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COINCIDENCE DEGREE METHODS IN ALMOST PERIODIC DIFFERENTIAL EQUATIONS

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Abstract. We consider the existence of almost periodic solutions to differential equations by using coincidence degree theory. A new equivalent spectral condition for the compactness of integral operators on almost periodic function spaces is established. It is shown that semigroup conditions are crucial in applications.

1. Introduction

The theory of almost periodic functions was mainly created by the Danish mathematician H. Bohr in 1920s. Almost periodic functions are intended to be a generalization of periodic functions in some sense. It is well known that almost periodic theory is interesting and at the same time difficult. In celestial mechanics, almost periodic solutions and stable solutions are intimately related. In the same way, stable electronic circuits exhibit almost periodic behavior. The methods to study the existence of almost periodic solutions can be found, e.g. in [13], [16], [27], [35]–[37].

The coincidence degree theory was established by Mawhin [25]. This theory, based on the Leray–Schauder degree theory, has a successful application in the study of the existence of periodic solutions and some boundary value problems

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of differential equations $x' = \psi(x,t)$, which is written in an abstract operator form as

$$\mathcal{L}x = \mathcal{N}x,$$

where $\mathcal{L}$ is a Fredholm linear operator of index zero. In the Continuation Theorem (Theorem 2.13), there is a condition: $\mathcal{N}$ is $\mathcal{L}$-compact, which is closely related to the compactness of the integral operator in the applications to differential equations. The $\mathcal{L}$-compactness is usually shown by using the Arzela–Ascoli theorem. The underlying reason is the compactness of the space on which the functions are defined.

A natural generalization of the study of periodic solutions could be application of the degree theory in almost periodic world. Once this is achieved, a new method will be available for almost periodic differential equations. However, this problem is very difficult. The underlying reason for this is the non-compactness of the space on which the functions are defined. There are no general theorems of analysis which yield uniform convergence on $\mathbb{R}$.

As it is commented in [5], the compactness is very difficult to exhibit, because the analog of the Arzela–Ascoli theorem for almost periodic functions, the so-called Lusternik theorem (Theorem 2.3), contains a condition of equi-almost periodicity that is practically unverifiable. In [28] the author provides several good examples to which the degree theory is not applicable. There exist both first and second order differential equations for which the associated operators on almost periodic function spaces have no fixed points but they map the closed unit ball into its interior ([28, Theorems 2.1 and 3.2]). It is also mentioned in [29] that it seems that the standard techniques (variational methods, continuation and degree theory, upper and lower solutions) are not applicable and that new phenomena appear. So, it is of significant interest to work out this problem and show these new phenomena. Indeed, we find that coincidence degree theory is applicable to complex almost periodic differential equations.

To our knowledge, there are a few papers that investigate the existence of almost periodic solutions by using the coincidence degree methods, see e.g. [2], [19]–[24], [32]–[34]. However, there exists a gap in these papers. The authors in these papers assume that the Arzela–Ascoli theorem and the module containment could imply $\mathcal{L}$-compactness, which means that the uniform convergence on any compact subsets of $\mathbb{R}$ could imply uniform convergence on $\mathbb{R}$, but this is not the case as pointed out by Zhou and Shao [38].

In the present paper, we continue to investigate such problems. We find that the mentioned above gap appears because the compactness of integral operators is not discussed in these papers. So, it is of great interest to study the compactness of integral operators and find almost periodic solutions to differential equations by involving coincidence degree theory.