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PERIODIC SOLUTIONS OF vdP AND vdP-LIKE SYSTEMS ON 3-TORI

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Abstract. Van der Pol equation (in short, vdP) as well as many its non-symmetric generalizations (the so-called van der Pol-like oscillators (in short, vdPl)) serve as nodes in coupled networks modeling real-life phenomena. Symmetric properties of periodic regimes of networks of vdP/vdPl depend on symmetries of coupling. In this paper, we consider N^3 identical vdP/vdPl oscillators arranged in a cubical lattice, where opposite faces are identified in the same way as for a 3-torus. Depending on which nodes impact the dynamics of a given node, we distinguish between $\mathbb{D}_N \times \mathbb{D}_N \times \mathbb{D}_N$-equivariant systems and their $\mathbb{Z}_N \times \mathbb{Z}_N \times \mathbb{Z}_N$-equivariant counterparts. In both settings, the local equivariant Hopf bifurcation together with the global existence of periodic solutions with prescribed period and symmetry, are studied. The methods used in the paper are based on the results rooted in both equivariant degree theory and (equivariant) singularity theory.

1. Introduction

1.1. Subject and goal. The van der Pol equation (in short, vdP, cf. (2.1)), originally introduced in [16] to study stable oscillations in electrical circuits, is

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very often considered as a starting point of applied nonlinear dynamics. An important feature of equation (2.1) is that it respects the antipodal symmetry. Different generalizations of vdP, which break this symmetry, have been considered by many authors in connection to a wide range of applied problems (in what follows, we will call these generalizations van der Pol-like equations (in short, vdPl)). Examples of particular importance include the FitzHugh–Nagumo model (see, for example, [14]) and the realistic kinetic model of the chlorite-iodide-malonic acid reaction (see, for example, [7]). To be more concrete about the importance of considering vdPl systems with quadratic terms, we refer to [1] and the references therein.

In real life models, vdP as well as vdPl serve as nodes in coupled networks. Symmetries of the coupling have an impact on the symmetries of the actual dynamics. In this paper, we will consider N^3 oscillators arranged in a cubical lattice, where opposite faces are identified in the same way as for a 3-torus. In such a configuration, two aspects of the coupling are important: (i) which nodes impact the dynamics of a given node (which we will call *coupling topology*), and (ii) how a neighbouring node impacts a given node (which we will call *coupling structure*). For the coupling topology, we consider the following two cases: (i) all six neighbors of a given node impact on that node's dynamics (we call such a coupling *bi-directional*); (ii) only three neighbors of a given node impact on that node's dynamics (we call such a coupling *uni-directional*). In the case of bi-directional coupling, the system respects a natural action of $\mathbb{D}_N \times \mathbb{D}_N \times \mathbb{D}_N$, while in the case of uni-directional coupling the symmetry generated by κ is destroyed, hence the total symmetry group is $\mathbb{Z}_N \times \mathbb{Z}_N \times \mathbb{Z}_N$ (cf. [5] and references therein). We will distinguish between two linear coupling structures, namely, for a given node, either the x-variable of a neighbor or the y-variable of a neighbor is coupled to the x variable of the specified node (compare (2.3) with (2.4)). We will call these x-coupling and y-coupling, respectively.

The *goal* of this paper is three-fold, namely, in the settings introduced above, we will: (i) establish the *occurrence* of the Hopf bifurcation, classify symmetric properties of the bifurcating branches and estimate their number; (ii) study *stability* of the corresponding periodic solutions, and (iii) investigate the *existence* of periodic solutions with prescribed period and symmetry.

1.2. Results. Keeping in mind a wide spectrum of potential applications in natural sciences and engineering, it is worthy to study the above mentioned problems (occurrence, stability and existence) in all possible settings. The usual dilemma of keeping a balance between Scylla of completeness and Charybdis of reasonable size of the manuscript, resulted in our paper as follows:

(i) Although, using the methods developed in this paper, the occurrence/multiplicity estimates/symmetry classification of the Hopf bifurcation can be estab-