \varrho, \quad \langle \nu_{(x,t)}; \mathbf{v}\rangle \equiv u$$

is a dissipative measure-valued solution of the Navier–Stokes system (3.3)–(3.5) in $\Omega \times (0,T)$, with the initial conditions $\nu_{(x,0)}$ and dissipation defect $\mathcal{D} \in L^\infty(0,T)$, $\mathcal{D} \ge 0$, if the following holds:

(a) (Equation of continuity) There exists a measure $r^C \in L^1(0,T;\mathcal{M}(\overline{\Omega}))$ and $\chi \in L^1(0,T)$ such that for almost all $\tau \in (0,T)$ and every $\psi \in C^1(\overline{\Omega} \times [0,T])$,

$$|\langle r^C(\tau); \nabla_x \psi\rangle| \le \chi(\tau)\mathcal{D}(\tau)\|\psi\|_{C^1(\overline{\Omega})}$$

and

$$\int_\Omega \langle \nu_{(x,\tau)}; s\rangle \psi(\,\cdot\,,\tau)\,dx - \int_\Omega \langle \nu_{(x,0)}; s\rangle \psi(0,\,\cdot\,)\,dx = \int_0^\tau \int_\Omega \big[\langle \nu_{(x,t)}; s\rangle \partial_t \psi + \langle \nu_{(x,t)}; s\mathbf{v}\rangle \cdot \nabla_x \psi\big]\,dx\,dt + \int_0^\tau \langle r^C; \nabla_x \psi\rangle\,dt. \tag{3.6}$$

(b) (Momentum equation)

$$u = \langle \nu_{(x,t)}; \mathbf{v}\rangle \in L^2(0,T; W_0^{1,2}(\Omega;\mathbb{R}^N)),$$

and there exists a measure $r^M \in L^1(0,T;\mathcal{M}(\overline{\Omega}))$ and $\xi \in L^1(0,T)$ such that for almost all $\tau \in (0,T)$ and every $\varphi \in \mathrm{C}^1(\overline{\Omega}\times[0,T];\mathbb{R}^N)$, $\varphi|_{\partial\Omega} = 0$,

$$\big|\langle r^M(\tau); \nabla_x \varphi\rangle\big| \le \xi(\tau)\mathcal{D}(\tau)\|\varphi\|_{C^1(\overline{\Omega})}$$

and

$$\begin{aligned}\int_\Omega &\langle \nu_{(x,\tau)}; s\mathbf{v}\rangle \cdot \varphi(\,\cdot\,,\tau)\,dx - \int_\Omega \langle \nu_{(x,0)}; s\mathbf{v}\rangle \cdot \varphi(0,\,\cdot\,)\,dx \\ &= \int_0^\tau \int_\Omega \big[\langle \nu_{(x,t)}; s\mathbf{v}\rangle \cdot \partial_t \varphi \\ &\qquad + \langle \nu_{(x,t)}; s(\mathbf{v}\otimes\mathbf{v})\rangle : \nabla_x \varphi + \langle \nu_{(x,t)}; p(s)\rangle \mathrm{div}_x \varphi\big]\,dx\,dt \\ &\quad - \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x u) : \nabla_x \varphi\,dx\,dt + \int_0^\tau \big\langle r^M; \nabla_x \varphi\big\rangle\,dt.\end{aligned} \tag{3.7}$$

(c) (Energy inequality)

$$\int_\Omega \left\langle \nu_{(x,\tau)}; \left(\frac{1}{2}s|\mathbf{v}|^2 + P(s)\right)\right\rangle dx + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x u) : \nabla_x u\,dx\,dt + \mathcal{D}(\tau) \le \int_\Omega \left\langle \nu_{(x,0)}; \left(\frac{1}{2}s|\mathbf{v}|^2 + P(s)\right)\right\rangle dx \tag{3.8}$$

for almost all $\tau \in (0,T)$. In addition, the following version of "Poincaré's inequality" holds for almost all $\tau \in (0,T)$:

$$\int_0^\tau \int_\Omega \big\langle \nu_{(x,t)}; |\mathbf{v} - u|^2\big\rangle\,dx\,dt \le c_P \mathcal{D}(\tau). \tag{3.9}$$

One can show (Theorem 2.1 in [18]) that if the pressure satisfies the following coercivity assumptions

$$\begin{gathered} p \in \mathrm{C}[0,\infty) \cap C^2(0,\infty), \qquad p(0) = 0, \qquad p'(\varrho) > 0 \quad \text{for } \varrho > 0, \\ \liminf_{\varrho\to\infty} p'(\varrho) > 0, \qquad \liminf_{\varrho\to\infty} \frac{P(\varrho)}{p(\varrho)} > 0, \end{gathered} \tag{3.10}$$

then there exists a dissipative mv solution with a prescribed finite-energy initial data. The following weak-strong uniqueness result can then be proven.

Theorem 3.6. *Let $\Omega \subset \mathbb{R}^N$, $N = 2,3$ be a bounded smooth domain. Suppose the pressure p satisfies* (3.10). *Let $\{\nu_{(x,t)}, \mathcal{D}\}$ be a dissipative measure-valued solution to the barotropic Navier–Stokes system* (3.3)–(3.5) *in $\Omega \times (0,T)$, with the initial state represented by $\nu_{(x,0)}$, in the sense specified in Definition* 3.5. *Let $[r, \mathbf{U}]$ be a strong solution of* (3.3)–(3.5) *in $\Omega \times (0,T)$ belonging to the class r, $\nabla_x r$, $\mathbf{U}$, $\nabla_x \mathbf{U} \in \mathrm{C}(\overline{\Omega} \times [0,T])$, $\partial_t \mathbf{U} \in L^2(0,T; C(\overline{\Omega}; \mathbb{R}^N))$, $r > 0$, $\mathbf{U}|_{\partial\Omega} = 0$. Then there is a constant $\Lambda = \Lambda(T)$, depending only on the norms of r, r^{-1}, $\mathbf{U}$, χ and ξ in the aforementioned spaces, such that*

$$\begin{aligned}\int_\Omega \Big\langle \nu_{(x,\tau)}; \frac{1}{2}\, s|\mathbf{v} - \mathbf{U}|^2 + P(s) - P'(r)(s-r) - P(r) \Big\rangle \, dx \\ + \int_0^\tau \int_\Omega |\nabla_x u - \nabla_x \mathbf{U}|^2 \, dx\, dt + \mathcal{D}(\tau) \\ \leq \Lambda(T) \int_\Omega \Big\langle \nu_{(x,0)}; \frac{1}{2} s|\mathbf{v} - \mathbf{U}(0,\,\cdot\,)|^2 + P(s) \\ - P'(r(0,\,\cdot\,))(s - r(0,\,\cdot\,)) - P(r(0,\,\cdot\,)) \Big\rangle \, dx\end{aligned}$$

for almost all $\tau \in (0,T)$. In particular, if the initial states coincide, meaning

$$\nu_{(x,0)} = \delta_{[r(x,0), \mathbf{U}(x,0)]} \quad \textit{for a.a. } x \in \Omega$$

then $\mathcal{D} = 0$, and

$$\nu_{(x,\tau)} = \delta_{[r(x,\tau), \mathbf{U}(x,\tau)]} \quad \textit{for a.a. } \tau \in (0,T),\ x \in \Omega.$$

4. Polyconvex elastodynamics

In this section we consider the *system of elasticity*

$$\frac{\partial^2 y}{\partial t^2} = \nabla \cdot S(\nabla y), \tag{4.1}$$

where $y\colon Q \times \mathbb{R}^+ \to \mathbb{R}^3$ stands for the motion, $F = \nabla y$, $v = \partial_t y$, and S stands for the Piola–Kirchhoff stress tensor obtained as the gradient of a stored energy function, $S = \partial W / \partial F$. Here we assume that W is polyconvex, that is $W(F) = G(\Phi(F))$ where $G\colon \mathrm{Mat}^{3\times3} \times \mathrm{Mat}^{3\times3} \times \mathbb{R} \to [0,\infty)$ is a strictly convex function and $\Phi(F) = (F, \operatorname{cof} F, \det F) \in \mathrm{Mat}^{3\times3} \times \mathrm{Mat}^{3\times3} \times \mathbb{R}$ stands for the vector of null-Lagrangians: F, the cofactor matrix $\operatorname{cof} F$ and the determinant $\det F$. It is observed in [13] and [14] that this system can be embedded into the following symmetrizable hyperbolic system in a new dependent variable $\Xi = (F, Z, w)$ taking values in $\mathrm{Mat}^{3\times3} \times \mathrm{Mat}^{3\times3} \times \mathbb{R}$

$$\frac{\partial v_i}{\partial t} = \frac{\partial}{\partial x^\alpha}\Big(\frac{\partial G}{\partial \Xi^A}(\Xi)\, \frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) \Big), \qquad \frac{\partial \Xi^A}{\partial t} = \frac{\partial}{\partial x^\alpha}\Big(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) v_i \Big). \tag{4.2}$$

This system admits the following entropy-entropy flux pair

$$\eta(v, F, Z, w) = \frac{1}{2}\, |v|^2 + G(F, Z, w), \qquad q_\alpha = v_i\, \frac{\partial G}{\partial \Xi^A}(\Xi)\, \frac{\partial \Phi^A}{\partial F_{i\alpha}}(F). \tag{4.3}$$

A strong solution to (4.1) is a function $w \in W^{1,\infty}$. It automatically satisfies

$$\partial_t \eta + \partial_\alpha q_\alpha = 0. \tag{4.4}$$

A weak solution which satisfies (4.4) as an inequality is called an entropy weak solution. The following definition of a dissipative mv solution is taken from [13].

Definition 4.1. Let the pair (y, ν) consist of a map y, with distributional time and space derivatives $(v, F) \in L^\infty(L^2) \oplus L^\infty(L^p)$ and a Young measure $\nu = \{\nu_{(x,t)}\}_{(x,t)\in\overline{Q}_T}$ generated by a sequence satisfying

$$\sup_{\varepsilon,t} \int \eta(v^\varepsilon, F^\varepsilon, Z^\varepsilon, w^\varepsilon)\, dx \, < \infty$$

which represents weak limits in the following way:

$$\text{wk-}\lim_{\varepsilon\to 0} f(v^\varepsilon, F^\varepsilon, Z^\varepsilon, w^\varepsilon) = \int f(\lambda_v, \lambda_\Xi)\, d\nu_{(x,t)}(\lambda_v, \lambda_\Xi) \tag{4.5}$$

$$\text{for all continuous } f = f(\lambda_v, \lambda_\Xi) \text{ with } \lim_{|\lambda_v|+|\lambda_\Xi|\to\infty} \frac{f(\lambda_v, \lambda_\Xi)}{|\lambda_v|^2/2 + G(\lambda_\Xi)} = 0$$

where $\lambda_v \in \mathbb{R}^3$, $\lambda_\Xi = (\lambda_F, \lambda_Z, \lambda_w) \in \mathrm{Mat}^{3\times 3} \times \mathrm{Mat}^{3\times 3} \times \mathbb{R} = \mathbb{R}^{19}$. The Young measure is connected with the map y through the requirements that (almost everywhere)

$$F = \langle \nu, \lambda_F \rangle, \qquad v = \langle \nu, \lambda_v \rangle, \qquad \Xi = \langle \nu, \lambda_\Xi \rangle. \tag{4.6}$$

The pair (y, ν) is a *measure-valued solution* to (4.1) if for $i = 1, 2, 3$

$$\partial_t v_i - \partial_\alpha \left\langle \nu, \frac{\partial G}{\partial \Xi^A}(\lambda_\Xi)\, \frac{\partial \Phi^A}{\partial F_{i\alpha}}(\lambda_F) \right\rangle = 0 \tag{4.7}$$

and, for $A = 1, \ldots, 19$,

$$\partial_t \Phi^A(F) - \partial_\alpha \left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) v_i \right) = 0 \tag{4.8}$$

in distributions with

$$\Xi = \Phi(\langle \nu, \lambda_F \rangle) = \Phi(F). \tag{4.9}$$

The solution is said to be a *dissipative measure-valued solution with concentration* if it is a measure-valued solution which verifies in addition:

$$\iint \frac{d\theta}{dt} \left(\langle \nu, \eta \rangle + \gamma \right) dx\, dt + \int \theta(0) \eta_0(x)\, dx \geq 0,$$

for all non-negative functions $\theta = \theta(t) \in \mathrm{C}^1_c([0, T))$ with $\theta \geq 0$. Here η_0 means the entropy η evaluated on the initial data and γ is the non-negative concentration measure.

Under the following additional growth assumptions on the function G:

(H1) $G \in \mathrm{C}^3(\mathrm{Mat}^{3\times 3} \times \mathrm{Mat}^{3\times 3} \times \mathbb{R}; [0, \infty))$ is a strictly convex function satisfying for some $C > 0$ the bound $D^2 G \geq C > 0$,

(H2) $G(F,Z,w) \geq c_1(|F|^p+|Z|^q+|w|^r+1)-c_2$ where $p \in (4,\infty)$, $q,r \in [2,\infty)$,
(H3) $G(F,Z,w) \leq c(|F|^p+|Z|^q+|w|^r+1)$,
(H4)' $|\partial_F G|+|\partial_Z G|^{p/(p-1)}+|\partial_w G|^{p/(p-2)} \leq o(1)(|F|^p+|Z|^q+|w|^r+1)$ where $o(1) \to 0$ as $|\Xi| \to \infty$,

the existence of dissipative mv solutions as well as a weak-strong uniqueness result are proven, cf. [14] and [13].

Theorem 4.2. *Let G satisfy* (H1)–(H3), (H4)' *and let* (y,ν,γ) *be a dissipative measure-valued solution in the sense of Definition* 4.1. *If the initial data equal those of a Lipschitz bounded solution* $(\widehat{v},\widehat{F}) \in W^{1,\infty}(\overline{Q}_T)$:

$$(v(x,0),\Xi(x,0)) = \big(\widehat{v}(x,0),\Phi\big(\widehat{F}(x,0)\big)\big)$$

then γ is zero, $(v,\Xi) = \big(\widehat{v},\Phi\big(\widehat{F}\big)\big)$ *and* $\nu = \delta_{\widehat{v},\Phi(\widehat{F})}$.

5. Weak-strong uniqueness for general hyperbolic conservation laws

In [9] the method of relative entropy is extended to a more general class of problems of hyperbolic and hyperbolic-parabolic type. In this section we describe the results of that paper. Consider the following hyperbolic problem

$$\partial_t A(u) + \partial_\alpha F_\alpha(u) = 0, \tag{5.1}$$

where $u(x,t)$ takes values in $\mathbb{R}^n$, $t \in \mathbb{R}^+$, $x \in \mathbb{R}^d$ and $A, F_\alpha\colon \mathbb{R}^n \to \mathbb{R}^n$ are given smooth functions with $\alpha = 1,\ldots,d$. It is assumed that this system is symmetrizable. The following hypotheses are assumed throughout:

(H_1) $A\colon \mathbb{R}^n \to \mathbb{R}^n$ is a C^2 globally invertible map,
(H_2) existence of an entropy-entropy flux pair (η,q), that is there exists G: $\mathbb{R}^n \to \mathbb{R}^n$, $G = G(u)$ smooth such that

$$\nabla\eta = G\cdot\nabla A, \qquad \nabla q_\alpha = G\cdot\nabla F_\alpha, \quad \alpha = 1,\ldots,d,$$

(H_3) the symmetric matrix $\nabla^2\eta(u) - G(u)\cdot\nabla^2 A(u)$ is positive definite.

Together with the following growth assumptions on the entropy $\eta(u)$, the functions F_α and A

(A_1) $\beta_1(|u|^p+1) - B \leq \eta(u) \leq \beta_2(|u|^p+1)$ for $u \in \mathbb{R}^n$ for some positive constants β_1, β_2, B and for some $p \in (1,\infty)$.
(A_2) $|F_\alpha(u)|/\eta(u) = o(1)$ as $|u| \to \infty$, $\alpha = 1,\ldots,d$.
(A_3) $|A(u)|/\eta(u) = o(1)$ as $|u| \to \infty$.
(A_4) $(|A(u)|^q+1)/C - B \leq \eta(u) \leq C(|A(u)|^q+1)$, $q > 1$, for some uniform constant $C > 0$ and $B > 0$.

Definition 5.1. A dissipative measure-valued solution $(u,\boldsymbol{\nu},\boldsymbol{\gamma})$ with concentration to (5.1) consists of $u \in L^\infty(L^p)$, a Young measure $\boldsymbol{\nu} = (\boldsymbol{\nu}_{(x,t)})_{\{(x,t)\in\overline{Q}_T\}}$

and a non-negative Radon measure $\boldsymbol{\gamma} \in \mathcal{M}^+(Q_T)$ such that $u(x,t) = \langle \boldsymbol{\nu}_{(x,t)}, \lambda \rangle$ and

$$\iint \langle \boldsymbol{\nu}_{(x,t)}, A_i(\lambda)\rangle \partial_t \varphi_i \, dx\, dt + \iint \langle \boldsymbol{\nu}_{(x,t)}, F_{i,\alpha}(\lambda)\rangle \partial_\alpha \varphi_i \, dx\, dt + \int \langle \boldsymbol{\nu}_{(x,0)}, A_i \rangle \varphi_i(x,0)\, dx = 0 \tag{5.2}$$

for $i = 1, \dots, n$ and any $\varphi \in \mathrm{C}^1_c(Q \times [0,T))$, and

$$\iint \frac{d\xi}{dt} \big[\langle \boldsymbol{\nu}_{(x,t)}, \eta(\lambda)\rangle \, dx\, dt + \boldsymbol{\gamma}(dx\, dt)\big] + \int \xi(0) \big[\langle \boldsymbol{\nu}_{(x,0)}, \eta \rangle \, dx + \boldsymbol{\gamma}_0(dx)\big] \geq 0, \tag{5.3}$$

for all $\xi = \xi(t) \in \mathrm{C}^1_c([0,T))$ with $\xi \geq 0$.

The following theorem provides weak measure-valued versus strong uniqueness in the L^p framework for $1 < p < \infty$.

Theorem 5.2. *Suppose that* (H1)–(H3) *hold, the growth properties* (A_1)–(A_4) *are satisfied, and the entropy $\eta(u) \geq 0$. Assume that $(u, \boldsymbol{\nu}, \boldsymbol{\gamma})$ is a dissipative measure-valued solution, $u = \langle \boldsymbol{\nu}_{(x,t)}, \lambda \rangle$, and $\overline{u} \in W^{1,\infty}(\overline{Q_T})$ is a strong solution to* (5.1)*. Then, if the initial data satisfy $\boldsymbol{\gamma}_0 = 0$ and $\boldsymbol{\nu}_{(x,0)} = \delta_{\overline{u}_0}(x)$, then $\boldsymbol{\nu} = \delta_{\overline{u}}$ and $u = \overline{u}$ almost everywhere on Q_T.*

Remark 5.3. The work of Christoforou and Tzavaras [9] is a generalization of the results presented in [3], where the case $A(u) = u$ was considered and concentration effects were ignored.

6. General relative entropy method in mathematical biology

In this section we give a short overview of an extension of relative entropy method to linear PDEs which was introduced in the context of biological systems in [34] and further developed in [32] and [33]. It was extended to measure solutions with measure initial data in [28]. Here the relative entropies are a family of renormalizations to the initial linear problem, obtained by multiplying the original equation by a nonlinear function.

The notation used in this section is intentionally inconsistent with the remainder of this paper — however it is consistent with the notation used in the above mentioned papers on the subject.

Following [37] and [28] we consider the McKendrick–Von Foerster equation

$$\begin{aligned} &\partial_t n(t,x) + \partial_x n(t,x) = 0 \quad \text{on } (\mathbb{R}^+)^2, \\ &n(t, x=0) = \int_0^\infty B(y) n(t,y)\, dy, \\ &n(t=0, x) = n^0(x). \end{aligned} \tag{6.1}$$

Here, $n(t,x)$ denotes the population density at time t with age x, and $B \in L^\infty(\mathbb{R}^+;\mathbb{R}^+)$ is a birth rate such that

$$\int_0^\infty B(x)\,dx > 1.$$

The associated primal and dual eigenvalue problems have the form

$$\tag{6.2} \begin{gathered} \partial_x N(x) + \lambda_0 N(x) = 0, \quad x \geq 0, \\ N(0) = \int_0^\infty B(y)\,N(y)\,dy, \\ N > 0, \qquad \int_0^\infty N(x)\,dx = 1, \end{gathered}$$

and

$$\begin{gathered} -\partial_x \phi(x) + \lambda_0 \phi(x) = \phi(0)B(x), \quad x \geq 0, \\ \phi \geq 0, \quad \int_0^\infty N(x)\phi(x)\,dx = 1, \end{gathered}$$

where $\lambda_0 > 0$. Under the above assumptions both these problems posses a unique solution. In fact, it can be seen that the solution of (6.2) is given by $N(x) = \lambda_0 e^{-\lambda_0 x}$. Since the death rate is ignored and the birth rate integrates to more than one, an exponential growth of the population is expected. In order to quotient out this growth, we set $\widetilde{n}(t,x) = n(t,x)e^{-\lambda_0 t}$. Then (6.1) becomes

$$\tag{6.3} \begin{gathered} \partial_t \widetilde{n}(t,x) + \partial_x \widetilde{n}(t,x) + \lambda_0 \widetilde{n}(t,x) = 0 \quad \text{on } (\mathbb{R}_+)^2, \\ \widetilde{n}(t, x=0) = \int_0^\infty B(y)\widetilde{n}(t,y)\,dy, \\ \widetilde{n}(t=0,x) = n^0(x). \end{gathered}$$

It can be shown that (6.3) has a unique solution in the weak sense for any $n^0 \in \mathcal{M}^+([0,\infty))$, see [25]. The following results on long-time asymptotics of this solution are proven in [28].

Theorem 6.1. *Let $n^0 \in \mathcal{M}^+([0;\infty))$. Then there is $y_0 > 0$, $\sigma > 0$ and a bounded function η, positive on $\operatorname{supp}\phi$, such that the solution of the renewal equation satisfies*

$$\tag{6.4} \begin{aligned} &\int_0^\infty \eta(x)\,d\left|\widetilde{n}(t,x) - m_0 N(x)\,\mathcal{L}^1\right| \\ &\qquad\qquad \leq e^{-\sigma(t-y_0)} \int_0^\infty \eta(x)\,d\left|\widetilde{n}^0(x) - m_0 N(x)\,\mathcal{L}^1\right|, \end{aligned}$$

where

$$m_0 = \int_0^\infty \phi(x)\,dn^0(x)$$

and $\mathcal{L}^1$ is one-dimensionsl Lebesgue measure.

THEOREM 6.2. *Assume in addition there exists $C > 0$ such that $B(x) \geq C\phi(x)$. Let $n^0 \in \mathcal{M}^+([0;\infty))$. Then the solution of the renewal equation satisfies*

$$\lim_{t\to\infty} \int_0^\infty \phi(x)\, d\left|\widetilde{n}(t,x) - m_0 N(x)\,\mathcal{L}^1\right| = 0, \quad \text{where} \quad m_0 = \int_0^\infty \phi(x)\, dn^0(x)$$

and $\mathcal{L}^1$ is one-dimensional Lebesgue measure.

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