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MULTIPLE POSITIVE SOLUTIONS FOR A CLASS OF VARIATIONAL SYSTEMS

Alfonso Castro — David Costa — Ratnasingham Shivaji

Abstract. We consider the variational system $-\Delta u = \lambda(\nabla F)(u)$ in Ω, $u = 0$ on $\partial\Omega$, where Ω is a bounded region in $\mathbb{R}^m$ $(m \geq 1)$ with C^1 boundary, λ is a positive parameter, $u\colon \Omega \to \mathbb{R}^N$ $(N > 1)$, and Δ denotes the Laplace operator. Here $F\colon \mathbb{R}^N \to \mathbb{R}$ is a function of class C^2. Using variational methods, we show how changes in the sign of F lead to multiple positive solutions.

1. Introduction

We study the existence of positive solutions to the variational system

$$\begin{cases} -\Delta u = \lambda(\nabla F)(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded region in $\mathbb{R}^m$ $(m \geq 1)$ with C^1 boundary, λ is a positive parameter, $u\colon \Omega \to \mathbb{R}^N$ $(N > 1)$, and Δ denotes the Laplacian operator. Here $F\colon \mathbb{R}^N \to \mathbb{R}$ is a function of class C^2. By a positive solution to (1.1) we mean a function $u = (u_1, \ldots, u_N)$ with each $u_j \in C^2(\Omega) \cap C^1(\overline{\Omega})$, $u_j(x) \geq 0$; Ω, $u_j(x) = 0$; $\partial\Omega$ and $u_l(x_0) > 0$ for some $l \in \{1, \ldots, N\}$, $x_0 \in \Omega$.

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Let $\Sigma = \{z = (z_1, \dots, z_N) \mid z_1 + \dots + z_N = 1$ with $z_i \geq 0$ for all $i = 1, \dots, N\}$ and $Q = \{z = (z_1, \dots, z_N) \mid z_i > 0$ for all $i = 1, \dots, N\}$. For $z \in \mathbb{R}^N$ we will denote $|z|_1 = |z_1| + \dots + |z_N|$ and $P(z) = z/|z|_1$, for $z \neq 0$. We assume:

(H1) $F(z_1, \dots, z_N) = 0$ if $z_1 \dots z_N = 0$, and $F(z_1, \dots, z_N) \leq 0$ if $z_i \leq 0$ for some $i = 1, \dots, N$.

(H2) There exist concave functions $\rho_1, \dots, \rho_{2k} \colon \Sigma \to [0, \infty)$ such that

$$0 < \rho_i(z) < \rho_{i+1}(z), \quad i = 1, \dots, 2k-1.$$

(H3) For $z \in Q$, $F(z) > 0$ if $|z|_1 \in (\rho_{2i+1}(P(z)), \rho_{2(i+1)}(P(z)))$, and $F(z) < 0$ if $|z|_1 \in (\rho_{2i}(P(z)), \rho_{2i+1}(P(z)))$ for $i = 0, 2, \dots, k-1$ or $|z|_1 \geq \rho_{2k}(P(z))$.

(H4) For $i = 0, \dots, k-1$ there exists $z^{(i)} \in Q$ such that $|z^{(i)}|_1 \in (\rho_{2i+1}(P(z^{(i)})), \rho_{2(i+1)}(P(z^{(i)})))$ and $F(z^{(i)}) > \max\{F(z) \mid |z|_1 < \rho_{2i+1}(P(z^{(i)}))\}$.

Our main result is the following:

Theorem 1.1. *There exist $\lambda_1 < \dots < \lambda_k$ such that if $\lambda > \lambda_i$, for $i = 1, \dots, k-1$, then the boundary value problem* (1.1) *has i positive solutions.*

In the single equation case ($N = 1$) there is a rich history on the study of such boundary value problems where the analysis of how the changes of sign of the nonlinear term give rise to the existence of multiple positive solutions. In particular see Brown–Budin [1] where a combination of variational and monotone iteration methods is used, Hess [8] where a combination of variational and topological degree arguments is applied, and De Figueiredo [7] where only variational methods are used. Clement and Sweers in [3] proved that if $f \colon [0, \infty) \to \mathbb{R}$ satisfies

$$\begin{aligned} f(s) &< 0 \quad \text{for } 0 < s < s_1 \text{ or } s > s_2, \\ f(s) &> 0 \quad \text{for } s_1 < s < s_2, \end{aligned} \tag{1.2}$$

for some $0 < s_1 < s_2$, then the (possibly *semipositone*) problem

$$-\Delta u = \lambda f(u) \quad \text{in } \Omega, \qquad u(x) = 0 \quad \text{for } x \in \partial\Omega, \tag{1.3}$$

has a positive solution u for λ large with $\|u\|_\infty \in (s_1, s_2)$ if and only if

$$\int_0^{s_2} f(s)\, ds > 0. \tag{1.4}$$

Also, Dancer and Schmitt in [5] used sub-supersolutions to show that (1.4) is necessary for existence of a positive solution to (1.3). In [6], De Figueiredo showed that for $f(u) = \sin(u)$ and $\lambda > 0$ in (1.3) this equation has only one positive solution. However, to date no extension of this study has been achieved for the system case (when $N > 1$), which we establish in this paper via variational methods.

For related results on 2×2 systems the reader is referred to [2] and [4]. In [2], extensive use of regularity properties of elliptic operators and the Krasnosel'skiĭ compression-expansion theorem is made to prove the existence of positive solutions. In [4], variational methods are employed in proving existence of solutions for non-cooperative systems.

In Section 2 we establish auxiliary lemmas needed for the proof of Theorem 1.1 in Section 3.

2. Auxiliary results

For $j=1,\dots,k$ let S_j be the closure of $\{z\in Q \mid |z|_1\le \rho_{2j}(P(z))\}$. Also, let S_0 be the closure of $\{z\in Q \mid |z|_1\le \rho_1(P(z))\}$. From (H2) it follows that S_j is convex. We will denote by P_j the projection of $\mathbb{R}^N$ onto the convex set S_j. Hence P_j is locally Lipschitzian. Let

$$F_j(u) := \begin{cases} 0 & \text{if } u\notin S_j, \\ F(u) & \text{if } u\in S_j. \end{cases} \tag{2.1}$$

Since F is differentiable, F_j is locally Lipschitzian and the functional

$$J_{j,\lambda}\colon H=(H_0^{1,2}(\Omega))^N\to\mathbb{R}$$

defined by

$$J_{j,\lambda}(u)=\int_\Omega\bigg(\frac12\sum_{i=1}^N|\nabla u_i|^2-\lambda F_j(u)\bigg)\,dx \tag{2.2}$$

attains its minimum value. Indeed, $J_{j,\lambda}$ is bounded below because F_j is bounded. If $\{J_{j,\lambda}(v_k)\}$ converges to $\inf\{J_{j,\lambda}(u)\mid u\in H\}$ then $\{v_k\}$ is bounded in H. Hence it has a subsequence (call it $\{v_k\}$ again) that converges in $L^2(\Omega)$ to some $\widehat{u}_j$. Hence $\{\int_\Omega F_j(v_k)\,dx\}$ converges to $\int_\Omega F_j(\widehat{u}_j)\,dx$. This and the convexity of the norm gives $J_{j,\lambda}(\widehat{u}_j)=\min\{J_{j,\lambda}(u)\mid u\in H\}$.

Lemma 2.1. *If $\widehat{u}_j:=(\widehat{u}_{j,1},\dots,\widehat{u}_{j,N})$ is as above, then $\widehat{u}_j(\zeta)\in\overline{S}_j$ for all $\zeta\in\Omega$.*

Proof. For the sake of simplicity in the notation, throughout this proof we denote $\widehat{u}_j=u$, $\widehat{u}_{j,l}=u_l$, and $J_{j,\lambda}=J$. Suppose first that $v=(u_l)_-$ is not zero for some $l\in\{1,\dots,N\}$. Hence, there exists $c>0$ such that

$$\int_{\Omega_c}|\nabla v|^2\,dx<\int_\Omega|\nabla v|^2\,dx,$$

where $\Omega_c=\{x\in\Omega\mid v(x)\le c\}$. Let $w=\min\{v,c\}$. Thus

$$w=(u_1,\dots,u_{l-1},v,u_{l+1},\dots,u_N)\in H.$$

By the definition of F_j we have $F_j(u(x)) = F_j(w(x))$ for all $x \in \Omega$. Therefore,

$$\begin{aligned} J(w) &= \frac{1}{2}\int_\Omega \sum_{k=1,\, k\neq l}^{N} |\nabla u_k|^2\, dx + \frac{1}{2}\int_\Omega |\nabla v|^2\, dx - \lambda \int_\Omega F_j(w)\, dx \\ &< \frac{1}{2}\int_\Omega \sum_{k=1}^{N} |\nabla u_k|^2\, dx - \lambda \int_\Omega F_j(u)\, dx = J(u), \end{aligned}$$

which contradicts the definition of u. Hence $u_l \geq 0$ for all $l = 1, \ldots, N$.

Suppose next that the Lebesgue measure of $W = \{x \in \Omega \mid u(x) \notin S_j\}$ is positive. Since S_j is convex there exist $\eta_1 \in \mathbb{R}^N$, $\mu \in (0, +\infty)$, such that the Lebesgue measure of $W_1 = \{x \in \Omega \mid u(x) \cdot \eta_1 > \mu\}$ is positive. Hence

$$\int_{W_1} \sum_{l=1, i=1}^{N,N} \left(\frac{\partial u_l}{\partial x_i}(x)\right)^2 dx > 0. \tag{2.3}$$

Let $w(x) = u(x) - ((u(x) \cdot \eta_1) - \mu)\eta_1$ for $(u(x) \cdot \eta_1) \geq \mu$ and $w(x) = u(x)$ for $(u(x)\eta_1) \leq \mu$. Thus $w \in H$. Since W_1 is a subset of the complement of S_j, $F_j(w(x)) = F_j(u(x))$ for all $x \in \Omega$. Let $\eta_2, \ldots, \eta_N$ be such that $\{\eta_1, \eta_2, \ldots, \eta_N\}$ is a complete orthonormal set in $\mathbb{R}^N$. Since

$$u(x) = (u(x) \cdot \eta_1)\eta_1 + \ldots + (u(x) \cdot \eta_N)\eta_N,$$

we have for $i = 1, \ldots, N$ that

$$\frac{\partial u}{\partial x_i}(x) = \left(\frac{\partial u}{\partial x_i}(x) \cdot \eta_1\right)\eta_1 + \ldots + \left(\frac{\partial u}{\partial x_i}(x) \cdot \eta_N\right)\eta_N.$$

Also, from the definition of w, for $x \in W_1$ and $i = 1, \ldots, N$ we have

$$\frac{\partial w}{\partial x_i}(x) = \left(\frac{\partial u}{\partial x_i}(x) \cdot \eta_2\right)\eta_2 + \ldots + \left(\frac{\partial u}{\partial x_i}(x) \cdot \eta_N\right)\eta_N.$$

Hence, letting $w_1, \ldots, w_N$ denote the components of w,

$$\sum_{l=1,k=1}^{N,N} \left(\frac{\partial w_l}{\partial x_i}(x)\right)^2 dx = \sum_{l=1,k=2}^{N,N} \left(\frac{\partial u}{\partial x_i} \cdot \eta_k\right)^2,$$

for al $x \in W_1$. Since W_1 is a set of positive measure,

$$\int_{W_1} \sum_{i=1}^{N} \left(\frac{\partial \widehat{u}_j}{\partial x_i} \cdot \eta_1\right)^2 dx > 0.$$

Thus

$$\int_{W_1} \sum_{i=1,k=1}^{N,N} \left(\frac{\partial u_k}{\partial x_i}\right)^2 dx = \int_{W_1} \sum_{i=1}^{N} \left|\frac{\partial u}{\partial x_i}\right|^2 dx \tag{2.4}$$

$$= \int_{W_1} \sum_{i=1,k=1}^{N,N} \left(\frac{\partial u}{\partial x_i}\cdot \eta_k\right)^2 dx > \int_{W_1} \sum_{i=1,k=2}^{N,N} \left(\frac{\partial u}{\partial x_i}\cdot \eta_k\right)^2 dx$$

$$= \int_{W_1} \sum_{i=1}^{N} \left|\frac{\partial w}{\partial x_i}\right|^2 dx = \int_{W_1} \sum_{i=1,k=1}^{N,N} \left(\frac{\partial w_k}{\partial x_i}\right)^2 dx.$$

Also, from the definition of w,

$$\int_{W_1} (\nabla w_i \cdot \eta_1)^2\, dx = 0$$

for all $i = 1, \ldots, N$. Letting $W_2 = \Omega - W_1$, we have by (2.4) that

$$J(w) = \int_{W_2} \left(\frac{1}{2}\sum_{k=1}^{N} |\nabla w_k|^2 - \lambda F_j(w)\right) dx + \int_{W_1} \left(\frac{1}{2}\sum_{k=1}^{N} |\nabla w_k|^2\right) dx$$

$$= \int_{W_2} \left(\frac{1}{2}\sum_{k=1}^{N} |\nabla u_k|^2 - \lambda F_j(\widehat{u}_j)\right) dx + \frac{1}{2}\int_{W_1} \sum_{i,k=1}^{N} \left(\frac{\partial u_i}{\partial x_k}\right)^2 dx$$

$$< \frac{1}{2}\int_{\Omega} \sum_{k=1}^{N} |\nabla u_k|^2\, dx - \lambda \int_{\Omega} F_j(\widehat{u}_j)\, dx = J(\widehat{u}_j),$$

Since this contradicts the definition of u, (2.3) is impossible. Therefore the Lebesgue measure of W is zero, which proves the lemma. □

Lemma 2.2. *If* $\widehat{u}_j := (\widehat{u}_{j,1}, \ldots, \widehat{u}_{j,N})$ *is as above, then* $\widehat{u}_j$ *is a solution to system* (1.1).

Proof. Let $\phi\colon \Omega \to \mathbb{R}^N$ be a function of compact support and class C^∞. By (H1) and Lemma 2.1, for $|t|$ sufficiently small we have

$$F_j((\widehat{u}_j + t\phi)(\zeta)) \geq F((\widehat{u}_j + t\phi)(\zeta))$$

for all $\zeta \in \Omega$. Therefore,

$$I_\phi(t) = J_\lambda(\widehat{u}_j + t\phi) = \frac{1}{2}\int_{\Omega} |\nabla(\widehat{u}_j + t\phi)|^2\, d\zeta - \lambda \int_{\Omega} F(\widehat{u}_j + t\phi)\, d\zeta$$

$$\geq \frac{1}{2}\int_{\Omega} |\nabla(\widehat{u}_j + t\phi)|^2\, d\zeta - \lambda \int_{\Omega} F_j(\widehat{u}_j + t\phi)\, d\zeta = J_{j,\lambda}(\widehat{u}_j) = J_\lambda(\widehat{u}_j) = I_\phi(0),$$

where

$$J_\lambda(v) = \frac{1}{2}\int_{\Omega} |\nabla(v)|^2\, d\zeta - \lambda \int_{\Omega} F(v)\, d\zeta.$$

Therefore I_ϕ has a local minimum at $t = 0$. Thus $0 = I'_\phi(0) = \langle \nabla J_\lambda(\widehat{u}_j), \phi\rangle$, which proves that $\widehat{u}_j$ is a weak solution to (1.1). Since ∇F is bounded in $\overline{S}_j$, by elliptic regularity for second order equations, $\widehat{u}_j$ is actually a solution to (1.1).□

3. Proof of Theorem 1.1

Proof. By Lemma 2.1, it is sufficient to show that

(A) There exists λ_i, $i = 1, \ldots, k$, such that $\lambda_1 < \ldots < \lambda_i$ and $J_{i,\lambda}(\widehat{u}_i) < \ldots < J_{2,\lambda}(\widehat{u}_2) < J_{1,\lambda}(\widehat{u}_1) < 0$ for $\lambda > \lambda_i$.

Let $N(x)$ denote the inward unit normal to $\partial\Omega$ at $x \in \partial\Omega$. Since $\partial\Omega$ is C^1 there exists $\varepsilon > 0$ such that $\{x + sN(x) \mid s \in [0, \varepsilon)\}$ is an open neighborhood of $\partial\Omega$ relatively to $\overline{\Omega}$ and $x + sN(x) \neq y + tN(y)$ for $(x, s) \neq (y, t)$, $x, y \in \partial\Omega$, $s, t \in [0, \varepsilon)$. Letting $\Omega_\delta = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) < \delta\}$ with $\delta \in (0, \varepsilon)$, we see that there exists a constant $M > 0$ such that

$$|\Omega_\delta| \leq M\delta. \tag{3.1}$$

Let $\widehat{u}_j$ be as in Lemma 2.1. Let us first see that there exists $\lambda_1 > 0$ such that $J_{1,\lambda}(\widehat{u}_1) < 0$ for $\lambda > \lambda_1$. Using (H4) there exists $z^{(1)} \in S_1$ such that $F(z^{(1)}) > \max\{F(u) \mid u \in S_0\}$. Let $c = F(z^{(1)}) - \max\{F(u) \mid u \in S_0\}$. Now fix $\delta \in (0, \min\{\varepsilon, c|\Omega|/(2M[c + M_1])\})$ where $M_1 = \max\limits_{u \in S_1} |F(u)|$ and let v be defined by:

$$v(x) = \begin{cases} z^{(1)} & \text{if } \operatorname{dist}(x, \partial\Omega) \geq \delta, \\ \dfrac{s}{\delta}\, z^{(1)} & \text{if } \operatorname{dist}(x, \partial\Omega) < \delta,\ x = y + sN(y),\ y \in \partial\Omega,\ s \in [0, \delta). \end{cases} \tag{3.2}$$

Since $\partial\Omega$ is C^1, v is a Lipschitzian function and there exists $K > 0$ such that

$$\int_\Omega \frac{1}{2}\sum_{i=1}^N |\nabla v_i|^2\, dx \leq K\delta^{-1}. \tag{3.3}$$

Then

$$\begin{aligned} J_{1,\lambda}(v) &= \int_\Omega \bigg(\frac{1}{2}\sum_{i=1}^N |\nabla v_i|^2 - \lambda F_1(v)\bigg)\, dx \\ &\leq K\delta^{-1} - \lambda\bigg(\int_{\Omega - \Omega_\delta} F_1(z^{(1)})\, dx + \int_{\Omega_\delta} F_1(v)\, dx\bigg) \\ &\leq K\delta^{-1} - \lambda[c(|\Omega| - |\Omega_\delta|) - |\Omega_\delta| M_1] \\ &\leq K\delta^{-1} - \lambda[c|\Omega| - cM\delta - M\delta M_1] \leq K\delta^{-1} - \lambda\frac{c|\Omega|}{2} \end{aligned}$$

since $\delta < c|\Omega|/(2M[c + M_1]) < 0$ for $\lambda > \lambda_1 = 2K/(c|\Omega|\delta)$. Hence $J_{1,\lambda}(\widehat{u}_1) < 0$ for $\lambda > \lambda_1$.

Next we show that there exists $\widetilde{\lambda}_2 > \lambda_1$ such that $J_{1,\lambda}(\widehat{u}_1) > J_{2,\lambda}(\widehat{u}_2)$ for $\lambda > \widetilde{\lambda}_2$. Using (H4) there exists $z^{(2)} \in S_2$ such that $F(z^{(2)}) > \max\{F(u) \mid u \in S_1\}$.

Now let $c = F(z^{(2)}) - \max\{F(u) \mid u \in S_1\}$. Fix $\delta \in (0, \min\{\varepsilon, (c|\Omega|/(2M[c + M_1 + M_2])\})$ where $M_2 = \max_{u \in S_1} |F(u)|$ and let

$$v(x) = \begin{cases} z^{(2)} & \text{if } \operatorname{dist}(x, \partial\Omega) \geq \delta, \\ \dfrac{s}{\delta} z^{(2)} & \text{if } \operatorname{dist}(x, \partial\Omega) < \delta,\ x = y + sN(y),\ y \in \partial\Omega,\ s \in [0, \delta). \end{cases} \tag{3.4}$$

Again, using that $\partial\Omega$ is C^1, we have a constant $K > 0$ such that

$$\int_\Omega \frac{1}{2} \sum_{i=1}^N |\nabla v_i|^2 \, dx \leq K\delta^{-1}.$$

Then

$$\begin{aligned} J_{2,\lambda}(v) &= \int_\Omega \left(\frac{1}{2} \sum_{i=1}^N |\nabla v_i|^2 - \lambda F_2(v) \right) dx \\ &\leq K\delta^{-1} - \lambda \left(\int_{\Omega - \Omega_\delta} F_2(z^{(2)}) \, dx + \int_{\Omega_\delta} F_2(v) \, dx \right) \\ &\leq K\delta^{-1} - \lambda \left(\int_{\Omega - \Omega_\delta} [F_1(\widehat{u}_1) + c] \, dx + \int_{\Omega_\delta} F_2(v) \, dx \right) \\ &= K\delta^{-1} - \lambda \left(\int_\Omega F_1(\widehat{u}_1) \, dx + c[|\Omega| - |\Omega_\delta|] + \int_{\Omega_\delta} [F_2(v) - F_1(\widehat{u}_1)] \, dx \right) \\ &< K\delta^{-1} + J_{1,\lambda}(\widehat{u}_1) - \lambda\{c[|\Omega| - |\Omega_\delta|] - |\Omega_\delta|[M_1 + M_2]\} \\ &< K\delta^{-1} + J_{1,\lambda}(\widehat{u}_1) - \lambda\{c|\Omega| - cM\delta - M\delta[M_1 + M_2]\} \\ &< K\delta^{-1} + J_{1,\lambda}(\widehat{u}_1) - \lambda \frac{c|\Omega|}{2} \quad \text{since } \delta < \frac{c|\Omega|}{2M[c + M_1 + M_2]} \\ &< J_{1,\lambda}(\widehat{u}_1) \quad \text{for } \lambda > \widetilde{\lambda}_2 = \frac{2K}{c|\Omega|\delta}. \end{aligned}$$

Hence $J_{2,\lambda}(\widehat{u}_2) < J_{1,\lambda}(\widehat{u}_1)$ for $\lambda > \widetilde{\lambda}_2$. Now choosing $\lambda_2 = \max\{\lambda_1, \widetilde{\lambda}_2\}$, for $\lambda > \lambda_2$ we have $J_{2,\lambda}(\widehat{u}_2) < J_{1,\lambda}(\widehat{u}_1) < 0$. Iterating this argument k times, (A) is proven and so is Theorem 1.1. □

Finally, in order to state a consequence of Theorem 1.1, we consider the following extensions of (H2)–(H4):

$\widehat{\text{(H2)}}$ There exist concave functions $\rho_i \colon \Sigma \to [0, \infty)$ such that $0 < \rho_i(z) < \rho_{i+1}(z)$ for $i \in \mathbb{N}$.

$\widehat{\text{(H3)}}$ For $z \in Q$, assume $F(z) > 0$ if $|z|_1 \in (\rho_{2i+1}(P(z)), \rho_{2(i+1)} l(P(z)))$, and $F(z) < 0$ if $|z|_1 \in (\rho_{2i}(P(z)), \rho_{2i+1}(P(z)))$ for each $i = 0, 1, \ldots$

$\widehat{\text{(H4)}}$ For each $i = 0, 1, \ldots$ there exists $z^{(i)} \in Q$ such that

$$|z^{(i)}|_1 \in (\rho_{2i+1}(P(z^{(2(i+1))})), \rho_{2(i+1)}(P(z^{(i)})))$$

and $F(z^{(i)}) > \max\{F(z) \mid |z|_1 < \rho_{2i+1}(P(z^{(i)}))\}$.

The following result is an immediate corollary of Theorem 1.1:

Theorem 3.1. *Assume* (H1) *and* $\widehat{(\mathrm{H2})}$–$\widehat{(\mathrm{H4})}$. *If we let* $N(\lambda)$ *denote the number of positive solutions of* (1.1) *then* $N(\lambda) \to \infty$ *as* $\lambda \to \infty$.

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Alfonso Castro
Department of Mathematics
Harvey Mudd College
Claremont, CA 91711, USA

E-mail address: castro@math.hmc.edu

David Costa
Department of Mathematics
University of Nevada Las Vegas
Las Vegas, NV 89154, USA

E-mail address: costa@unlv.nevada.edu

Ratnasingham Shivaji
Department of Mathematics and Statistics
University of North Carolina at Greensboro
Greensboro, NC 27402, USA

E-mail address: shivaji@uncg.edu