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MULTIPLE POSITIVE SOLUTIONS FOR FRACTIONAL ELLIPTIC SYSTEMS INVOLVING SIGN-CHANGING WEIGHT

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Abstract. We study multiplicity results for positive solutions for a fractional elliptic system involving both concave-convex and critical growth terms. With the help of Nehari manifold and Ljusternik–Schnirelmann category, we investigate how the coefficient h of the critical nonlinearity affects the number of positive solutions to this problem and get a relationship between the number of positive solutions and the topology of the global maximum set of h.

1. Introduction and the main result

In this paper, we are concerned with the number of positive solutions to the following fractional elliptic system:

$$(\mathrm{E}_{f,g})\qquad \begin{cases} (-\Delta)^{s/2}u = f(x)|u|^{q-2}u + \dfrac{\alpha}{\alpha+\beta}\, h(x)|u|^{\alpha-2}u|v|^{\beta} & \text{in } \Omega, \\ (-\Delta)^{s/2}v = g(x)|v|^{q-2}v + \dfrac{\beta}{\alpha+\beta}\, h(x)|u|^{\alpha}|v|^{\beta-2}v & \text{in } \Omega, \\ u = v = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where Ω is a bounded set in $\mathbb{R}^N$ with smooth boundary, $N > s$ with $s \in (0,2)$ fixed, $1 < q < 2$, $\alpha, \beta > 1$ satisfy $\alpha + \beta = 2_s^* = 2N/(N-s)$, 2_s^* is the fractional

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Sobolev critical exponent, and $(-\Delta)^{s/2}$ is the fractional Laplacian. Moreover, f, g, h are continuous functions.

In recent years, problems involving fractional operators have received a special attention since they have important applications in many sciences. We limit here ourselves with a non-exhaustive list of fields and papers in which these operators are used: obstacle problem [20], [26], optimization and finance [2], [13], phase transition [1], [28], material science [4], anomalous diffusion [18], [19], conformal geometry and minimal surfaces [5], [7], [8]. The list may continue with applications in crystal dislocation, soft thin films, multiple scattering, quasi-geostrophic flows, water waves, and so on. The interested reader may consult also references in the cited papers. Set $\alpha+\beta=p\le 2_s^*$, $f(x)\equiv g(x)$, $h(x)\equiv 1$ and $u=v$, then $(\mathrm{E}_{f,g})$ reduces to the following fractional elliptic equation with concave-convex nonlinearities:

$$(\mathrm{E}_\lambda)\qquad \begin{cases}(-\Delta)^{s/2}u=\lambda|u|^{q-2}u+|u|^{p-2}u & \text{in } \Omega,\\ u=0 & \text{on } \mathbb{R}^N\setminus\Omega.\end{cases}$$

Goyal and Sreenadh [16] studied the existence and multiplicity of positive solutions to (E_λ). Moreover, involving the Nehari manifold and Fibering maps, Chen and Deng [9] obtained the existence of multiple solutions to (E_λ) for the subcritical and critical cases. For the fractional Laplacian system with concave-convex nonlinearities, He, Squassina, and Zou [17] proved that $(\mathrm{E}_{\lambda,\mu})$ ($(\mathrm{E}_{f,g})$ with $f(x)\equiv\lambda$ and $g(x)\equiv\mu$) possesses at least two positive solutions when λ and μ are small enough. Similar results were achieved by Chen and Deng [10]. Their tool was the decomposition of the Nehari manifold.

There are several existence results for the following problem:

$$\varepsilon^s(-\Delta)^{s/2}u+V(x)u=f(u),\quad x\in\mathbb{R}^N, \tag{1.1}$$

where ε is a positive parameter, f has a subcritical growth, V possesses a local minimum. For $\varepsilon=1$, we would like to cite [22], [24] for the existence of one positive solution imposing a global condition on V. For ε a small positive constant, several scholars established existence and concentration of positive solutions to (1.1), by imposing different conditions on V and f (see [23], [14], [29], [15]). In particular, with the help of Nehari manifold and Lusternik–Schnirelmann category, Figueiredo and Siciliano [15] obtained a relationship between the number of positive solutions and the topology of the minimum set of V.

An interesting question now is how the weight potential h of a critical term affects the number of positive solutions to $(\mathrm{E}_{f,g})$ involving critical nonlinearity and sign-changing weight potentials. Furthermore, we wonder if there is a similar relationship between the number of positive solutions to $(\mathrm{E}_{f,g})$ and the topology of the global maximum set of h as that in [15]. To state our main results, we introduce precise conditions on f,g and h: