

PERIODIC SOLUTIONS OF vdP AND vdP-LIKE SYSTEMS ON 3-TORI

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ABSTRACT. Van der Pol equation (in short, vdP) as well as many its non-symmetric generalizations (the so-called van der Pol-like oscillators (in short, vdPl)) serve as nodes in coupled networks modeling real-life phenomena. Symmetric properties of periodic regimes of networks of vdP/vdPl depend on symmetries of coupling. In this paper, we consider N^3 identical vdP/vdPl oscillators arranged in a cubical lattice, where opposite faces are identified in the same way as for a 3-torus. Depending on which nodes impact the dynamics of a given node, we distinguish between $\mathbb{D}_N \times \mathbb{D}_N \times \mathbb{D}_N$ -equivariant systems and their $\mathbb{Z}_N \times \mathbb{Z}_N \times \mathbb{Z}_N$ -equivariant counterparts. In both settings, the local equivariant Hopf bifurcation together with the global existence of periodic solutions with prescribed period and symmetry, are studied. The methods used in the paper are based on the results rooted in both equivariant degree theory and (equivariant) singularity theory.

1. Introduction

1.1. Subject and goal. The van der Pol equation (in short, vdP, cf. (2.1)), originally introduced in [16] to study stable oscillations in electrical circuits, is

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very often considered as a starting point of applied nonlinear dynamics. An important feature of equation (2.1) is that it respects the antipodal symmetry. Different generalizations of vdP, which break this symmetry, have been considered by many authors in connection to a wide range of applied problems (in what follows, we will call these generalizations van der Pol-like equations (in short, vdPl)). Examples of particular importance include the FitzHugh–Nagumo model (see, for example, [14]) and the realistic kinetic model of the chlorite-iodide-malonic acid reaction (see, for example, [7]). To be more concrete about the importance of considering vdPl systems with quadratic terms, we refer to [1] and the references therein.

In real life models, vdP as well as vdPl serve as nodes in coupled networks. Symmetries of the coupling have an impact on the symmetries of the actual dynamics. In this paper, we will consider N^3 oscillators arranged in a cubical lattice, where opposite faces are identified in the same way as for a 3-torus. In such a configuration, two aspects of the coupling are important: (i) which nodes impact the dynamics of a given node (which we will call *coupling topology*), and (ii) how a neighbouring node impacts a given node (which we will call *coupling structure*). For the coupling topology, we consider the following two cases: (i) all six neighbors of a given node impact on that node's dynamics (we call such a coupling *bi-directional*); (ii) only three neighbors of a given node impact on that node's dynamics (we call such a coupling *uni-directional*). In the case of bi-directional coupling, the system respects a natural action of $\mathbb{D}_N \times \mathbb{D}_N \times \mathbb{D}_N$, while in the case of uni-directional coupling the symmetry generated by κ is destroyed, hence the total symmetry group is $\mathbb{Z}_N \times \mathbb{Z}_N \times \mathbb{Z}_N$ (cf. [5] and references therein). We will distinguish between two linear coupling structures, namely, for a given node, either the x -variable of a neighbor or the y -variable of a neighbor is coupled to the x variable of the specified node (compare (2.3) with (2.4)). We will call these x -coupling and y -coupling, respectively.

The *goal* of this paper is three-fold, namely, in the settings introduced above, we will: (i) establish the *occurrence* of the Hopf bifurcation, classify symmetric properties of the bifurcating branches and estimate their number; (ii) study *stability* of the corresponding periodic solutions, and (iii) investigate the *existence* of periodic solutions with prescribed period and symmetry.

1.2. Results. Keeping in mind a wide spectrum of potential applications in natural sciences and engineering, it is worthy to study the above mentioned problems (occurrence, stability and existence) in all possible settings. The usual dilemma of keeping a balance between Scylla of completeness and Charybdis of reasonable size of the manuscript, resulted in our paper as follows:

(i) Although, using the methods developed in this paper, the occurrence/multiplicity estimates/symmetry classification of the Hopf bifurcation can be estab-

lished for any combination of bi-directional/uni-directional vdP/vdPl x -coupled/ y -coupled systems, we only treat two cases, namely, that of bi-directionally y -coupled vdP oscillators and uni-directionally x -coupled vdPl oscillators.

(ii) We provide an instability result for bi-directionally y -coupled vdP oscillators and stability results for uni-directionally x -coupled vdPl oscillators.

(iii) We establish the existence of periodic solutions with prescribed period and symmetry only in the case of bi-directionally y -coupled vdP oscillators.

1.3. Methods. The methods used in this paper are based on the results rooted in both equivariant degree theory and (equivariant) singularity theory. To be more specific:

(i) To treat the occurrence/multiplicity estimates/symmetric classification of the Hopf bifurcation, we appeal to the abstract results presented in [4] (see also [3], [13]).

(ii) The stability results are obtained in the framework of the theory presented in [11] (see also [8], [9]).

(iii) The main ingredient to establish the existence of periodic solutions with prescribed period and symmetry is Theorem 6.2 which was presented in [4] and [2] (see also [12]).

For the representation theory background, we refer to [6], [15].

1.4. Overview. After the introduction, the paper is organized as follows. In Section 2, we formulate main results of the paper. Section 4 is devoted to the proof of the main occurrence/multiplicity/symmetry results (Theorem 2.3 and 2.4). The proof (see Subsection 4.2) is based on an abstract Theorem 4.2 and equivariant spectral information collected in Subsections 3.1–3.3. We believe that several algebraic observations related to the computation of maximal twisted orbit types in complexifications of tensor product representations (see Subsection 3.3) may be interesting in their own. Section 5 contains the proof of the instability result for system (2.3) (see Theorem 2.6) and stability result for system (2.4) (see Theorem 2.7). The proof follows the standard lines (see [11], Theorem 3.4.2, and [9]), and combines the spectral equivariant data from Subsections 3.1–3.3 with the computations of the first Lyapunov coefficient from Subsection 3.4. The proof of the existence result (see Theorem 2.8) is given in Section 6, where one can also find an adapted version of the abstract existence result given in [4], Theorem 12.7, and [2] (cf. Theorem 6.2). We conclude with a short Appendix where several symbols frequently used in this paper to denote some twisted groups are explained (cf. [4]).

2. Main results

In this paper, we are interested in networks of identical vdP oscillators

$$(2.1) \quad \begin{aligned} \dot{x} &= \nu(ax - x^3) - y, \\ \dot{y} &= bx, \end{aligned}$$

and vdP-like oscillators

$$(2.2) \quad \begin{aligned} \dot{x} &= -y - x^3 - x^2 + ax, \\ \dot{y} &= bx, \end{aligned}$$

coupled in the symmetric configuration of a three-dimensional torus. To be more precise, we consider N^3 oscillators, where N is an *odd* number, with both bi-directional coupling

$$(2.3) \quad \begin{cases} \dot{x}_{(\alpha,\beta,\gamma)} = \nu(ax_{(\alpha,\beta,\gamma)} - x_{(\alpha,\beta,\gamma)}^3) - y_{(\alpha,\beta,\gamma)} \\ \quad + \delta(2y_{(\alpha,\beta,\gamma)} - y_{(\alpha+1,\beta,\gamma)} - y_{(\alpha-1,\beta,\gamma)}) \\ \quad + \zeta(2y_{(\alpha,\beta,\gamma)} - y_{(\alpha,\beta+1,\gamma)} - y_{(\alpha,\beta-1,\gamma)}) \\ \quad + \varepsilon(2y_{(\alpha,\beta,\gamma)} - y_{(\alpha,\beta,\gamma+1)} - y_{(\alpha,\beta,\gamma-1)}), \\ \dot{y}_{(\alpha,\beta,\gamma)} = bx_{(\alpha,\beta,\gamma)}, \end{cases}$$

and uni-directional coupling

$$(2.4) \quad \begin{cases} \dot{x}_{(\alpha,\beta,\gamma)} = -y_{(\alpha,\beta,\gamma)} - x_{(\alpha,\beta,\gamma)}^3 - x_{(\alpha,\beta,\gamma)}^2 + ax_{(\alpha,\beta,\gamma)} \\ \quad + \delta(x_{(\alpha,\beta,\gamma)} - x_{(\alpha+1,\beta,\gamma)}) + \zeta(x_{(\alpha,\beta,\gamma)} - x_{(\alpha,\beta+1,\gamma)}) \\ \quad + \varepsilon(x_{(\alpha,\beta,\gamma)} - x_{(\alpha,\beta,\gamma+1)}), \\ \dot{y}_{(\alpha,\beta,\gamma)} = bx_{(\alpha,\beta,\gamma)}. \end{cases}$$

Here $x, y \in \mathbb{R}^{N^3}$ and their entries are indexed by the triple (α, β, γ) where $\alpha, \beta, \gamma \in \{1, \dots, N\}$, $\delta, \zeta, \varepsilon \in \mathbb{R}$ and $\nu, b > 0$.

REMARK 2.1. To avoid distinctions occurring due to the parity of N , we will only consider the case when N is odd.

DEFINITION 2.2. We will say that a periodic function $x: \mathbb{R} \rightarrow U$ with period T has a (*spatio-temporal*) *symmetry* $H < G \times S^1$, if for every $(g, e^{i\theta}) \in H$ and

$$g \cdot x(t + \theta T/2\pi) = x(t), \quad \text{for every } t.$$

THEOREM 2.3. For each fixed $\mathbf{t} = (t_1, t_2, t_3)$, where $t_1, t_2, t_3 \in \{1, \dots, n\}$, put

$$(2.5) \quad K_{\mathbf{t}} = 1 + 2\delta(1 - \cos(2\pi t_1/N)) + 2\zeta(1 - \cos(2\pi t_2/N)) + 2\varepsilon(1 - \cos(2\pi t_3/N)).$$

If $K_{\mathbf{t}} > 0$, then system (2.3) undergoes Hopf bifurcation as a passes 0. Furthermore, for each $(H^\varphi) \in \mathcal{S}(\mathbf{t})$, there exist $8N^3/|H_1 \times H_2 \times H_3|$ branches of bifurcating non-constant periodic solutions of (2.3) with limit frequency $\omega_{\mathbf{t}} = \sqrt{bK_{\mathbf{t}}}$

and minimal symmetry (H^φ) (here $\mathcal{S}(\mathbf{t})$ is the set of spatio-temporal symmetries associated to \mathbf{t} which is described in Subsection 3.3, formula (3.11)).

THEOREM 2.4. For each fixed $\mathbf{t} = (t_1, t_2, t_3)$, where $t_1, t_2, t_3 \in \{1, \dots, N\}$, system (2.4) undergoes Hopf bifurcation as a passes

$$(2.6) \quad a_{\mathbf{t}}^z = \delta(1 - \cos(2t_1\pi/N)) + \zeta(1 - \cos(2t_2\pi/N)) + \varepsilon(1 - \cos(2t_3\pi/N)).$$

Furthermore, there exist a branch of bifurcating non-constant periodic solutions of (2.4) with symmetry $(\mathbb{Z}_n \times \mathbb{Z}_n \times \mathbb{Z}_n)^{(t_1, t_2, t_3)}$ and limit period

$$P_{\mathbf{t}}^1 = \left| \frac{4\pi}{H + \sqrt{H^2 + 4b}} \right|,$$

and a branch with the same symmetry and limit period

$$P_{\mathbf{t}}^2 = \left| \frac{4\pi}{H - \sqrt{H^2 + 4b}} \right|,$$

where $H = \delta \sin(2t_1\pi/N) + \zeta \sin(2t_2\pi/N) + \varepsilon \sin(2t_3\pi/N)$ and the symbol $(\mathbb{Z}_n \times \mathbb{Z}_n \times \mathbb{Z}_n)^{(t_1, t_2, t_3)}$ is described in Appendix.

REMARK 2.5. We do not guarantee that $P_{\mathbf{t}}^1$ or $P_{\mathbf{t}}^2$ are minimal periods. For this reason, it is possible for a single solution to have both $P_{\mathbf{t}}^1$ and $P_{\mathbf{t}}^2$ as a period, in which case we only guarantee the existence of a single branch. This can occur in the case when $P_{\mathbf{t}}^1/P_{\mathbf{t}}^2 \in \mathbb{Q}$.

sign(δ)	sign(ζ)	sign(ε)	a_*	$\mathbf{t} = (t_1, t_2, t_3)$
-	-	-	0	(0, 0, 0)
+	-	-	$\delta(1 - \cos(\theta_N))$	$(\frac{N-1}{2}, 0, 0)$
-	+	-	$\zeta(1 - \cos(\theta_N))$	$(0, \frac{N-1}{2}, 0)$
-	-	+	$\varepsilon(1 - \cos(\theta_N))$	$(0, 0, \frac{N-1}{2})$
+	+	-	$(\delta + \zeta)(1 - \cos(\theta_N))$	$(\frac{N-1}{2}, \frac{N-1}{2}, 0)$
+	-	+	$(\delta + \varepsilon)(1 - \cos(\theta_N))$	$(\frac{N-1}{2}, 0, \frac{N-1}{2})$
-	+	+	$(\zeta + \varepsilon)(1 - \cos(\theta_N))$	$(0, \frac{N-1}{2}, \frac{N-1}{2})$
+	+	+	$(\delta + \zeta + \varepsilon)(1 - \cos(\theta_N))$	$(\frac{N-1}{2}, \frac{N-1}{2}, \frac{N-1}{2})$

TABLE 1. Details of the Hopf bifurcations from a stable equilibrium (here $\theta_N = (N-1)\pi/N$)

THEOREM 2.6. Put $\theta_N := (N-1)\pi/N$. Suppose $k_1\delta + k_2\zeta + k_3\varepsilon$ is less than

$$\frac{1}{2(\cos(\theta_N) - 1)}$$

for some $k_1, k_2, k_3 \in \{0, 1\}$. Then, all branches of bifurcating non-constant periodic solutions of (2.3) guaranteed by Theorem 2.3 are unstable.

THEOREM 2.7. *For any fixed δ, ζ and ε , the equilibrium of system (2.4) is stable for $a < a^*$ and unstable for $a > a^*$, where a^* is described in Table 1. Furthermore, the fully synchronized branch of system (2.4), born at $a = 0$, is stable if and only if δ, ζ and ε are negative.*

THEOREM 2.8. *Assume that $K_{\mathbf{t}} \neq 0$ for any $\mathbf{t} = (t_1, t_2, t_3)$ (cf. (2.5)). Choose $p \notin \{2\pi(2k-1)/\sqrt{bd} : k \in \mathbb{Z}, d > 0, d = K_{\mathbf{t}} \text{ for some } \mathbf{t}\}$. Then, for each \mathbf{t} with $K_{\mathbf{t}} > (2\pi/p)^2$, there exists a value of ν such that for each $(H^\varphi) \in \mathcal{S}(\mathbf{t})$, system (2.3) admits $8N^3/|H_1 \times H_2 \times H_3|$ p -periodic solutions with minimal symmetry (H^φ) (here $\mathcal{S}(\mathbf{t})$ is the set of symmetries associated to \mathbf{t} which is described in Subsection 3.3, formula (3.11)).*

3. Equivariant spectral data and first Lyapunov coefficient

3.1. Isotypical decomposition of the phase space. Although (2.3) and (2.4) have different symmetry groups, they have the same phase space (\mathbb{R}^{N^3}) as a geometric set. Since it is usually unambiguous, we will use the same notation for the representations of $G_1 := \mathbb{Z}_N \times \mathbb{Z}_N \times \mathbb{Z}_N$ and $G_2 := \mathbb{D}_N \times \mathbb{D}_N \times \mathbb{D}_N$. Put $V := \mathbb{R}^{N^3}$ and denote by $W := V \oplus V$ the phase space of systems (2.3) and (2.4). To describe spatial symmetries of system (2.4), we will consider $G_1 = \mathbb{Z}_N \times \mathbb{Z}_N \times \mathbb{Z}_N$ as a subgroup of $S^1 \times S^1 \times S^1$ and define the G_1 -action on V by specifying how each of its generators acts, namely:

$$\begin{aligned} ((e^{2\pi i/N}, 1, 1) \cdot x)_{(\alpha, \beta, \gamma)} &= x_{(\alpha+1, \beta, \gamma)}, \\ ((1, e^{2\pi i/N}, 1) \cdot x)_{(\alpha, \beta, \gamma)} &= x_{(\alpha, \beta+1, \gamma)}, \\ ((1, 1, e^{2\pi i/N}) \cdot x)_{(\alpha, \beta, \gamma)} &= x_{(\alpha, \beta, \gamma+1)}. \end{aligned}$$

Here $+$ is taken modulo N . By direct verification, the right-hand side of system (2.4) is G_1 -equivariant. To extend this action to a G_2 -action, we need to specify how the remaining generators act, namely:

$$\begin{aligned} ((\kappa, 1, 1) \cdot x)_{(\alpha, \beta, \gamma)} &= x_{(-\alpha, \beta, \gamma)}, \\ ((1, \kappa, 1) \cdot x)_{(\alpha, \beta, \gamma)} &= x_{(\alpha, -\beta, \gamma)}, \\ ((1, 1, \kappa) \cdot x)_{(\alpha, \beta, \gamma)} &= x_{(\alpha, \beta, -\gamma)}, \end{aligned}$$

where $-$ is again taken modulo N . It is clear that the right-hand side of (2.3) is G_2 -equivariant.

To describe the isotypical decomposition of V as a G_1 -space, we need to classify all (real) irreducible G_1 -representations. For each $0 \leq t_1, t_2, t_3 \leq N-1$, put $\mathbf{t} := (t_1, t_2, t_3)$ and denote by $\mathcal{V}_{\mathbf{t}}^z$ an irreducible representation of G_1 associated with \mathbf{t} . We have

$$(3.1) \quad \mathcal{V}_{0,0,0}^z = \mathbb{R}$$

is the trivial real G_1 -representation. For $(t_1, t_2, t_3) \neq (0, 0, 0)$, put

$$(3.2) \quad \mathcal{V}_{\mathbf{t}}^z = \mathbb{R}^2$$

and define the G_1 -action as follows:

$$(e^{2k_1\pi i/N}, e^{2k_2\pi i/N}, e^{2k_3\pi i/N}) \cdot \begin{bmatrix} x \\ y \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix},$$

where

$$A := \begin{bmatrix} \cos((k_1t_1 + k_2t_2 + k_3t_3)(2\pi/N)) & \sin((k_1t_1 + k_2t_2 + k_3t_3)(2\pi/N)) \\ -\sin((k_1t_1 + k_2t_2 + k_3t_3)(2\pi/N)) & \cos((k_1t_1 + k_2t_2 + k_3t_3)(2\pi/N)) \end{bmatrix}.$$

By direct verification, $\mathcal{V}_{\mathbf{t}}^z$ is G_1 -equivalent to $\mathcal{V}_{-\mathbf{t}}^z$ where $-$ is taken modulo N . Hence, there is one one-dimensional trivial representation and $(N^3 - 1)/2$ non-trivial two-dimensional non-equivalent G_1 -representations.

For each fixed \mathbf{t} , we define vectors $x_{\mathbf{t}}^1$ and $x_{\mathbf{t}}^2$ by specifying them component-wisely as follows:

$$\begin{aligned} (x_{\mathbf{t}}^1)_{\alpha, \beta, \gamma} &= \cos(\alpha t_1 + \beta t_2 + \gamma t_3), \\ (x_{\mathbf{t}}^2)_{\alpha, \beta, \gamma} &= \sin(\alpha t_1 + \beta t_2 + \gamma t_3). \end{aligned}$$

Define a family of subspaces of $V_{\mathbf{t}}^z$ by

$$V_{\mathbf{t}}^z = \text{span}(x_{\mathbf{t}}^1, x_{\mathbf{t}}^2).$$

Notice that $V_{\mathbf{t}}^z$ is a G_1 -irreducible component of V and is G_1 -equivalent to $\mathcal{V}_{\mathbf{t}}^z$ (cf. (3.1), (3.2)).

REMARK 3.1. V admits a primary G_1 -decomposition which includes every G_1 -irreducible representation.

Let us denote by \mathcal{U}_0^d the trivial one-dimensional real \mathbb{D}_N -representation and by \mathcal{U}_t^d the natural 2-dimensional real \mathbb{D}_N -representation, where the action is defined by

$$e^{2k\pi i/N} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos(kt(2\pi/N)) & \sin(kt(2\pi/N)) \\ -\sin(kt(2\pi/N)) & \cos(kt(2\pi/N)) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

and

$$\kappa \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}.$$

Denote $\mathcal{V}_{\mathbf{t}}^d := \mathcal{U}_{t_1}^d \otimes \mathcal{U}_{t_2}^d \otimes \mathcal{U}_{t_3}^d$. Since \mathcal{U}_t^d is of real type for any $t = 0, \dots, N$, it is easy to see that $\mathcal{V}_{\mathbf{t}}^d$ is a real *irreducible* $G_2 = (\mathbb{D}_N \times \mathbb{D}_N \times \mathbb{D}_N)$ -representation. Furthermore, the dimension of $\mathcal{V}_{\mathbf{t}}^d$ is either 1, 2, 4 or 8 depending on how many non-zero components \mathbf{t} has. Put $\mathbf{t}^\dagger := (t_1, -t_2, t_3)$, $\mathbf{t}^\# := (t_1, t_2, -t_3)$ and $\mathbf{t}^* := (t_1, t_2, -t_3)$. Put

$$V_{\mathbf{t}}^d := \text{span}(x_{\mathbf{t}}^1, x_{\mathbf{t}}^2, x_{\mathbf{t}^\dagger}^1, x_{\mathbf{t}^\dagger}^2, x_{\mathbf{t}^\#}^1, x_{\mathbf{t}^\#}^2, x_{\mathbf{t}^*}^1, x_{\mathbf{t}^*}^2).$$

By simple but lengthy computations, one can easily show that $V_{\mathbf{t}}^d$ is G_2 -invariant and equivalent to $\mathcal{V}_{\mathbf{t}}^d$.

REMARK 3.2. V admits a primary G_2 -decomposition, however, unlike its decomposition as a G_1 -space (cf. Remark 3.1), the decomposition as a G_2 -space does not include every G_2 -irreducible representation.

3.2. Equivariant spectral decomposition. The linearization of system (2.3) at the origin restricted to the isotypical component $V_{\mathbf{t}}^d \oplus V_{\mathbf{t}}^d$ is given by $A_{\mathbf{t}}^d(a): V_{\mathbf{t}}^d \oplus V_{\mathbf{t}}^d \rightarrow V_{\mathbf{t}}^d \oplus V_{\mathbf{t}}^d$, where

$$(3.3) \quad A_{\mathbf{t}}^d(a) = \begin{pmatrix} \nu a & -K_{\mathbf{t}} \\ b & 0 \end{pmatrix} \otimes \text{Id}_{\mathbf{t}},$$

$$(3.4) \quad K_{\mathbf{t}} = 1 + 2\delta(1 - \cos(2\pi t_1/N)) + 2\zeta(1 - \cos(2\pi t_2/N)) + 2\varepsilon(1 - \cos(2\pi t_3/N))$$

and $\text{Id}_{\mathbf{t}}$ is the matrix of the identity operator on $V_{\mathbf{t}}^d$ (here \otimes stands for the Kronecker product of matrices). It is clear that the eigenvalues of $A_{\mathbf{t}}^d(a)$ are given by

$$(3.5) \quad \lambda_{\mathbf{t}}^d(a) = \frac{\nu a \pm \sqrt{(\nu a)^2 - 4bK_{\mathbf{t}}}}{2}.$$

The linearization of system (2.4) at the origin restricted to the isotypical component $V_{\mathbf{t}}^z \oplus V_{\mathbf{t}}^z$ is given by $A_{\mathbf{t}}^z(a): V_{\mathbf{t}}^z \oplus V_{\mathbf{t}}^z \rightarrow V_{\mathbf{t}}^z \oplus V_{\mathbf{t}}^z$, where

$$A_{\mathbf{t}}^z(a) = \begin{pmatrix} H_{\mathbf{t}}(a) & G_{\mathbf{t}} & -1 & 0 \\ -G_{\mathbf{t}} & H_{\mathbf{t}}(a) & 0 & -1 \\ b & 0 & 0 & 0 \\ 0 & b & 0 & 0 \end{pmatrix},$$

$$H_{\mathbf{t}}(a) = a - \delta(1 - \cos(2\pi t_1/N)) - \zeta(1 - \cos(2\pi t_2/N)) - \varepsilon(1 - \cos(2\pi t_3/N))$$

and

$$(3.6) \quad G_{\mathbf{t}} = \delta \sin(2\pi t_1/N) + \zeta \sin(2\pi t_2/N) + \varepsilon \sin(2\pi t_3/N).$$

To compute the eigenvalues of $A_{\mathbf{t}}^z(a)$, we should notice that $A_{\mathbf{t}}^z(a) = {}^{\mathbb{R}}L_{\mathbf{t}}(a)$, where $L_{\mathbf{t}}(a)$ is the complex matrix

$$(3.7) \quad L_{\mathbf{t}}(a) = \begin{pmatrix} H_{\mathbf{t}}(a) - iG_{\mathbf{t}} & -1 \\ b & 0 \end{pmatrix}$$

(here the symbol ${}^{\mathbb{R}}L_{\mathbf{t}}$ stands for the realification of $L_{\mathbf{t}}$). It is well known that $\sigma({}^{\mathbb{R}}L_{\mathbf{t}}(a)) = \sigma(L_{\mathbf{t}}(a)) \cup \overline{\sigma(L_{\mathbf{t}}(a))}$. It is easy to see that the eigenvalues of $L_{\mathbf{t}}(a)$ are given by

$$(3.8) \quad \lambda_{\mathbf{t}}^z(a) = \frac{H_{\mathbf{t}}(a) - iG_{\mathbf{t}} \pm \sqrt{(H_{\mathbf{t}}(a) - iG_{\mathbf{t}})^2 - 4b}}{2}.$$

REMARK 3.3. For a generic choice of parameters $\gamma, \zeta, \varepsilon$, all eigenvalues $\lambda_{\mathbf{t}}^z(a)$ are distinct and in the case $\lambda_{\mathbf{t}}^z(a)$ is purely imaginary, not in resonance.

3.3. Isotropies in $(V_t^z)^c$ and $(V_t^d)^c$. Note. For an explanation of all symbols used in this section, see Appendix.

(a) Since (V_t^z) is a real irreducible G_1 -representation of complex type, $(V_t^z)^c$ decomposes into the direct sum of two non-equivalent conjugate irreducible $(G_1 \times S^1)$ -representations: $V_1 \oplus \bar{V}_1$. Since V_1 is one-dimensional, it has only two orbit types, namely $(G_1 \times S^1)$ and $(\mathbb{Z}_n \times \mathbb{Z}_n \times \mathbb{Z}_n)^{(t_1, t_2, t_3)}$. Similarly, \bar{V}_1 has only $G_1 \times S^1$ and $(\mathbb{Z}_n \times \mathbb{Z}_n \times \mathbb{Z}_n)^{(-t_1, -t_2, -t_3)}$.

(b) Let us recall the following

DEFINITION 3.4. We will call an orbit type $H < G \times S^1$ maximal in \mathcal{U}_j^c if for any $\tilde{H} \neq G \times S^1$ which is an orbit type in \mathcal{U}_j^c , one has $\tilde{H} < H$.

To restrict candidates for maximal isotropies in $(V_t^d)^c$, we use the following simple observation.

LEMMA 3.5. Let \mathcal{G}_1 (resp. \mathcal{G}_2) be a finite group and let U_1 (resp. U_2) be a unitary $\mathcal{G}_1 \times S^1$ -representation (resp. $G_2 \times S^1$ -representation), where S^1 acts on U_1 and U_2 by complex multiplication.

(a) If $H_1^{\varphi_1}$ is a twisted isotropy in U_1 and $H_2^{\varphi_2}$ is a twisted isotropy in U_2 , then $(H_1 \times H_2)^{(\varphi_1, \varphi_2)}$ is an isotropy in $U_1 \otimes U_2$.

(b) If $(H_1 \times_{K_1} H_2)^{(\varphi_1, \varphi_2)}$ is an isotropy in $U_1 \otimes U_2$, then for some $v_1 \in U_1$ and $v_2 \in U_2$, one has $\mathcal{G}_{v_1} \geq K_1^{\varphi_1}$ and $\mathcal{G}_{v_2} \geq K_2^{\varphi_2}$ (cf. (7.3)).

PROOF. (a) Take $v_1 \in (\mathcal{U}_1)_{H_1^{\varphi_1}}$ and $v_2 \in (\mathcal{U}_2)_{H_2^{\varphi_2}}$. Then, for any

$$g := (h_1, h_2, \varphi_1(h_1)\varphi_2(h_2)) \in (H_1 \times H_2)^{(\varphi_1, \varphi_2)},$$

one has $g(v_1 \otimes v_2) = v_1 \otimes v_2$, i.e. $\mathcal{G}_{v_1 \otimes v_2} \geq (H_1 \times H_2)^{(\varphi_1, \varphi_2)}$. On the other hand, if $\mathcal{G}_{v_1 \otimes v_2} \ni (h_1, h_2, e^{i\theta})$, then $(e^{i\theta}T_{h_1} \otimes T_{h_2})v_1 \otimes v_2 = v_1 \otimes v_2$, which implies that for some $\hat{\theta}$ and $\tilde{\theta}$ with $\theta = \hat{\theta} + \tilde{\theta}$ one has $e^{i\hat{\theta}}T_{h_1}v_1 = v_1$ and $e^{i\tilde{\theta}}T_{h_2}v_2 = v_2$. Hence, $(h_1, e^{i\hat{\theta}}) \in \mathcal{G}_{v_1} = H_1^{\varphi_1}$, $(h_1, e^{i\tilde{\theta}}) \in \mathcal{G}_{v_2} = H_2^{\varphi_2}$ and $(h_1, h_2, e^{i\theta}) = (h_1, h_2, e^{i\hat{\theta}} \cdot e^{i\tilde{\theta}}) \in (H_1 \times H_2)^{(\varphi_1, \varphi_2)}$, i.e. $\mathcal{G}_{v_1 \otimes v_2} \leq (H_1 \times H_2)^{(\varphi_1, \varphi_2)}$.

(b) Take $v \in (U_1 \times U_2)_{(H_1 \times H_2)^{(\varphi_1, \varphi_2)}}$ and decompose it as

$$v = \sum_{i=1}^n v_i \otimes e_i.$$

For any $g := (k_1, 1_{H_1}, \varphi_1(k_1)) \in (H_1 \times H_2)^{(\varphi_1, \varphi_2)}$, one has

$$g \cdot v = \sum_{i=1}^n e^{\varphi_i(k_1)} T_{h_1} v_i \otimes e_i = \sum_{i=1}^n v_i \otimes e_i,$$

hence, $v_i \in U_1^{K_1^{\varphi_1}}$ for any $i = 1, \dots, n$, i.e. $(\mathcal{G}_1)_{v_i} \leq H_1^{\varphi_1}$. A similar argument shows that $(\mathcal{G}_2)_{v_i} \leq H_2^{\varphi_2}$. \square

A direct consequence of Lemma 3.5 is the following:

COROLLARY 3.6. *Under the notations of Lemma 3.5, assume that $H_1^{\varphi_1}$ and $H_2^{\varphi_2}$ are maximal isotropies in U_1 and U_2 , respectively, and $N_{\mathcal{G}_i}(H_i) = H_i$. Then, $(H_1 \times H_2)^{(\varphi_1, \varphi_2)}$ is a maximal isotropy in the tensor product.*

PROOF. From Lemma 3.5 (a) it immediately follows that $(H_1 \times H_2)^{(\varphi_1, \varphi_2)}$ is an isotropy. Assume for contradiction that it is submaximal. Then

$$(3.9) \quad (H_1 \times H_2)^{(\varphi_1, \varphi_2)} \leq (\tilde{H}_{1K_1} \times_{K_2} \tilde{H}_2)^{(\tilde{\varphi}_1, \tilde{\varphi}_2)},$$

where

$$(3.10) \quad \tilde{H}_i \supset K_i > H_i$$

and $\tilde{\varphi}_i$ is an extension of φ_i to \tilde{H}_i . If $K_i \not\geq H_i$ then by Lemma 3.5 (b), $K_i^{\varphi_i}$ is contained in an isotropy, which contradicts maximality of $H_i^{\varphi_i}$. Since, by assumption, $N_{\mathcal{G}_i}(H_i) = H_i$ it follows from (3.10) that $\tilde{H}_i = H_i$ which contradicts (3.9). \square

Returning to the particular situation, where $\mathcal{G}_i = \mathbb{D}_N$ and $U_i = (\mathcal{U}_{t_i}^d)^c$, we have the following:

LEMMA 3.7. *If $H_1^{\varphi_1}, H_2^{\varphi_2}$ and $H_3^{\varphi_3}$ are maximal isotropies in $(\mathcal{U}_{t_1}^d)^c, (\mathcal{U}_{t_2}^d)^c$ and $(\mathcal{U}_{t_3}^d)^c$ respectively, then $(H_1 \times H_2 \times H_3)^{\varphi_1 \varphi_2 \varphi_3}$ is a maximal isotropy in $(\mathcal{V}_{\mathbf{t}}^d)^c$.*

PROOF. Let us begin by observing that the maximal orbit types in $(\mathcal{U}_{t_i}^d)^c$ are either $\mathbb{D}_N \times \{1\}$ in the case when $t_i = 0$, or $(\mathbb{Z}_N^{t_i}, (\mathbb{D}_1^+), (\mathbb{D}_1^-))$ if $t_i \neq 0$. By assumption, N is odd, therefore $N(\mathbb{D}_1) = \mathbb{D}_1$. This means that Lemma 3.5 and Corollary 3.6 exclude the possibility that $(H_1 \times H_2 \times H_3)^{\varphi_1 \varphi_2 \varphi_3}$ is not a maximal isotropy except in the case when $(H_1 \times H_2 \times H_3)^{\varphi_1 \varphi_2 \varphi_3} = (\mathbb{Z}_N \times \mathbb{Z}_N \times \mathbb{Z}_N)^{t_1 t_2 t_3}$. However, it follows from Lemma 3.5 that the only candidate for a twisted subgroup of $\mathbb{D}_N \times \mathbb{D}_N \times \mathbb{D}_N \times S^1$, which is an isotropy in $(\mathcal{V}_{\mathbf{t}}^d)^c$ and contains $(\mathbb{Z}_N \times \mathbb{Z}_N \times \mathbb{Z}_N)^{t_1 t_2 t_3}$, is of the form \mathcal{H}^φ , where

$$\mathcal{H} = \{(g_1, g_2, g_3) \in \mathbb{D}_N \times \mathbb{D}_N \times \mathbb{D}_N : \psi(g_1) = \psi(g_2) = \psi(g_3)\}$$

(here $\psi: \mathbb{D}_N \rightarrow \mathbb{Z}_2$ is the homomorphism with kernel \mathbb{Z}_N). On the other hand, it can be easily seen that any vector $x \in (\mathcal{V}_{\mathbf{t}}^d)^c$ which is fixed by $(\mathbb{Z}_N \times \mathbb{Z}_N \times \mathbb{Z}_N)^{t_1 t_2 t_3}$ cannot be fixed by the element $(\kappa, \kappa, \kappa, e^{i\theta})$ for any $e^{i\theta} \in S^1$, hence $(\mathbb{Z}_N \times \mathbb{Z}_N \times \mathbb{Z}_N)^{t_1 t_2 t_3}$ is also maximal. \square

Being motivated by Lemma 3.7, we introduce the following notations:

$$S(t_i) = \begin{cases} \{(\mathbb{D}_N \times \{1\})\} & \text{if } t_i = 0, \\ \{(\mathbb{Z}_N^{t_i}, (\mathbb{D}_1^+), (\mathbb{D}_1^-))\} & \text{if } t_i \neq 0 \end{cases}$$

and

$$(3.11) \quad \mathcal{S}(\mathbf{t}) = \{(H_1 \times H_2 \times H_3)^{\varphi_1 \varphi_2 \varphi_3} : (H_i^{\varphi_i}) \in S(t_i)\}.$$

3.4. Stability analysis of system (2.2). In this subsection, we analyze stability of Hopf branches of periodic solutions to system (2.2). This information will be used later for the stability analysis of the fully synchronized Hopf branches of periodic solutions to system (2.4).

LEMMA 3.8. *For the parameter a crossing zero, system (2.2) undergoes a supercritical Hopf bifurcation.*

PROOF. By inspection, the eigenvalues of the linearization of (2.2) at the origin have the form $\lambda = (a \pm \sqrt{a^2 - 4b})/2$, in particular, the Hopf bifurcation takes place when a crosses zero. To analyze the character of the bifurcation, take the linear change of coordinates:

$$(3.12) \quad \begin{cases} \tilde{x} = \frac{a}{\sqrt{4b-a^2}}x - \frac{2}{\sqrt{4b-a^2}}y, \\ \tilde{y} = x, \end{cases} \quad \Rightarrow \quad \begin{cases} \dot{\tilde{x}} = \frac{a}{2}\tilde{x} - \frac{\sqrt{4b-a^2}}{2}\tilde{y} - \frac{a\tilde{y}^2 - a\tilde{y}^3}{\sqrt{4b-a^2}}, \\ \dot{\tilde{y}} = \frac{\sqrt{4b-a^2}}{2}\tilde{x} + \frac{a}{2}\tilde{y} - \tilde{y}^2 - \tilde{y}^3. \end{cases}$$

By direct computation, near $a = 0$, one has:

$$(3.13) \quad \frac{\partial}{\partial a}(\operatorname{Re}(\lambda)) = \frac{1}{2} > 0 \quad \text{and} \quad l_1 = -\frac{3}{8} < 0,$$

where l_1 stands for the first Lyapunov coefficient. Combining (3.13) with [11], Theorem 3.4.2, completes the proof. \square

4. Occurrence of Hopf bifurcations

4.1. Abstract result. Let G be a finite group and U be an orthogonal G -representation which admits a G -isotypical decomposition

$$(4.1) \quad U = U_0 \oplus \dots \oplus U_k,$$

where U_j is modeled on the irreducible representation \mathcal{U}_j . We will denote by \mathcal{U}_j^c the complexification of U_j which is a $G \times S^1$ -representation.

Suppose $f: \mathbb{R} \oplus U \rightarrow U$ is a C^1 -smooth function and consider the system

$$(4.2) \quad \dot{x}(t) = f(\alpha, x).$$

DEFINITION 4.1. We will say that $(\alpha_0, 0)$ is an *isolated center* with limit frequency β_0 of (4.2) if:

- (a) $(\alpha_0, 0)$ is a center of (4.2) with limit frequency β_0 , that is $D_x f(\alpha_0, 0)$ admits $i\beta_0$ as a purely imaginary eigenvalue;
- (b) $(\alpha_0, 0)$ is the only center in a neighborhood of $(\alpha_0, 0)$ in $\mathbb{R} \oplus U$.

We are now in a position to formulate the abstract occurrence result which we will apply to the system considered in this paper.

THEOREM 4.2 (cf. [4]). *Suppose f in system (4.2) satisfies the following conditions:*

- (P1) f is a C^1 -smooth equivariant map (we assume G acts trivially on \mathbb{R});
- (P2) $f(\alpha, 0) = 0$ for all $\alpha \in \mathbb{R}$;
- (P3) $(\alpha_0, 0)$ is an isolated center for (4.2) (cf. Definition 4.1);
- (P4) $\det D_x f(\alpha_0, 0) \neq 0$;
- (P5) $D_x f(\alpha_0, 0)|_{U_j}$ decreases stability as α passes α_0 , while the stability of $D_x f(\alpha_0, 0)|_{U_k}$ does not increase for any U_k (cf. (4.1)).

Then, for every maximal orbit type (H^φ) in \mathcal{U}_j^c , there exist $|(G \times S^1)/H^\varphi|_{S^1}$ branches of non-constant periodic solutions of (4.2) bifurcating from the origin with (spatio-temporal) symmetry (H^φ) and limit period $2\pi/\beta_0$ (cf. Definition 2.2). Here $|(G \times S^1)/H^\varphi|_{S^1}$ is the number of S^1 -orbits in the space $(G \times S^1)/H$.

REMARK 4.3. Using the concept of isotypical crossing number one can relax condition (P5) (see, for example, [4], [3], [13]).

4.2. Proofs of Theorems 2.3 and 2.4. To detect the occurrence of the equivariant Hopf bifurcation in systems (2.3) and (2.4) and to classify symmetric properties of the resulting branches, we will combine the equivariant spectral data collected in Subsections 3.1–3.3 with Theorem 4.2.

PROOF OF THEOREM 2.3. We begin by observing that conditions (P1) and (P2) are obvious. It follows immediately from (3.3)–(3.5) and $b > 0$ that system (2.3) can only have a center $(a, 0)$ when $a = 0$. Also, since $b, K_t > 0$, formula (3.3) implies (P4). Finally, all eigenvalues cross the imaginary axis in the same direction (see (3.5)), meaning that (P5) is also satisfied. Combining this with the description of isotropies in Subsection 3.3(a) completes the proof of Theorem 2.3. \square

PROOF OF THEOREM 2.4. Observe that (P1) and (P2) are obvious. Plugging $\lambda = i\omega$ into the characteristic equation of matrix $L_t(a)$ (cf. (3.7)) and separating real and imaginary parts yields that all centers $(a_t^z, 0)$ of system (2.4) are given by (2.6). From this (P3) follows immediately, while (P4) is provided by $b > 0$. Finally, differentiating (3.8) with respect to a at $a = a_t^z$ shows that all eigenvalues cross the imaginary axis in the same direction. Combining this with the description of isotropies in Subsection 3.3(b) (in particular, formula (3.11)) completes the proof of Theorem 2.4. \square

5. Stability of bifurcating branches

PROOF OF THEOREM 2.6. Recall that if the equilibrium is unstable then any bifurcating branch of non-constant periodic solutions will also be unstable. Since the eigenvalues of the linearization of system (2.3) at the origin are given by $\sqrt{-bK_{\mathbf{t}}}$, it is easy to see that if $K_{\mathbf{t}}$ is negative for some \mathbf{t} , then the theorem is proved. Take k_1, k_2, k_3 provided by the hypothesis of Theorem 2.6. Clearly, $K_{\mathbf{t}} < 0$ for $\mathbf{t} = (k_1(n-1)/2, k_2(n-1)/2, k_3(n-1)/2)$ which completes the proof. \square

PROOF OF THEOREM 2.7. It is clear from formulas (3.6) and (3.8) that the sign of $\operatorname{Re}(\lambda_{\mathbf{t}}^z(a))$ is given by the sign of $H_{\mathbf{t}}(a)$. Observe that the largest value of $H_{\mathbf{t}}(a)$ is achieved by the value of \mathbf{t} specified in the Table 1 and changes its sign at $a = a^*$. Therefore, for any \mathbf{t} and $a < a^*$, one has $\operatorname{Re}(\lambda_{\mathbf{t}}^z(a)) < 0$. Therefore, for $a < a^*$, the equilibrium is stable while for $a > a^*$, the equilibrium is unstable.

In the case when $\delta, \zeta, \varepsilon < 0$, only one pair of eigenvalues crosses the imaginary axis at $a^* = 0$. Observe that the central manifold coincides with the central space. To complete the proof we combine Lemma 3.8 with the fact that all other eigenvalues have negative real part (cf. [11], Theorem 3.4.2). \square

6. Existence of periodic solutions with prescribed period and symmetry

Abstract result. To prove Theorem 2.8, we will use a slight modification of the main result from [4], Chapter 12 (cf. Theorem 12.7). To begin with, we need the following

DEFINITION 6.1. Let G be a finite group and let V be a real orthogonal G -representation. Assume $A: V \rightarrow V$ is an equivariant linear operator and $\lambda \in \sigma(A^c)$. Take the eigenspace $E(\lambda) \subset V^c$ and denote by $\mathcal{O}(\lambda)$ the set of all maximal $G \times S^1$ -orbit types occurring in $E(\lambda)$.

Let $V := \mathbb{R}^n$ be an orthogonal permutational G -representation and consider the system

$$(6.1) \quad \begin{cases} \dot{x}_i = \nu \left(ax_i - \frac{x_i^3}{3} \right) - \sum_{j=1}^n C_{ij} y_j, \\ \dot{y}_i = bx_i, \end{cases} \quad (i = 1, \dots, n).$$

THEOREM 6.2. Assume C is a non-singular G -equivariant symmetric matrix. Then, for each real positive $p \notin \{2\pi(2k-1)/\sqrt{\mu} : \mu \in \sigma^+(bC), k \in \mathbb{N}\}$ and each $\mu \in \sigma(bC)$ satisfying $0 < (2\pi/p)^2 < \mu$, there exists $\nu > 0$ such that for every $(H^\varphi) \in \mathcal{O}(\mu)$, system (6.1) admits $|G|/|H|$ p -periodic solutions with minimal symmetry (H^φ) .

PROOF OF THEOREM 2.8. The linearization of system (6.1) at the origin has the form:

$$\mathfrak{A} = \begin{bmatrix} \nu a \operatorname{Id} & -C \\ b \operatorname{Id} & 0 \end{bmatrix}$$

where Id stands for the $n \times n$ -identity matrix. Since in the case of system (2.3), the isotypical decomposition of V contains only irreducible representations of real type, then $\mathfrak{A}|_{V_i}$ is of the form

$$\begin{bmatrix} \nu a & -\lambda \\ b & 0 \end{bmatrix} \otimes \operatorname{Id}|_{V_i}$$

where λ is an eigenvalue of $C|_{V_i}$. It follows from Remark 3.2 and formula (3.3) that

$$\sigma(C) = \{K_t : 0 < t_i < (N-1)/2\}.$$

Also note that $\mathcal{O}(K_t) = \mathcal{S}(t)$. Combining this with Theorem 6.2 completes the proof of Theorem 2.8. \square

7. Appendix

If W is a G -representation, then for any function $x: S^1 \rightarrow W$ the spatio-temporal symmetry of x is a group $\mathfrak{H} < G \times S^1$ such that $g \cdot x(t - \theta) = x(t)$ for any $t \in \mathbb{R}/2\pi\mathbb{Z} \simeq S^1$ and any $(g, e^{i\theta}) \in \mathfrak{H}$. If x is non-constant, then its spatio-temporal symmetry group will have the structure of a graph of a homomorphism from some subgroup $H < G$ to S^1 . In our general discussion, we used the following notation:

$$H^\varphi := \{(h, \varphi(h)) : h \in H\}.$$

We call this a *twisted* symmetry group with twisting homomorphism φ . If the domain of the twisting homomorphism is a direct product of groups, we can describe the twisting homomorphism by its restrictions to each of the components. Therefore, we use the following notation:

$$(7.1) \quad (H_1 \times H_2 \times H_3)^{\varphi_1 \varphi_2 \varphi_3} := \{(h_1, h_2, h_3, \varphi_1(h_1)\varphi_2(h_2)\varphi_3(h_3)) : (h_1, h_2, h_3) \in H_1 \times H_2 \times H_3\}.$$

Given two groups \mathcal{G}_1 and \mathcal{G}_2 , to describe subgroups of $\mathcal{G}_1 \times \mathcal{G}_2$, define the projection homomorphisms:

$$\begin{aligned} \pi_1: \mathcal{G}_1 \times \mathcal{G}_2 &\rightarrow \mathcal{G}_1, & \pi_1(g_1, g_2) &= g_1; \\ \pi_2: \mathcal{G}_1 \times \mathcal{G}_2 &\rightarrow \mathcal{G}_2, & \pi_2(g_1, g_2) &= g_2. \end{aligned}$$

The following result, being a reformulation of the well-known Goursat's Lemma (cf. [10]), provides the desired description of subgroups \mathcal{H} of the product group $\mathcal{G}_1 \times \mathcal{G}_2$.

THEOREM 7.1. *Let \mathcal{H} be a subgroup of the product group $\mathcal{G}_1 \times \mathcal{G}_2$. Put $H := \pi_1(\mathcal{H})$ and $K := \pi_2(\mathcal{H})$. Then, there exist a group L and two epimorphisms $\varphi: H \rightarrow L$ and $\psi: K \rightarrow L$, such that*

$$(7.2) \quad \mathcal{H} = \{(h, k) \in H \times K : \varphi(h) = \psi(k)\}.$$

For the needs of our paper, it is enough to characterize such subgroups up to the kernels of the homomorphisms φ and ψ . For this reason, we put

$$(7.3) \quad \mathcal{H} =: H_{1K_1} \times_{K_2} H_2$$

where K_1 is the kernel of φ and K_2 is the kernel of ψ .

In the particular case of \mathbb{D}_N , we denote by \mathbb{Z}_N the subgroup generated by $e^{2\pi i/N}$, and by \mathbb{D}_1 the subgroup generated by κ . Furthermore, we put

$$\begin{aligned} \mathbb{Z}_N^{t_i} &:= \mathbb{Z}_N^\varphi \quad \text{where } \varphi: e^{k2\pi i/N} \mapsto e^{kt_i2\pi i/N}, \\ \mathbb{D}_1^+ &:= \mathbb{D}_1^\varphi \quad \text{where } \varphi: \kappa \mapsto 1, \\ \mathbb{D}_1^- &:= \mathbb{D}_1^\varphi \quad \text{where } \varphi: \kappa \mapsto -1. \end{aligned}$$

We combine this with the notation for twisted groups given in (7.1) in the obvious way, for example:

$$(\mathbb{Z}_N \times \mathbb{D}_1 \times \mathbb{Z}_N)^{(t_1, -, t_2)} := (\mathbb{Z}_N \times \mathbb{D}_1 \times \mathbb{Z}_N)^{\varphi_1 \varphi_2 \varphi_3},$$

where $\varphi_1: e^{k2\pi i/N} \mapsto e^{kt_12\pi i/N}$, $\varphi_2: \kappa \mapsto -1$ and $\varphi_3: e^{k2\pi i/N} \mapsto e^{kt_22\pi i/N}$.

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