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SOME PROPERTIES OF SETS, FIXED POINT THEOREMS IN ORDERED PRODUCT SPACES AND APPLICATIONS TO A NONLINEAR SYSTEM OF FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. We study a partial order in product spaces and then present some new properties of sets via the partial order. Based on these properties and monotone iterative technique, we establish some new fixed point theorems in product spaces. As an application, we utilize the main fixed point theorem to study a nonlinear system of fractional differential equations. We get the existence-uniqueness of positive solutions for this system, which complements the existing results of positive solutions for this nonlinear problem in the literature.

1. Introduction

During the past several decades, nonlinear functional analysis has been an active area of research. As an important content of nonlinear functional analysis, nonlinear operator theory has attracted much attention and has been widely studied (see for example [1]–[3], [6], [7], [10], [16], [18], [27], [31]). As we know, nonlinear operator theory is an important theoretical foundation and basic tool of nonlinear sciences, and it is a research field of modern mathematics which

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has profound theories and extensive applications. Nonlinear operator theory has been extensively used to study nonlinear differential equations, integral equations, matrix equations and boundary value problems, etc. In most cases, we deal with these nonlinear problems by means of some fixed point theorems for single operator. However, multiple operators in product spaces can be regarded as an integration, this motivates to consider multiple operators and establish some nonlinear operator theories in product spaces. Let us mention the following papers concerned with fixed point theorems in product spaces: Fora [9], Kuczumow [14], Tan and Xu [21], Ding et al. [8], Wiśnicki [23], Kohlenbach and Leustean [13]. These operators' results have not been widely utilized to study equation problems. The reason is that the conditions are difficult to verify for particular operators.

In this article, we first study one partial order in product spaces, and then present some properties of sets. Using these properties and monotone iterative technique, we establish some new fixed point theorems in product spaces. Here we mainly consider the following operator equation:

$$(x,y) = (A(x,y), B(x,y)). \tag{1.1}$$

Motivated by our works [27], [31], we will establish some existence and uniqueness results of positive solutions for operator equation (1.1), which extend the results of [31] to some degree.

In the last section of this paper, we study a nonlinear system of fractional differential equations. We give the existence-uniqueness of positive solutions for this system, which complements the existence results of positive solutions for this nonlinear problem. Moreover, we note that our main fixed point theorems can be applied easily to many nonlinear problems.

2. Preliminaries and one partial order

Let E be a linear space on a scalar field K. Then the product space $E \times E$ is a linear space with

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2),$$
$$\lambda(x,y) = (\lambda x, \lambda y), \quad \lambda \in K.$$

If E is a Banach space with the norm $\|\cdot\|_E$, then the product space $E \times E$ is also a Banach space with the norm

$$\|(u,v)\|_{E\times E} = \|u\|_E + \|v\|_E, \quad (u,v) \in E \times E; \tag{2.1}$$

or

$$\|(u,v)\|_{E\times E} = \max\{\|u\|_E, \|v\|_E\}, \quad (u,v) \in E \times E. \tag{2.2}$$

Moreover, the norms (2.1) and (2.2) are equivalent.